The ε–uniform convergence of the discrete derivatives for singularly perturbed problems*

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Abstract. *The derivatives of the solution of singularly perturbed differential equations become unbounded as the singular perturbation parameter ε tends to zero. Therefore to approximate such derivatives, it is required to scale the derivatives in such a way that they are of order one for all values of the perturbation parameter. In practice, derivatives are related to the flux or drag and, hence, it is desirable to have ε–uniform approximations to the scaled derivatives. In this paper, singularly perturbed convection–diffusion problems are considered. The use of standard scaled discrete derivatives to approximate the scaled continuous derivatives of the solution of singularly perturbed problems is examined. Standard scaled discrete derivatives generated from exact numerical methods on a uniform mesh are shown to be not ε–uniformly convergent. On the other hand, standard scaled discrete derivatives computed from a numerical method based on an appropriately fitted piecewise–uniform mesh are shown to be ε–uniformly convergent. Numerical results are presented and discussed to illustrate the significance of these theoretical results.*

1 Introduction

To illustrate the problem we consider the following singularly perturbed constant coefficient problem

$$\varepsilon u_\varepsilon'' + a u_\varepsilon' = f, \quad x \in (0,1), \qquad 0 < \varepsilon \le 1, \tag{1a}$$

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(1) = 0 \tag{1b}$$

where a and f are positive constants. In addition to finding an approximation to the solution u_ε of a differential equation, an accurate approximation to the scaled first

*This research was supported in part by the National Science Foundation grant DMS-9627244, by the Enterprise Ireland grant SC-98-612 and by the Russian Foundation for Basic Research under grant No. 98-01-00362.

derivative $\varepsilon u'_\varepsilon$ of the solution is often required. An obvious first choice is the scaled discrete derivative $\varepsilon D^+ U_\varepsilon$. For problem (1) we have

$$\varepsilon u'_\varepsilon(x) = \frac{f}{a}\Big(\varepsilon - a\frac{e^{-ax/\varepsilon}}{1-e^{-a/\varepsilon}}\Big)$$

and for any finite difference scheme, which is exact at the nodes, we have

$$\varepsilon D^+ U_\varepsilon(x_i) = \frac{f}{a}\Big(\varepsilon - a\frac{e^{-ax_i/\varepsilon}}{1-e^{-a/\varepsilon}}(\frac{1-e^{-\rho_i}}{\rho_i})\Big) \quad \text{with} \quad \rho_i = a(x_{i+1}-x_i)/\varepsilon$$

where $\{U_\varepsilon(x_i)\}_{i=0}^N$ is the solution of the finite difference scheme. Thus, on a uniform mesh, the error in these scaled quantities at the point $x = 0$ is

$$\varepsilon|D^+ U_\varepsilon(0) - u'_\varepsilon(0)| = \frac{f}{1-e^{-a/\varepsilon}}\big(1 - \frac{1-e^{-\rho}}{\rho}\big), \quad \text{where} \quad \rho = a/N\varepsilon.$$

Taking $\varepsilon N = a$, and letting $N \to \infty$ we get

$$\lim_{N\to\infty} \varepsilon|D^+ U_\varepsilon(0) - u'_\varepsilon(0)| = fe^{-1} \neq 0.$$

In other words, for this constant coefficient problem, using a uniform mesh, the scaled discrete derivative $\varepsilon D^+ U_\varepsilon$ does not converge ε–uniformly to $\varepsilon u'_\varepsilon$ as the mesh is refined, even though the underlying numerical method is exact at the mesh points. Note also that, on a non–uniform mesh

$$\varepsilon|D^+ u_\varepsilon(x_i) - u'_\varepsilon(x_i)| = \frac{fe^{-ax_i/\varepsilon}}{1-e^{-a/\varepsilon}}\Big(1 - \frac{1-e^{-\rho_i}}{\rho_i}\Big) \leq C\frac{x_{i+1}-x_i}{\varepsilon}e^{-ax_i/\varepsilon}$$

and so if $x_1 = \varepsilon\psi(N)$, where $\psi(N) \to 0$ as $N \to \infty$, the scaled discrete derivative $\varepsilon D^+ u_\varepsilon(0)$ converges ε–uniformly to $\varepsilon u'_\varepsilon(0)$.

2 Convergence results

More generally, we now consider the following class of singularly perturbed problems

$$L_\varepsilon u_\varepsilon \equiv \varepsilon u''_\varepsilon + a(x)u'_\varepsilon = f(x) \quad x \in \Omega = (0,1), \tag{2a}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B, \tag{2b}$$

$$a, f \in C^2(\Omega), \quad a(x) > \alpha > 0, \quad x \in \overline{\Omega}. \tag{2c}$$

The solution of any such problem can be decomposed into a sum of smooth and singular components of the form

$$u_\varepsilon = v_\varepsilon + w_\varepsilon$$

where v_ε is the solution of

$$L_\varepsilon v_\varepsilon = f, \quad v_\varepsilon(0) = v_0(0) + \varepsilon v_1(0),\ v_\varepsilon(1) = u_\varepsilon(1) \tag{3a}$$

with $av_0' = f$, $v_0(1) = u_\varepsilon(1)$, $av_1' = -v_0''$, $v_1(1) = 0$ and consequently w_ε is the solution of the homogeneous problem

$$L_\varepsilon w_\varepsilon = 0, \quad w_\varepsilon(0) = u_\varepsilon(0) - v_\varepsilon(0),\ w_\varepsilon(1) = 0. \tag{3b}$$

The components $v_\varepsilon, w_\varepsilon$ and their derivatives satisfy the bounds (see, for example, [1])

$$\|v_\varepsilon^{(k)}\| \leq C(1+\varepsilon^{2-k}), \quad k = 0,1,2,3, \tag{4a}$$

$$|w_\varepsilon^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\alpha x/\varepsilon}, \quad k = 0,1,2,3, \quad x \in \Omega. \tag{4b}$$

To solve problem (2) we use numerical methods of the form

$$L_\varepsilon^N U_\varepsilon \equiv \varepsilon\delta^2 U_\varepsilon + a(x_i)D^+U_\varepsilon = f(x_i), \quad x_i \in \Omega_\varepsilon^N, \tag{5a}$$

$$U_\varepsilon(0) = u_\varepsilon(0), \quad U_\varepsilon(1) = u_\varepsilon(1), \tag{5b}$$

where $\overline{\Omega}_\varepsilon^N$ is the piecewise–uniform fitted mesh defined by

$$\overline{\Omega}_\varepsilon^N = \{x_i | x_i = ih,\ i \leq N/2;\ x_i = x_{i-1} + H,\ N/2 < i\} \tag{5c}$$

with $h = 2\sigma/N$, $H = 2(1-\sigma)/N$ and

$$\sigma = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\}. \tag{5d}$$

We note that $h \leq N^{-1},\ H \leq 2N^{-1}$. Just as the solution u_ε of the continuous problem (2) can be decomposed, the discrete solution U_ε of (5) can be decomposed into the sum $U_\varepsilon = V_\varepsilon + W_\varepsilon$, where V_ε and W_ε are the solutions of the problems

$$L_\varepsilon^N V_\varepsilon = f(x_i), \quad x_i \in \Omega_\varepsilon^N, \qquad V_\varepsilon(0) = v_\varepsilon(0), \quad V_\varepsilon(1) = v_\varepsilon(1), \tag{6a}$$

$$L_\varepsilon^N W_\varepsilon = 0, \quad x_i \in \Omega_\varepsilon^N, \qquad W_\varepsilon(0) = w_\varepsilon(0), \quad W_\varepsilon(1) = 0. \tag{6b}$$

The error in the numerical solutions can then be written in the form $U_\varepsilon - u_\varepsilon = \big(V_\varepsilon - v_\varepsilon\big) + \big(W_\varepsilon - w_\varepsilon\big)$.

Lemma 1 *At each mesh point $x_i \in \Omega_\varepsilon^N$ the smooth component of the error satisfies the estimate*

$$|\big(V_\varepsilon - v_\varepsilon\big)(x_i)| \leq CN^{-1}(1-x_i)$$

where v_ε and V_ε, respectively, is the solution of (3a) and (6a).

Proof. Use the following bound on the local truncation error

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i)| \le \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})\|v_\varepsilon^{(3)}\| + \frac{\|a\|}{2}(x_{i+1}-x_i)\|v_\varepsilon^{(2)}\| \le CN^{-1}$$

and the two mesh functions $\psi^\pm(x_i) = CN^{-1}(1-x_i) \pm (V_\varepsilon - v_\varepsilon)(x_i)$ to complete the proof with the discrete minimum principle.

Lemma 2 *For all $N \ge 4$, the singular component of the error satisfies the estimate*

$$\text{(a)} \quad |\big(W_\varepsilon - w_\varepsilon\big)(x_i)| \le CN^{-1}\ln N, \quad x_i \in \Omega_\varepsilon^N$$

where w_ε is the solution of (3b) and W_ε is the solution of (6b) and, in the case of $\sigma = \frac{\varepsilon}{\alpha}\ln N$,

$$\text{(b)} \quad |W_\varepsilon(x_i)| \le CN^{-1}(1-x_i), \quad x_i \ge \sigma.$$

Proof. (a) The proof follows the argument given in [1], except that a more sophisticated barrier function is used in the subinterval $[0,\sigma]$ to get a sharper error bound. Consider first the case when $\sigma = \frac{\varepsilon}{\alpha}\ln N$. For all $x_i \in [\sigma, 1]$, it is shown in [1] that $|W_\varepsilon(x_i)| \le CN^{-1}$ and $|w_\varepsilon(x_i)| \le CN^{-1}$. For all $x_i \in (0,\sigma)$, we use the bounds on the derivatives and a standard local truncation error estimate to obtain

$$\begin{aligned} |L_\varepsilon^N(W_\varepsilon - w_\varepsilon)(x_i)| &\le \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})\|w_\varepsilon^{(3)}\|_{[x_{i-1},x_{i+1}]} + \frac{\|a\|}{2}(x_{i+1}-x_i)\|w_\varepsilon^{(2)}\|_{[x_{i-1},x_{i+1}]} \\ &\le C\sigma\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/\varepsilon}. \end{aligned}$$

We note also that $(W_\varepsilon - w_\varepsilon)(0) = 0$ and $|(W_\varepsilon - w_\varepsilon)(\sigma)| \le CN^{-1}$. Instead of the linear barrier functions used in [1], we define the mesh functions $\Psi^\pm$ on the subinterval $[0,\sigma]$ by

$$\Psi^\pm(x_i) = C_1\sigma e^{2\gamma h/\varepsilon}\varepsilon^{-1}N^{-1}Y_i + CN^{-1} \pm (W_\varepsilon - w_\varepsilon)(x_i)$$

where Y is the solution of the constant coefficient problem

$$\varepsilon\delta^2 Y + \gamma D^+ Y = 0, \; x_i \in \Omega_\varepsilon^N \cap (0,\sigma)$$

with the boundary conditions $Y_0 = 1$ and $Y_{N/2} = 0$; and γ is an arbitrary parameter such that $0 < \gamma < \alpha$. It is easy to check that

$$L_\varepsilon^N\Psi^\pm(x_i) \le C_1(a(x_i)-\gamma)\varepsilon D^+Y_i\sigma\varepsilon^{-2}N^{-1}e^{2\gamma h/\varepsilon} + C\sigma\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/(2\varepsilon)} \le 0.$$

From the discrete minimum principle, it follows that

$$|W_\varepsilon(x_i) - w_\varepsilon(x_i)| \le C_1\sigma\varepsilon^{-1}N^{-1}e^{2\gamma h/\varepsilon}Y_0 + CN^{-1} \le CN^{-1}\ln N$$

as required. For the case $\sigma = 1/2$, apply the argument used above for the subinterval $[0, \sigma]$ to the entire interval $[0, 1]$, and observe, from the definition of σ, that, for all values of ε and N,

$$\frac{\sigma}{\varepsilon} \leq \frac{\ln N}{\alpha} .$$

To prove (b) use the mesh functions

$$\psi^{\pm}(x_i) = |W_\varepsilon(\sigma)| \frac{1 - x_i}{1 - \sigma} \pm W_\varepsilon(x_i)$$

on the interval $[\sigma, 1]$ to get the required bound on $|W_\varepsilon(x_i)|$, for all $i \geq N/2$.

Combining the estimates in Lemmas 1 and 2 gives an ε–uniform error estimate at each point of the mesh Ω_ε^N. As in [1], we can extend these results to each point of the domain Ω, which leads to the global ε–uniform error estimate

$$\sup_{0<\varepsilon\leq 1} \|\bar{U}_\varepsilon - u_\varepsilon\|_{\overline{\Omega}} \leq CN^{-1} \ln N \qquad (7)$$

where $\bar{U}_\varepsilon$ is the piecewise linear interpolant of U_ε over Ω and C is a constant independent of N and ε.

We now establish the ε–uniform convergence of the scaled discrete derivatives to the corresponding scaled derivatives of the solution u_ε.

Theorem 1 *At each mesh point $x_i \in \Omega_\varepsilon^N \cup \{0\}$*

$$\|\varepsilon D^+ u_\varepsilon(x_i) - \varepsilon u'_\varepsilon\|_{[x_i, x_{i+1}]} \leq CN^{-1} \ln N,$$

where u_ε is the solution of (2).

Proof. Note that, for all $x_i \in \Omega_\varepsilon^N \cup \{0\}$,

$$\|D^+ v_\varepsilon(x_i) - v'_\varepsilon\|_{[x_i, x_{i+1}]} \leq C(x_{i+1} - x_i)\|v''_\varepsilon\|_{[x_i, x_{i+1}]} \leq CN^{-1} \qquad (8)$$

and, when $\sigma = 1/2$, we see from the argument leading to (8) that

$$\varepsilon\|D^+ w_\varepsilon(x_i) - w'_\varepsilon\|_{[x_i, x_{i+1}]} \leq CN^{-1} \ln N. \qquad (9)$$

For all $x_i \in [0, \sigma)$, we have that

$$\varepsilon\|D^+ w_\varepsilon(x_i) - w'_\varepsilon\|_{[x_i, x_{i+1}]} \leq C(x_{i+1} - x_i)\varepsilon\|w''_\varepsilon\|_{[x_i, x_{i+1}]} \leq C\sigma(\varepsilon N)^{-1} \leq CN^{-1} \ln N.$$

Also, for all $x_i \in [\sigma, 1)$, the local truncation error can be written in the form

$$D^+ w_\varepsilon(x_i) - w'_\varepsilon(x) = \frac{1}{x_{i+1} - x_i} \int_{s=x_i}^{x_{i+1}} \int_{t=x_i}^{s} w''_\varepsilon(t) \text{ d}t \text{ d}s - \int_{t=x_i}^{x} w''_\varepsilon(t) \text{ d}t.$$

Then, using the fact that $L_\varepsilon w_\varepsilon = 0$ and integration by parts, we obtain

$$\begin{aligned}\varepsilon \int_{t=x_i}^{s} w''_\varepsilon(t)\ \mathrm{d}t &= -\int_{t=x_i}^{s} a(t) w'_\varepsilon(t)\ \mathrm{d}t = \int_{t=x_i}^{s} a'(t) w_\varepsilon(t)\ \mathrm{d}t - a(t) w_\varepsilon(t)\Big|_{t=x_i}^{s} \\ &\leq C\|w_\varepsilon\|_{[x_i, x_{i+1}]}\ .\end{aligned}$$

Note also that, when $\sigma = \frac{\varepsilon}{\alpha}\ln N$, we have for all $x \geq \sigma$

$$|w_\varepsilon(x)| \leq |w_\varepsilon(\sigma)| \leq CN^{-1}.$$

Combining these inequalities completes the proof.

Lemma 3 *At each mesh point $x_i \in \Omega^N_\varepsilon \cup \{0\}$*

$$|\varepsilon D^+\big(V_\varepsilon(x_i) - v_\varepsilon(x_i)\big)| \leq CN^{-1},$$

where v_ε is the solution of (3a) and V_ε is the solution of (6a).

Proof. Denote the error and the local truncation error, respectively, at each mesh point by

$$e_i = V_\varepsilon(x_i) - v_\varepsilon(x_i) \quad \text{and} \quad \tau_i = L^N_\varepsilon e_i.$$

From Lemma 1 $|e_{N-1}| \leq CN^{-2}$ and, since $e_N = 0$,

$$|\varepsilon D^- e_N| = \varepsilon\Big|\frac{e_N - e_{N-1}}{H}\Big| \leq C\varepsilon N^{-1}. \tag{10}$$

For mesh points in the coarse mesh, using the expression for the local truncation error we have

$$\varepsilon D^- e_i = (\varepsilon + a(x_i)H) D^+ e_i - H\tau_i,\ \ N/2 < i < N,$$

and this can be rewritten in the form

$$\varepsilon D^- e_i = \varepsilon D^+ e_i + a(x_i)(e_{i+1} - e_i) - H\tau_i,\ \ N/2 < i < N. \tag{11}$$

Summing these equations from $j = i$ to $N-1$ yields, for each $i,\ N/2 < i < N$,

$$\varepsilon D^- e_i = \varepsilon D^- e_N + \sum_{j=i}^{N-1} a(x_j)(e_{j+1} - e_j) - H\sum_{j=i}^{N-1}\tau_j.$$

Note that

$$\sum_{j=i}^{k} a(x_j)(e_{j+1} - e_j) = a(x_k)e_{k+1} - a(x_{i-1})e_i - \sum_{j=i}^{k}(a(x_j) - a(x_{j-1}))e_j. \tag{12}$$

Now from Lemma 1 we have that $|e_i| \leq CN^{-1}$ and from its proof we have $|\tau_i| \leq CN^{-1}$, and (10). From these inequalities we have the following bound for all i such that $N/2 \leq i < N$

$$|\varepsilon D^+ e_i| \leq CN^{-1}.$$

At the transition point $x_{N/2} = \sigma$,

$$\varepsilon D^- e_{N/2} = \varepsilon D^+ e_{N/2} + a(x_{N/2})(\frac{h+H}{2})D^+ e_{N/2} - (\frac{h+H}{2})\tau_{N/2},$$

and noting that

$$|D^+ e_{N/2}| = |\frac{e_{N/2+1} - e_{N/2}}{H}| \leq C$$

results in $|\varepsilon D^- e_{N/2}| \leq CN^{-1}$. In the fine mesh, $x_i < \sigma$, we sum (11) from $j = i$ to $N/2 - 1$ to get

$$\varepsilon D^- e_i = \varepsilon D^- e_{N/2} + \sum_{j=i}^{N/2-1} a(x_j)(e_{j+1} - e_j) - h \sum_{j=i}^{N/2-1} \tau_j.$$

Repeat the argument that was used in the coarse mesh area, to get

$$|\varepsilon D^- e_i| \leq CN^{-1}, \quad 1 \leq i < N/2$$

which completes the proof.

Lemma 4 *In the case of $\sigma = \varepsilon \ln N/\alpha$,*

$$|\varepsilon D^+ W_\varepsilon(x_i)| \leq CN^{-1} \quad , \qquad \sigma \leq x_i < 1, \tag{13a}$$

$$\varepsilon |D^+(W_\varepsilon(x_i) - w_\varepsilon(x_i))| \leq CN^{-1} \quad , \qquad \sigma \leq x_i < 1, \tag{13b}$$

$$\varepsilon |D^-(W_\varepsilon(\sigma) - w_\varepsilon(\sigma))| \leq CN^{-1} \ln N \quad , \tag{13c}$$

where w_ε is the solution of (3b) and W_ε is the solution of (6b).

Proof. From Lemma 2, $|W_\varepsilon(x_i)| \leq CN^{-1}(1-x_i)$ for $i \geq N/2$ and thus $|D^- W_\varepsilon(x_N)| \leq CN^{-1}$. Since $L_\varepsilon^N W_\varepsilon = 0$, we can write

$$\varepsilon D^- W_\varepsilon(x_i) = (\varepsilon + a(x_i)H) D^+ W_\varepsilon(x_i), \quad N/2 < i < N,$$

which after summation yields, for all $\sigma < x_i < 1$,

$$\varepsilon D^- W_\varepsilon(x_i) = \varepsilon D^- W_\varepsilon(x_N) - a(x_{i-1}) W_\varepsilon(x_i) - \sum_{j=i}^{N-1} W_\varepsilon(x_j)(a(x_j) - a(x_{j-1})),$$

which completes the proof of (13a).

Note that

$$\begin{aligned}\varepsilon|D^+(W_\varepsilon(x_i)-w_\varepsilon(x_i))| &= \varepsilon|D^+W_\varepsilon(x_i)-w'_\varepsilon(x_i)+w'_\varepsilon(x_i)-D^+w_\varepsilon(x_i)|\\ &\le \varepsilon|D^+W_\varepsilon(x_i)|+\varepsilon|w'_\varepsilon(x_i)|+\varepsilon|w'_\varepsilon(x_i)-D^+w_\varepsilon(x_i)|\\ &\le CN^{-1}\end{aligned}$$

which completes the proof of (13b).

At the transition point

$$\begin{aligned}\varepsilon D^-W_\varepsilon(\sigma) &= \varepsilon D^+W_\varepsilon(\sigma)+a(\sigma)\big(\frac{h+H}{2}\big)D^+W_\varepsilon(\sigma),\\ &= (1+\frac{a(\sigma)h}{2\varepsilon})\varepsilon D^+W_\varepsilon(\sigma)+\frac{a(\sigma)}{2}\big(W_\varepsilon(\sigma+H)-W_\varepsilon(\sigma)\big)\\ &\le CN^{-1}.\end{aligned}$$

Note also that,

$$|\varepsilon w'_\varepsilon(\sigma-h)|\le Ce^{-\frac{\alpha\sigma}{\varepsilon}}e^{\frac{\alpha h}{\varepsilon}}\le CN^{-1}e^{\frac{\ln N}{N}}\le CN^{-1}$$

and complete the proof of (13c) using Theorem 1 which gives

$$\varepsilon|D^+w_\varepsilon(\sigma-h)-w'_\varepsilon(\sigma-h)|\le CN^{-1}\ln N.$$

Theorem 2 *For all* $x_i\in\Omega^N_\varepsilon\cup\{0\}$

$$\varepsilon|D^+(W_\varepsilon(x_i)-w_\varepsilon(x_i))|\le CN^{-1}\ln N$$

where w_ε *is the solution of (3b) and* W_ε *is the solution of (6b).*

Proof. Consider first the case of $\sigma=\varepsilon\ln N/\alpha$. Use the previous lemma in the case of $x_i\ge\sigma$. Denote the error and the local truncation error, respectively, at each mesh point by

$$\hat{e}_i=W_\varepsilon(x_i)-w_\varepsilon(x_i)\quad\text{and}\quad\hat{\tau}_i=L^N_\varepsilon\hat{e}_i.$$

For $1\le i<N/2-1$

$$\varepsilon D^-\hat{e}_i=\varepsilon D^-\hat{e}_{N/2}+\sum_{j=i}^{N/2-1}a(x_j)(\hat{e}_{j+1}-\hat{e}_j)-h\sum_{j=i}^{N/2-1}\hat{\tau}_j.$$

Note that

$$\big|h\sum_{j=i}^{N/2-1}\hat{\tau}_j\big|\le C\frac{h^2}{\varepsilon^2}\sum_{j=i}^{N/2-1}e^{-\frac{\alpha x_{j-1}}{\varepsilon}}\le C\frac{(h/\varepsilon)^2}{1-e^{-\frac{\alpha h}{\varepsilon}}}\le C\frac{h}{\varepsilon}\max\{1,\frac{h}{\varepsilon}\}\le CN^{-1}\ln N.$$

Finish using the argument given in Lemma 3. In the case of $\sigma=0.5$, use the argument from Lemma 3 to first establish that $\varepsilon|W_\varepsilon(x_i)-w_\varepsilon(x_i)|\le CN^{-1}\ln N(1-x_i)$ which implies that $\varepsilon|D^-\hat{e}_N|\le CN^{-1}\ln N$; then repeat the argument given above on the entire interval $[0,1]$.

Theorem 3 *Let u_ε be the solution of (2) and let U_ε be the corresponding numerical solution generated by (5). Then the discrete and exact scaled derivatives satisfy the estimate*

$$\varepsilon|D^+U_\varepsilon(x_i) - u'_\varepsilon(x)| \le CN^{-1}\ln N, \qquad \text{for all } x \in [x_i, x_{i+1}]$$

where C is a constant independent of N and ε.

Proof. Use the triangle inequality

$$\varepsilon|D^+U_\varepsilon(x_i) - u'_\varepsilon(x)| \le \varepsilon|D^+U_\varepsilon(x_i) - D^+u_\varepsilon(x_i))| + \varepsilon|D^+u_\varepsilon(x_i) - u'_\varepsilon(x)|$$

and the previous lemmas to complete the proof.

Remark Note that the bound

$$\varepsilon|D^+U_\varepsilon(0) - u'_\varepsilon(0)| \le CN^{-1}\ln N$$

together with Theorem 1 imply that $\varepsilon|D^+\big(U_\varepsilon(0) - u_\varepsilon(0)\big)| \le CN^{-1}\ln N$, which in turn implies that at the first internal mesh point

$$|U_\varepsilon(x_1) - u_\varepsilon(x_1)| \le C(N^{-1}\ln N)^2.$$

We define the piecewise constant function $\overline{D}^+U_\varepsilon$ on $[0,1]$ by

$$\varepsilon\overline{D}^+U_\varepsilon(x) = \varepsilon D^+U_\varepsilon(x_i), \quad \text{for} \quad x \in [x_i, x_{i+1}), \quad i = 0, 1...., N-1$$

and at the point $x = 1$ by

$$\varepsilon\overline{D}^+U_\varepsilon(1) = \varepsilon D^+U_\varepsilon(x_{N-1}).$$

Then, from the last theorem, $\overline{D}^+U_\varepsilon$ is an ε–uniform global approximation to $\varepsilon u'_\varepsilon$ in the sense that

$$\sup_{0<\varepsilon\le 1} \|\varepsilon\overline{D}^+U_\varepsilon - \varepsilon u'_\varepsilon\|_{\overline{\Omega}} \le CN^{-1}\ln N.$$

In other words the numerical method (5) is ε–uniform in an appropriately scaled global C^1 norm. That is,

$$\sup_{0<\varepsilon\le 1} \varepsilon\|\overline{D}^+U_\varepsilon - u'_\varepsilon\|_{\overline{\Omega}} + \|\overline{U}_\varepsilon - u_\varepsilon\|_{\overline{\Omega}} \le CN^{-1}\ln N. \tag{14}$$

Note that $\varepsilon u''_\varepsilon = f - au'_\varepsilon$. Hence, for all $x \in [x_i, x_{i+1}]$,

$$\varepsilon^2\delta^2U_\varepsilon(x_i) - \varepsilon^2u''_\varepsilon(x) = \varepsilon(f(x_i) - f(x)) + (a(x) - a(x_i))\varepsilon u'_\varepsilon(x) - a(x_i)\varepsilon\big(D^+U_\varepsilon - u'_\varepsilon\big)$$

which implies that, with the obvious definition for the extension $\overline{\delta}^2U_\varepsilon$,

$$\sup_{0<\varepsilon\le 1} \varepsilon^2\|\overline{\delta}^2U_\varepsilon - u''_\varepsilon\|_{\overline{\Omega}} + \varepsilon\|\overline{D}^+U_\varepsilon - u'_\varepsilon\|_{\overline{\Omega}} + \|\overline{U}_\varepsilon - u_\varepsilon\|_{\overline{\Omega}} \le CN^{-1}\ln N. \tag{15}$$

In other words the numerical method (5) applied to problem (2) is ε–uniform in an appropriately scaled global C^2 norm.

3 Numerical results

Consider the following two dimensional convection–diffusion problem on the unit square $\Omega = (0,1) \times (0,1)$

$$\varepsilon \Delta u_\varepsilon + \frac{\partial u_\varepsilon}{\partial x} = 0, \qquad (x,y) \in \Omega, \tag{16a}$$

$$u_\varepsilon(x,0) = 64x^3(1-x)^3,\ u_\varepsilon(1,y) = 64y^3(1-y)^3,\ u_\varepsilon(x,1) = u_\varepsilon(0,y) = 0. \tag{16b}$$

A regular boundary layer occurs at the edge $x = 0$ and a parabolic boundary layer at the edge $y = 0$. We use the piecewise–uniform partially fitted mesh

$$\Omega_\sigma^N \times \Omega_u^N \tag{17}$$

which is fitted to the regular boundary layer but not to the parabolic boundary layer. The mesh Ω_u^N is a uniform mesh along the y–axis and Ω_σ^N is the one–dimensional fitted mesh defined in (5c). We use the numerical method based on the upwind finite difference operator

$$\varepsilon\delta_x^2 U_\varepsilon + \varepsilon\delta_y^2 U_\varepsilon + D_x^+ U_\varepsilon = 0, (x_i, y_j) \in \Omega_\sigma^N \times \Omega_u^N,$$

where δ_x^2, D_x^+ are the standard centered and forward difference operators, and the partially fitted mesh (17), to solve problem (16). It is known from the theoretical results in [2] that this method is not ε–uniform. This may be seen from the numerical results in Table 1. There is an error ridge along a diagonal of the table, typical of a non ε–uniform method, but in this case the maximum pointwise errors are at worst only 6% for $N \geq 32$. For many, this level of accuracy may be acceptable despite the fact that the method is not ε–uniform. However, it is when this method is used to generate approximations to a scaled first derivative of the solution, that its inadequacy becomes more pronounced. This is of practical importance, because often the physical quantities of interest, such as the flux, involve scaled first order derivatives of the solution. A small error in the approximation of the solution may be tolerable, but in practice we often want comparably small errors in the numerical approximations of the scaled derivatives $(\varepsilon u_x, \sqrt{\varepsilon} u_y)$. The computed maximum pointwise errors of the numerical approximation to the scaled derivatives generated by this partially fitted method and the upwind finite difference operator are given in Table 2. These results demonstrate that the numerical approximations of the scaled derivatives do not converge ε–uniformly in the boundary layer regions where the mesh is not fitted. In fact, there is an error of 100% for small values of ε.

We now consider the numerical method based on the same upwind finite difference operator and the fitted mesh $\Omega_{\sigma_1}^N \times \Omega_{\sigma_2}^N$ where $\overline{\Omega}_{\sigma_1}^N$ is the piecewise–uniform fitted mesh defined in (5c) and $\overline{\Omega}_{\sigma_2}^N$ is defined by

$$\overline{\Omega}_{\sigma_2}^N = \{y_i |\ \ y_i = ih_2,\ i \leq N/4;\ \ y_i = y_{i-1} + H_2,\ N/4 < i \leq 3N/4; \\ y_i = y_{i-1} + h_2,\ 3N/4 < i\} \tag{18a}$$

Table 1: Computed maximum pointwise errors E_ε^N for problem (16) using an upwind finite difference operator and the partially fitted mesh (17) with no refinement in the parabolic layers for various values of ε and N.

	Number of intervals N					
ε	8	16	32	64	128	256
1	0.185D-01	0.737D-02	0.301D-02	0.126D-02	0.514D-03	0.169D-03
2^{-2}	0.325D-01	0.168D-01	0.836D-02	0.399D-02	0.174D-02	0.599D-03
2^{-4}	0.991D-01	0.504D-01	0.251D-01	0.128D-01	0.624D-02	0.235D-02
2^{-6}	0.116D+00	0.682D-01	0.379D-01	0.200D-01	0.960D-02	0.354D-02
2^{-8}	0.152D+00	0.886D-01	0.484D-01	0.251D-01	0.118D-01	0.428D-02
2^{-10}	0.164D+00	0.945D-01	**0.650D-01**	0.323D-01	0.126D-01	0.455D-02
2^{-12}	0.167D+00	0.960D-01	0.568D-01	**0.590D-01**	0.267D-01	0.725D-02
2^{-14}	0.168D+00	0.963D-01	0.526D-01	0.556D-01	**0.558D-01**	0.238D-01
2^{-16}	0.168D+00	0.964D-01	0.526D-01	0.273D-01	0.551D-01	0.540D-01
2^{-18}	0.168D+00	0.965D-01	0.527D-01	0.273D-01	0.216D-01	**0.547D-01**
2^{-20}	0.168D+00	0.965D-01	0.527D-01	0.273D-01	0.129D-01	0.216D-01
2^{-22}	0.168D+00	0.965D-01	0.527D-01	0.273D-01	0.129D-01	0.561D-02
.	.	.	.	.	.	.
	.	.	.	.	.	.
2^{-34}	0.168D+00	0.965D-01	0.527D-01	0.273D-01	0.129D-01	0.465D-02

with $h_2 = 4\sigma_2/N$, $H_2 = 2(1-\sigma_2)/N$ and

$$\sigma_2 = \min\{\frac{1}{2}, \sqrt{\varepsilon}\ln N\}. \tag{18b}$$

The results in Tables 3 and 4 show that in this case the numerical approximations to the scaled derivatives converge ε–uniformly on this piecewise uniform mesh. We remark especially that the ε–uniform convergence displayed in Table 4 is in stark contrast to the corresponding behaviour of the entries in Table 2.

References

[1] Miller J. J. H., O'Riordan E. and Shishkin G.I. (1996). *Fitted Numerical Methods for Singular Perturbation Problems.* World Scientific Publishing Co.. Singapore.

[2] Shishkin G. I. (1989). Approximation of solutions of singularly perturbed boundary value problems with a parabolic boundary layer. *USSR Comput. Maths. Math. Phys.*, **29**, (4), 1–10.

Table 2: Computed maximum pointwise errors $E_\varepsilon^N(\sqrt{\varepsilon}D_y^+U)$ in approximating $\sqrt{\varepsilon}u_y$ for problem (16) using the upwind finite difference operator and the partially fitted mesh (17) for various values of ε and N.

	Number of intervals N					
ε	8	16	32	64	128	256
1	0.430D+00	0.210D+00	0.102D+00	0.469D-01	0.205D-01	0.690D-02
2^{-2}	0.430D+00	0.230D+00	0.129D+00	0.627D-01	0.275D-01	0.925D-02
2^{-4}	0.650D+00	0.305D+00	0.154D+00	0.742D-01	0.325D-01	0.110D-01
2^{-6}	0.934D+00	0.482D+00	0.230D+00	0.113D+00	0.498D-01	0.169D-01
2^{-8}	0.123D+01	0.775D+00	0.433D+00	0.219D+00	0.999D-01	0.346D-01
2^{-10}	**0.124D+01**	0.102D+01	0.756D+00	0.423D+00	0.210D+00	0.846D-01
2^{-12}	0.123D+01	**0.104D+01**	0.988D+00	0.754D+00	0.428D+00	0.215D+00
2^{-14}	0.123D+01	0.104D+01	**0.101D+01**	0.983D+00	0.754D+00	0.431D+00
2^{-16}	0.123D+01	0.103D+01	0.100D+01	**0.100D+01**	0.986D+00	0.755D+00
2^{-18}	0.123D+01	0.103D+01	0.100D+01	0.100D+01	**0.100D+01**	0.985D+00
2^{-20}	0.123D+01	0.103D+01	0.100D+01	0.100D+01	0.100D+01	**0.100D+01**
2^{-22}	0.123D+01	0.103D+01	0.100D+01	0.100D+01	0.100D+01	0.100D+01
	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-34}	0.123D+01	0.103D+01	0.100D+01	0.100D+01	0.100D+01	0.100D+01

Table 3: Computed maximum pointwise errors $E_\varepsilon^N(\varepsilon D_x^+U)$ and $E^N(\varepsilon D_x^+U)$ in approximating εu_x for problem (16) using $\Omega_{\sigma_1}^N \times \Omega_{\sigma_2}^N$ for various values of ε and N.

	Number of intervals N					
ε	8	16	32	64	128	256
1	0.444D+00	0.204D+00	0.989D-01	0.473D-01	0.204D-01	0.681D-02
2^{-2}	0.375D+00	0.219D+00	0.104D+00	0.481D-01	0.205D-01	0.684D-02
2^{-4}	0.142D+00	0.101D+00	0.536D-01	0.284D-01	0.134D-01	0.480D-02
2^{-6}	0.120D+00	0.805D-01	0.488D-01	0.266D-01	0.128D-01	0.465D-02
2^{-8}	0.101D+00	0.767D-01	0.463D-01	0.254D-01	0.123D-01	0.452D-02
2^{-10}	0.942D-01	0.755D-01	0.459D-01	0.251D-01	0.122D-01	0.448D-02
2^{-12}	0.934D-01	0.748D-01	0.457D-01	0.250D-01	0.121D-01	0.446D-02
2^{-14}	0.932D-01	0.744D-01	0.455D-01	0.250D-01	0.121D-01	0.446D-02
2^{-16}	0.931D-01	0.742D-01	0.453D-01	0.250D-01	0.121D-01	0.446D-02
.	.	.	.	.	.	.
	.	.	.	.	.	.
2^{-34}	0.931D-01	0.740D-01	0.453D-01	0.250D-01	0.121D-01	0.445D-02
$E^N(\varepsilon D_x^+U)$	0.444D+00	0.219D+00	0.104D+00	0.481D-01	0.205D-01	0.684D-02

Table 4: Computed maximum pointwise errors $E_\varepsilon^N(\sqrt{\varepsilon}D_y^+U)$ and $E^N(\sqrt{\varepsilon}D_y^+U)$ in approximating $\sqrt{\varepsilon}u_y$ for problem (16) using $\Omega_{\sigma_1}^N \times \Omega_{\sigma_2}^N$ for various values of ε and N.

	Number of intervals N					
ε	8	16	32	64	128	256
1	0.430D+00	0.210D+00	0.102D+00	0.469D-01	0.205D-01	0.690D-02
2^{-2}	0.430D+00	0.230D+00	0.129D+00	0.627D-01	0.275D-01	0.925D-02
2^{-4}	0.650D+00	0.305D+00	0.154D+00	0.742D-01	0.325D-01	0.110D-01
2^{-6}	0.934D+00	0.482D+00	0.230D+00	0.113D+00	0.498D-01	0.169D-01
2^{-8}	0.736D+00	0.587D+00	0.370D+00	0.219D+00	0.999D-01	0.346D-01
2^{-10}	0.680D+00	0.562D+00	0.347D+00	0.208D+00	0.108D+00	0.411D-01
.	.	.	.	.	.	.
	.	.	.	.	.	.
2^{-34}	0.671D+00	0.560D+00	0.347D+00	0.208D+00	0.108D+00	0.410D-01
$E^N(\sqrt{\varepsilon}D_y^+U)$	0.934D+00	0.587D+00	0.370D+00	0.219D+00	0.108D+00	0.411D-01