A technique for computing realistic values of the error parameters for the numerical solutions of singular perturbation problems [1]

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Abstract. *In this paper we describe an experimental technique to determine approximate values of the error parameters associated with a parameter–uniform numerical method for solving singularly perturbed convection–diffusion problems. We employ the technique to compute realistic values of these parameters for the numerical solutions generated by a monotone parameter–uniform numerical method applied to an elliptic boundary value problem with different types of boundary layers such as regular, parabolic and corner layers. Such error parameters allow us effectively to evaluate actual error bounds for the numerical solutions and to determine the parameter-uniformity of new numerical methods and, therefore, their applicability in practice.*

1 Introduction

The numerical solution of a singularly perturbed problem and its actual maximum pointwise error depend on the singular perturbation parameter ε and the number N of mesh points in each coordinate direction of the discrete problem. A standard criterion for assessing the quality of a numerical method for a singularly perturbed problem is a theoretical error bound of the following form: there exist positive constants N_0, $C = C(N_0)$ and $p = p(N_0)$, all independent of N and ε, such that for all $N \geq N_0$

$$\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N} \leq C_p N^{-p} \tag{1}$$

where u_ε is the exact solution of the continuous problem, U_ε^N is a numerical approximation, for example, on a piecewise–uniform fitted mesh Ω_ε^N (for details see, e.g., [1]).

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The maximum pointwise error $\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N}$ of the numerical method on the mesh Ω_ε^N is bounded above by the error bound C_pN^{-p}, which is determined, for a given N, by the ε–uniform error constant C_p and the ε–uniform order of convergence p. The quantities p and C_p are said to be the ε–uniform error parameters.

In practice numerical analysts establish a sharp estimate of p, while the error constant C_p is normally left unspecified. It is clear that without an estimate of the error constant C_p, an error bound of this form gives information only about the asymptotic rate of convergence of the numerical solutions. In other words, it tells us that if we use more mesh points, and N is sufficiently large then the error decreases like N^{-p}. However, if the value of the error constant C_p is unknown, we cannot compute the magnitude of the right hand side of (1) and, consequently, we have no estimate of the magnitude of the actual errors in our numerical approximations. It is possible that the error may be large for a reasonable value of N. In practice, for a specific computation, the values of N and ε lie in finite ranges $R_N = \{N : \underline{N} \le N \le \overline{N}\}$ and $R_\varepsilon = [\underline{\varepsilon}, \overline{\varepsilon}]$ respectively, and so we want to choose p and its associated C_p in such a way that, for all $N \in R_N$ and all $\varepsilon \in R_\varepsilon$, the resulting error bound C_pN^{-p} is as small as possible. In other words, we want to obtain error bounds that are both practical and useful for the appropriate ranges of N and ε .

Note that the theoretical analysis often does not give sharp estimates for the order p of ε-uniform convergence, for example, in the case of problems with a rather complicated behaviour of the singular solution. Such low theoretical orders would seem to imply that the numerical method will yield errors which are too large for this method to be of practical use. Also, as has been said above, the theory cannot give a reasonable estimate for the error constant C_p, so the theoretical *a priori* analysis does not allow us to evaluate error bounds for the numerical solution. Thus, we are forced to apply an *a posteriori* experimental technique to estimate the actual ε-uniform errors. Such a technique can be crucial in many applied problems for which the theoretical analysis of ε-uniform convergence is absent.

The main goal here is to describe an experimental computational technique for determining realistic estimates of p and C_p and to illustrate its use by estimating the maximum pointwise error in numerical approximations to the solution of a singularly perturbed convection–diffusion problem with regular, parabolic and corner boundary layers. It is clear that the actual maximum pointwise error $\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N}$ in the numerical solution of a singularly perturbed problem depends on the two variables N and ε. But if the numerical method is ε–uniform for the particular problem being solved, then the error parameters are ε–uniform.

On the other hand, for any standard non–ε–uniform numerical method, the error constant C_p increases as ε decreases. Worse still, given a range R_N, for such numerical methods there are always values of ε for which the errors grow as N increases, that is the errors become larger as the mesh is refined. Such behaviour is normally regarded as unacceptable in the numerical solutions of singularly perturbed problems.

2 Algorithm for estimating error parameters

A systematic determination of *a posteriori* estimates of the ε–uniform error constant C_p, and the ε–uniform order of convergence p, of a numerical method for solving a given singularly perturbed problem, is described for cases when the numerical solutions can be computed for several values of N and ε. The arguments are heuristic rather than rigorous, which is typical for *a posteriori* techniques of this type.

We assume that, on the appropriate meshes Ω_ε^N, the piecewise linear interpolants $\overline{U}_\varepsilon^N$ of the numerical solutions U_ε^N have been determined. Then, for all integers N satisfying $N, 2N \in R_N$ and for a finite set of values $\varepsilon \in R_\varepsilon$, the maximum pointwise two–mesh differences

$$D_\varepsilon^N = ||U_\varepsilon^N - \overline{U}_\varepsilon^{2N}||_{\Omega_\varepsilon^N} \tag{2}$$

are computed. From these values the ε–uniform maximum pointwise two–mesh differences

$$D^N = \max_{\varepsilon \in R_\varepsilon} D_\varepsilon^N \tag{3}$$

are formed for each available value of N satisfying $N, 2N \in R_N$. Approximations to the ε–uniform order of local convergence are defined, for all $N,\ 4N \in R_N$, by

$$p^N = \log_2 \frac{D^N}{D^{2N}} \tag{4}$$

and we take the computed ε–uniform order of convergence to be

$$p^* = \min_N p^N. \tag{5}$$

Note that

$$\begin{aligned} D_\varepsilon^N &= ||U_\varepsilon^N - \overline{U}_\varepsilon^{2N}||_{\Omega_\varepsilon^N} \geq \left| \ ||U_\varepsilon^N - u_\varepsilon||_{\Omega_\varepsilon^N} - ||u_\varepsilon - \overline{U}_\varepsilon^{2N}||_{\Omega_\varepsilon^N} \ \right| \\ &\approx CN^{-p}(1 - 2^{-p}). \end{aligned}$$

This motivates the following definitions. Corresponding to the value of p^* in (5) we calculate the quantities

$$C_{p^*}^N = \frac{D^N N^{p^*}}{1 - 2^{-p^*}} \tag{6}$$

and we take the computed ε–uniform error constant to be

$$C_{p^*}^* = \max_N C_{p^*}^N. \tag{7}$$

The above definitions of the computed error parameters supersede the similar definitions given in [2].

3 Application of the algorithm

We consider the following singular perturbation convection–diffusion problem on the unit square $\Omega = (0,1)^2$ with boundary Γ and edges $\Gamma_{\mathrm{B}} = \{(x,0)|0 \le x \le 1\}$, $\Gamma_{\mathrm{R}} = \{(1,y)|0 \le y \le 1\}$, $\Gamma_{\mathrm{T}} = \{(x,1)|0 \le x \le 1\}$ and $\Gamma_{\mathrm{L}} = \{(0,y)|0 \le y \le 1\}$

$$\varepsilon\Delta u_\varepsilon + \frac{\partial u_\varepsilon}{\partial x} = 0, \quad (x,y) \in \Omega, \tag{8a}$$

$$u_\varepsilon(x,0) = 64x^3(1-x)^3, \quad (x,y) \in \Gamma_{\mathrm{B}}, \tag{8b}$$

$$u_\varepsilon(1,y) = 64y^3(1-y)^3, \quad (x,y) \in \Gamma_{\mathrm{R}}, \tag{8c}$$

$$u_\varepsilon(x,y) = 0, \quad (x,y) \in \Gamma \setminus (\Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{R}}). \tag{8d}$$

A rich variety of boundary layers is present in its solution, involving regular, parabolic and corner boundary layers. The regular boundary layer is on the edge Γ_{L}, the parabolic boundary layers are on the edges Γ_{B} and Γ_{T} and the corner boundary layers are at the corners C_{BL} and C_{TL} on Γ_{L}.

We apply the following monotone numerical method based on a standard upwind finite difference operator on a piecewise–uniform fitted mesh to generate numerical solutions of problem (8)

$$[\varepsilon(\delta_x^2 + \delta_y^2) + D_x^+]U_\varepsilon = 0, \quad (x_i, y_j) \in \Omega_\varepsilon^N, \tag{9a}$$

$$U_\varepsilon = u_\varepsilon, \quad (x_i, y_j) \in \Gamma^N \tag{9b}$$

where

$$\overline{\Omega}_\varepsilon^N = \overline{\Omega}_{\sigma_1}^N \times \overline{\Omega}_{\sigma_2}^N, \tag{9c}$$

$$\overline{\Omega}_{\sigma_1}^N = \{x_i : 0 \le i \le N\} \quad \text{and} \quad \overline{\Omega}_{\sigma_2}^N = \{y_j : 0 \le j \le N\}, \tag{9d}$$

$$x_i = \begin{cases} 2i\frac{\sigma_1}{N}, & 0 \le i \le N/2, \\ \sigma_1 + 2(i-(N/2))\frac{(1-\sigma_1)}{N}, & N/2 \le i \le N, \end{cases} \tag{9e}$$

$$y_j = \begin{cases} 4i\frac{\sigma_2}{N}, & 0 \le j \le N/4, \\ \sigma_2 + 2(i-(N/4))\frac{(1-2\sigma_2)}{N}, & N/4 \le j \le 3N/4, \\ 1-\sigma_2 + 4(i-(3N/4))\frac{\sigma_2}{N}, & 3N/4 \le j \le N, \end{cases} \tag{9f}$$

$$\sigma_1 = \min\{1/2,\ c\varepsilon \ln N\}, \quad c \ge \alpha_1^{-1}, \tag{9g}$$

$$\sigma_2 = \min\{1/4,\ \varepsilon^{\frac{1}{2}} \ln N\}. \tag{9h}$$

Table 1: The values of $D_\varepsilon^N, D^N, p^N, p^*$ and $C_{p^*}^N$ given by the algorithm in §2 applied to the numerical solutions of problem (8) obtained by the numerical method (9) for various values of ε and N

	Number of intervals N					
ε	8	16	32	64	128	256
1	0.117D-01	0.439D-02	0.174D-02	0.751D-03	0.345D-03	0.169D-03
2^{-2}	0.163D-01	0.847D-02	0.439D-02	0.225D-02	0.114D-02	0.599D-03
2^{-4}	0.538D-01	0.282D-01	0.130D-01	0.670D-02	0.388D-02	0.235D-02
2^{-6}	0.525D-01	0.337D-01	0.177D-01	0.105D-01	0.607D-02	0.354D-02
2^{-8}	0.650D-01	0.403D-01	0.233D-01	0.133D-01	0.756D-02	0.428D-02
2^{-10}	0.723D-01	0.434D-01	0.249D-01	0.142D-01	0.806D-02	0.454D-02
2^{-12}	0.746D-01	0.443D-01	0.253D-01	0.144D-01	0.819D-02	0.462D-02
2^{-14}	0.752D-01	0.446D-01	0.254D-01	0.144D-01	0.822D-02	0.464D-02
2^{-16}	0.754D-01	0.446D-01	0.254D-01	0.144D-01	0.823D-02	0.465D-02
.	.	.	.	.	.	.
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2^{-34}	0.754D-01	0.446D-01	0.254D-01	0.144D-01	0.823D-02	0.465D-02
D^N	0.754D-01	0.446D-01	0.254D-01	0.144D-01	0.823D-02	0.465D-02
p^N	0.76	0.81	0.82	0.81	0.82	$p^* = 0.76$
$C_{0.76}^N$	0.894	0.897	0.865	0.832	0.803	0.768

In [3] this method is shown theoretically to be ε–uniform for problems of this kind.

The results of applying the algorithm of the previous section to the resulting numerical solutions are summarized in Table 1. The computed estimates of p and C_p are $p^* = 0.76$ and $C_{0.76}^* = 0.897$.

The robustness for $\varepsilon \in R_\varepsilon$ and $N \in R_N$ of these computed values of p and C_p is now examined. Starting from the values of D^N for $N = 8, 16, 32, 64, 128, 256$, given in Table 1, the quantities C_p^N are computed from the formula

$$C_p^N = \frac{D^N N^p}{1 - 2^{-p}}$$

for the same values of N and for a convenient range of values of p surrounding and including the computed value p^* given by (5). The results are set out in Table 2. The rows of the table contain the values of C_p^N for fixed p and the columns correspond to the values for fixed N. The bold entry in each row of the table is the maximum value in the row. We take the computed ε–uniform order of convergence p^* to be the maximum value of p for which this bold entry is the first entry of the row, and we take the value of that entry to be the computed ε–uniform error constant $C_{p^*}^*$. Thus $p^* = 0.75$ and $C_{0.75}^* = 0.885$ for $N \in R_N = [8, 512]$ and $p^* = 0.81$, $C_{0.81}^* = 0.981$ for $N \in R_N = [16, 512]$, which is a more restricted range. If the p^* and $C_{p^*}^*$ obtained from this tabular technique are either equal to or close to the computed values obtained with the above algorithm, this indicates their insensitivity with respect to N and their robustness with respect to ε for all $\varepsilon \in R_\varepsilon$ and all $N \in R_N$. This is indeed

Table 2: Values of C_p^N defined by (6) for the numerical solutions of problem (8) obtained by the numerical method (9) for various values of p and N

	Number of intervals N					
p	8	16	32	64	128	256
0.1	**1.39**	0.880	0.537	0.327	0.200	0.121
0.2	**0.883**	0.600	0.393	0.256	0.168	0.109
0.3	**0.750**	0.546	0.383	0.268	0.188	0.131
0.5	**0.728**	0.610	0.491	0.395	0.318	0.254
0.7	**0.841**	0.809	0.748	0.691	0.639	0.587
0.75	**0.885**	0.881	0.844	0.806	0.773	0.734
0.76	0.894	**0.897**	0.865	0.832	0.803	0.768
0.8	0.935	**0.964**	0.956	0.946	0.938	0.922
0.81	0.946	**0.981**	0.979	0.973	0.975	0.966
0.9	1.06	1.17	1.24	1.31	1.40	**1.47**
1.0	1.21	1.43	1.63	1.85	2.11	**2.38**

Table 3: Computed ε–uniform maximum pointwise error bound $C^*_{p^*}N^{-p^*}$ from the algorithm for the numerical method (9) applied to problem (8) for various values of N

	Number of intervals N						
	8	16	32	64	128	256	512
$0.897N^{-0.76}$	0.185	0.110	0.064	0.038	0.022	0.013	0.008
$0.981N^{-0.81}$	–	0.104	0.059	0.034	0.019	0.011	0.006

the case for the numerical solutions of problem (8) obtained by method (9), because the algorithm yields $p^* = 0.76$ and $C^*_{0.76} = 0.897$ and the tabular technique yields $p^* = 0.75$ and $C^*_{0.75} = 0.885$ for $N \in R_N = [8, 512]$. We may take the values $p^* = 0.75$ and $C^*_{0.75} = 0.897$ as our 'safest' estimates of the error parameters.

4 Practical uses of ε–uniform error parameters

The first obvious use of the computed ε–uniform error parameters is to compute estimates of the ε–uniform maximum pointwise error bound for a range of values of N. The values p^* and $C^*_{p^*}$ of the ε–uniform error parameters obtained from the above algorithm are used to obtain the realistic ε–uniform error bound $C^*_{p^*}N^{-p^*}$ for all $N \in R_N$. In particular the resulting computed ε–uniform error bound for method (9) applied to problem (8) is $0.897N^{-0.76}$ for $N \in R_N = [8, 512]$ or $0.981N^{-0.81}$ for $N \in R_N = [16, 512]$, values of which are given in Table 3 for various values of N.

We can also use the computed ε–uniform error parameters to answer questions of the following kind: *given a required guaranteed ε–uniform maximum pointwise error no greater than δ say, what is the number N of mesh points that must be used?* It follows immediately from (1) that this accuracy is attained if

$$N \geq \max\{N_0, (\frac{C_p}{\delta})^{1/p}\}.$$

To apply this in a specific case we now consider the solution of problem (8) by method (9). With the values $p^* = 0.76$ and $C^*_{0.76} = 0.897$ obtained above, and taking $\delta = 0.01$, it suffices to choose N such that

$$N \geq \left[\left(\frac{0.897}{0.01} \right)^{1/0.76} \right] + 1 = 373$$

where the notation $[x]$ denotes the greatest integer less than or equal to the real number x. This shows that, in this case, if we take $N = 373$, we are likely to have an ε–uniform maximum pointwise error no greater than 0.01, for all values of ε in the range R_ε.

Finally, we make the important observation that once we have a numerical method that is ε–uniform of some positive order for a class of problems, then we can compute an approximation at each point of the domain $\overline{\Omega}$ to the exact solution to whatever accuracy we wish by using a sufficiently large value of N. This approximate solution can then be used as a benchmark solution for testing the performance of any other numerical method proposed for solving this problem, whether or not any theoretical error estimates are available for that method.

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