Mathematics in primary schools

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Abstract

The primary school mathematics Syllabus is currently being revised. This paper considers some aspects of the outgoing Syllabus (1971). The author believes that some of the textbooks are confusing and unrewarding, and that primary-school children should rely much more on a fixed set of 'structural materials.' The use of structural materials is recommended in the 1971 Syllabus, but not universally adopted.

1 Aims

This article discusses part of the (now obsolete) 1971 mathematics Syllabus for Primary schools. It makes some suggestions which might make some of the curriculum easier to learn. There are suggestions for 'structural materials' which the author believes could be useful and important. Few changes, or none, are suggested for the curriculum itself. Indeed, little is suggested which did not appear in some form in the Syllabus.

The aim is not to make the curriculum more ambitious or up-to-date, or less, but simply to make it easier to learn. If this is possible then the social benefits are obvious, and supported by various studies, such as are mentioned at the end of the present article.

2 Arithmetic, algebra, geometry, and textbooks

The 1971 Syllabus is a long and complex document. (It actually describes two alternatives, an ordinary and an experimental syllabus. Only the ordinary syllabus is considered here.) There are not many things in the Syllabus which which one could find fault. Here is one (quoted verbatim):

> Traditionally, Mathematics, as taught in the Primary School, has been limited mainly to arithmetic. Modern mathematics, however, tends to do away with the boundaries between the different branches and no sharp dividing line can now be drawn between them.

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If the writers of a Primary mathematics textbook take this to heart, then arithmetic, algebra, and geometry could appear on any page, without distinction. If there is some truth in the above quotation, it is exaggerated. For teaching purposes, at least, even advanced mathematics is clearly divided into a few subjects, and to jumble them together would be confusing.

My impression is that some primary textbooks, at least, have done away with the boundaries between different branches of Mathematics. As a result, one gets a feeling of *overload* — questions seem to fly at you from all directions. I can't see how this helps the young student, nor how algebra is made easier by lumping it together with arithmetic.

Returning to the quotation from the Syllabus: despite what it says, the Syllabus is mostly concerned with arithmetic — nothing wrong with that. There is some algebra, quite early: solving equations such as $x + 2 = 5$ (though the letter x is not used). Geometry is introduced late in the course. The 1971 curriculum is satisfactory as it stands.

3 Piaget

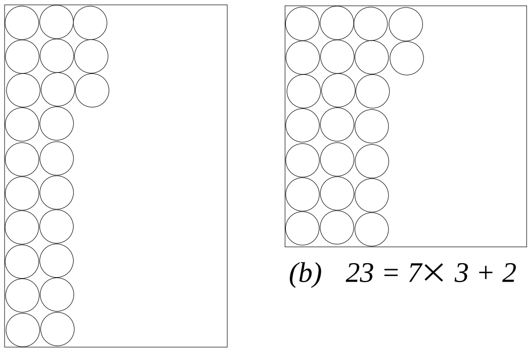
The Syllabus quotes some observations of the Swiss educationalist Jean Piaget (1896–1980), namely,

1. Children develop mathematical concepts more slowly than had been thought previously.
2. Children aged 4–7 years generally think intuitively and their opinions are governed mainly by appearances.
3. Although all children go through certain stages of development in concept formation some move at a faster rate than others.
4. Children up to c.11 years of age develop concepts best through experience at first hand and in concrete situations.
5. A child forms mathematical concepts as a result of his actions on objects and not on the objects themselves.

It is difficult to quarrel with these findings. The fifth observation is striking: indeed, it could apply to students of all ages. Most of us prefer things to be simple, concrete, and definite. Most of us prefer to learn by hands-on experience. As a student once put it, very neatly, 'I hear, I forget; I see, I remember; I do, I understand.'

As for children going by appearances (second in the list), sometimes the appearances are the reality and what the adult perceives is secondary. I learnt this when trying to teach a young infant a few words of English, using a picture-book illustrating the alphabet.

On the first page, the book showed an apple together with the letter 'A' in the corner, on the second a bee with the letter 'B' in the corner, and so on. I showed the infant the 'apple' picture, saying 'apple,' and then realised that when I pointed to the picture, the infant thought I meant the book. Efforts to correct the misunderstanding led to friction, which can't be described for reasons of space. For some time afterwards the infant believed that 'apple' meant 'book.'



(b) 23 = 7× 3 + 2

(a) 23 = 10× 2 + 3

Figure 1: 23 tokens arranged.

Seemingly it takes some time before a child will look at pictures as distinct from the book containing them. In the 'apple' episode the child was less focused on the appearances than on the underlying object.

The conclusion to be drawn is that teaching for *everybody, especially* children, should be as concrete as possible. This is where the mathematics textbooks fall short — they are attractive to look at, being crammed with *pictures*, but pictures are not good enough — not nearly good enough. They are certainly no substitute for 'structural materials.'

4 Structural materials.

'Structural materials' are things such as counting frames, stencils, and so forth, with which the pupils can experiment. The Syllabus mentions them frequently and I feel that they are *essential* to learning mathematics. Unfortunately the Syllabus cautions against becoming dependent upon them.

The curriculum begins, of course, with *counting*. The children learn to count up to 10, then to 100, and so forth. The Syllabus recommends that they count different kinds of thing, such as pegs, or apples, or schoolchildren. It is important that children learn to count almost anything: however, I believe that children should use a fixed counting 'model' — a set of tokens the size of a 2p piece, say. Initially they can count up to 9 tokens, say, and write down the result. After that they should arrange the tokens into columns of 10, with the remainder on the right. See Figure 1 (a). (Part (b) of the figure illustrates division by 7, discussed later).

It seems more natural to arrange the tokens into rows of 10 tokens, not columns, but there is good reason to count by columns. The later section on memorising tables gives reasons.

A child should find it easy enough to write down the digits 23 by counting the columns of 10s and the number left over. Given a frame of the correct size, it should be easy to count up to fifty or so tokens in this way.

If these tokens had an appreciable weight then they could also be counted by weighing.

These tokens could be blank or they could all be labelled '1,' showing that they represented units; one could then introduce tokens labelled 10, on the understanding that ten of the former

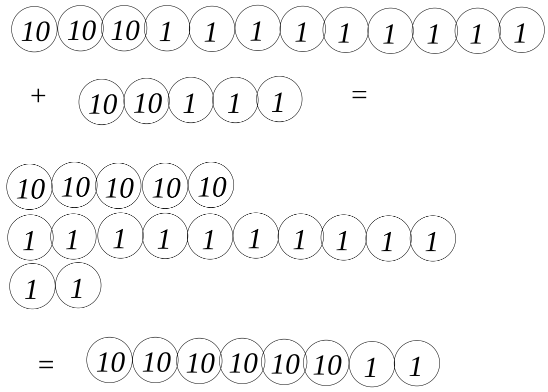


Figure 2: calculating 23 + 39.

was equivalent to one of the latter. The children could then begin addition exercises.[1] For example, to add 23 to 39, one converts the numbers to tokens, rearranges, and replaces ten unit tokens by 1 ten token, and gets the correct result (62) as illustrated in Figure 2.

Using labelled tokens has the following advantages:

- It makes addition comprehensible. In the example given, exchanging ten 1-tokens for one 10-token corresponds directly to 'carrying' a digit.
- The tokens can be used by the pupil at school and at home as a simple calculator. It is unnecessary to learn tables at this stage, or to add by laborious counting. I feel that the child should continue to use them for calculations until he or she has fully memorised the addition table.
- The same goes for subtraction. The tokens make sense of 'borrowing,' which involves exchanging one 10-token for ten 1-tokens.
- Including 100-tokens allows 3-digit calculations. Also, the child should be aware of their similarity to money, which is sure to make them interesting.
- The tokens are not so useful for multiplication, but they do help to make the process more comprehensible.
- Labelling tokens '1, 0.1, 0.01' allows decimal numbers to be fitted into the same scheme.
- The analogy to money could also help introduce the pupils to fractions; one could produce tokens labelled 1/2 and say that two of them had the same value as a unit token; similarly for 1/3 and so on.
- Simple algebra problems could be constructed by having some tokens colour-coded or labelled with an 'x,' say, to represent unknown quantities. It would be easy to represent equations which they satisfy such as $x + 2 = 5$, or $3x = 6$. The pupils could solve the first literally by *taking away* two units from each side and the second by *dividing* the 6 units into 3 groups of 2.

[1] The author has a faint memory of this in Montessori school, where blocks rather than tokens were used.

- The children should find it easier to handle addition and subtraction in other units, such as hours, minutes, and seconds, pints and fluid ounces, or metres, centimetres. and millimetres, or binary numbers, and so on.

These tokens should be used by all children. Other structural materials should, of course, be introduced. Their value is summarised in the following quotation.

> "Nine- and ten-year-olds are quite capable of dealing with concepts involving such things as weight, number, area, distance, or temperature provided they can operate in the presence of concrete referents." [1, p. 200]

5 Adding, taking away, multiplying, and dividing.

This section is about the words used for arithmetic operations. Arithmetic operations are based on everyday activities. If we take 2 marbles, and add 3 marbles to it, we have 2+3, or 5, marbles. If we take 1 marble away, we get $5 - 1 = 4$ marbles. Addition is about adding groups of things together; taking away (subtraction) is about taking things away.

In order to learn arithmetic properly one must bear such activities in mind.

Multiplication is about adding together several groups of the same size. The Syllabus is explicit about this: $3 \times 2 = 3 + 3$.

Order of factors in multiplication. At this point I should like to mention a curiosity about English usage which I think has some significance, though it looks trivial. It is about the order in which words are used.

Suppose we stretch the addition metaphor to idiotic lengths and say 'potato + potato = 2 potatoes.' An archaic way of saying the same thing is 'potato + potato = potato times 2.' I think that the way we do arithmetic implicitly uses the archaic convention. So 3×2 should be read either as 'three times two' or as 'two threes.' To say 'three twos' is wrong, though the result is the same. Three twos means $2 + 2 + 2$.

There are three reasons for mentioning this. First, the Syllabus dwells heavily on the fact that multiplication is commutative — so $3 \times 2 = 2 \times 3$. However, one cannot expect a child to understand this property if he or she doesn't know how to interpret it in terms of sums.

The second reason is that in calculating 3×2, the child should view the factors differently. 3 is the number being multiplied; 2 is the multiple. In practice, this is how one performs multiplication. In calculating 123×7, one 'processes' the 123 by multiplying each digit and carrying. So the factor 123 is more like an 'object.'

The third reason is that paying attention to the order of factors should help when learning fractions.

'Nought' versus 'Zero.' I feel that 'nought,' or better, 'nothing,' should be preferred to 'zero.' According to the dictionary, 'zero' is 'nought,' more-or-less, in Arabic. A child learning the tables may imagine that 'five times zero is five,' while agreeing that 'no fives are nothing.'

6 Memorising tables.

The 1971 Syllabus says at one point that, while children must memorise multiplication facts (i.e., tables) and recall them at will, that the memorisation must *follow* and not precede

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3						

Figure 3: addition table.

-	0	1	2	3	4	5
0	0					
1	1	0				
2	2	1	0			
3	3	2	1	0		

Figure 4: subtraction table.

... [certain exercises]. This must be excellent advice, but its effect could be that children learn their tables 2 years later than they should.

Let us begin with addition. I think that every child should build up a full addition table on a single sheet of paper with a 10×10 grid of numbers as illustrated in Figure 3.

The first row gives $0+1$, $0+2$, etcetera, and should be recited in this way. As a homework exercise, the children should fill in a row of the table — using counters, or the labelled tokens mentioned previously, to do the calculations. Each row gives one of the addition tables as we know them. The tables should then be memorised. Somehow it seems easier for children to memorise a table they have worked out and written themselves, rather than read in a book.

The subtraction tables (see Figure 4) can be constructed and learnt in the same way, likewise the multiplication table (Figure 5).

Initially, children should learn multiplication by forming rectangular arrangements of tokens. An array with three columns of tokens, two per column, shows $2+2+2=6$, or $2\times 3=6$.

The multiplication table should be built up row by row, like the addition table. The fourth row, for example, gives the '$3 \times \ldots$' table (the first row is entirely 0). The child can build it by repeated addition: $3 \times 0 = 0$, $3 \times 1 = 3$, $3 \times 2 = 3+3 = 6$, $3 \times 3 = 3+3+3 = 9\ldots$

Recitation of tables in unison is essential, and not unpleasant. As already mentioned, 2×3 is either 'three twos' or 'two times 3,' and never 'two threes.' (Of course, there should be no attempt to explain this. The children's knowledge of the distinction can be subconscious.)

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15

Figure 5: multiplication table.

Children should recite tables both ways: three times nought is nought, three times one is three, three times two is six ..., or, no threes are nothing, one three is three, two threes are six, ...

They can also recite the $\times 3$ row together with the $3\times$ column together, thereby learning the commutative property of multiplication.

There seems to be no reason to memorise 'times 11' or 'times 12' tables. They seem to be a hangover from pre-decimal currency.

7 Division

A child should begin studying multiplication by counting rectangular arrays of tokens. Conversely, division can be studied by fitting tokens into rectangular arrays. Division by ten is easy: for example, $23 \div 10 = 2$, remainder 3 (Figure 1 (a)). Figure 1 (b) illustrates $23 \div 7 = 3$, remainder 2. A child can solve simple division problems by activities of this kind. The work can be speeded up if frames of different sizes are prepared, to accommodate columns of height 9, 8, ... down to 5, say.

The addition and multiplication tables have 100 entries each; the subtraction table has 55 entries. But the division tables would have 440 entries, and not fit on one sheet. If we confine ourselves to division with no remainder, the multiplication table provides the necessary information. It might help if children were given a way to calculate remainders.

We are interested in calculating the remainder on dividing a 2-digit number by 2, 3, ... 9. The remainder on division by 9 is easily computed: add the tens and units; repeat until the result is less than 10.

For example, under this rule $49 \to 4 + 9 = 13 \to 1 + 3 = 4$. The remainder on dividing 49 by 9 is 4. This is the old trick of 'casting out the nines.' Figure 6 should make it less mysterious. Adding the tens t to the units u is the same as subtracting $9 \times t$ from the number; $49 - 9 \times 4 = 13$. Repeating: $13 - 9 \times 1 = 4$. The remainder is 4.

Now division by 9 is made easier. First calculate the remainder, and subtract it; then divide the result by 9. It is easy to divide a number by 9, if the number is at most 90 and an exact multiple of 9: just add 1 to the 'tens' digit. For example, $45 \div 9 = 4 + 1 = 5$.

Remainder on division by 8 is nearly as simple. We can 'cast out the eights.' This involves

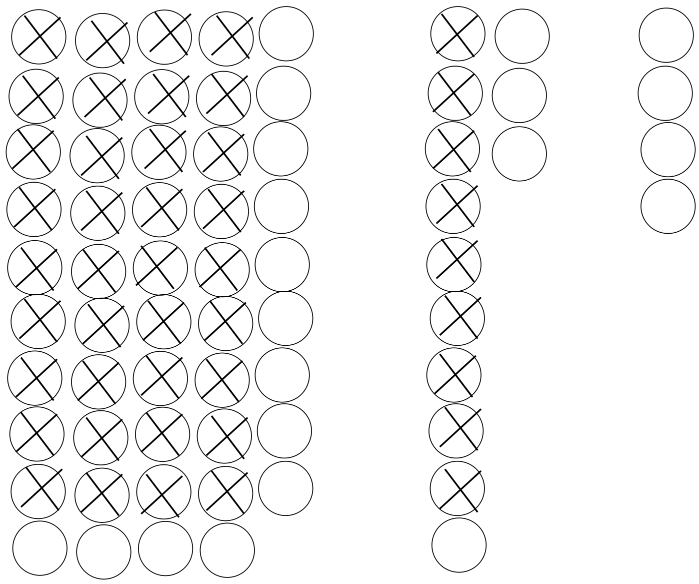


Figure 6: remainder on dividing 49 by 9.

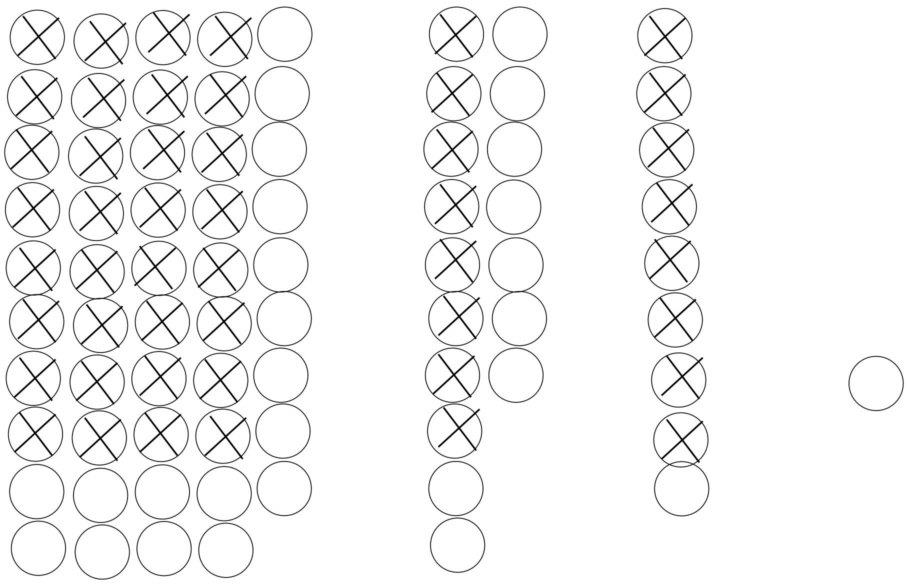


Figure 7: remainder on dividing 49 by 8.

adding *twice* the tens to the units and repeating if necessary. For example , $49 \to 2 \times 4 + 9 = 17 \to 2 \times 1 + 7 = 9 \to 9 - 8 = 1$. The answer is 1. See Figure 7.

Remainder on division by 7 can be calculated similarly: add *three times* the tens to the units and repeat if necessary. The rationale is the same as for 8 and 9.

Remainder on division by 3: calculate the remainder on division by 9, and take *its* remainder on division by 3. For example, $49 \to 4 \to 1$.

Remainder on division by 2 or by 5: work only with the units, discarding the tens. So remainder on dividing 49 by 2 (respectively, 5) is same as when dividing 9: 1 (respectively, 4).

Remainder on division by 4: calculate remainder on division by 8, then calculate *its* remainder on division by 4. For example $49 \to 1 \to 1$.

Remainder on division by 6: calculate the remainder on dividing by 9. Call the result r. Then calculate the quotient on dividing by 9. If the quotient is even, return the remainder on dividing r by 6. Otherwise return the remainder on dividing $r + 3$ by 6.

For example, $49 \div 9 = 5$, remainder 4. The quotient is odd, so calculate the remainder on dividing $4 + 3 = 7$ by 6. The answer is 1.

8 Fractions

The Syllabus mentions that children find it hard to grasp the rules of arithmetic for fractions. This is reasonable enough, since fractions are a very different kind of number. I think that fractions are best introduced using new categories of tokens, perhaps labelled $1/2, 1/3, \ldots$, or just coloured differently and interpreted as $1/2, 1/3, \ldots$ as the occasion demands.

Just as ten 1-tokens equal one ten-token, fractions can be introduced by $1/2 + 1/2 = 1$, i.e., $(1/2) \times 2 = 1$, $1/3 + 1/3 + 1/3 = 1$, i.e., $(1/3) \times 3 = 1, \ldots$

More general fractions can be defined. For example, $2/3 = 1/3 + 1/3 = (1/3) \times 2$; $23/7 = (1/7) \times 23$, and so on. Also mixed fractions can be defined in the obvious way: $3\frac{2}{7} = 3 + 2/7$. Without any further understanding of fractions, a child should be able to convert between improper fractions and mixed fractions. Figure 8 can be compared with Figure 1 (b). Each column can be replaced by 1, so $23/7 = 3\frac{2}{7}$. Having learnt division, the child should find it easy to convert improper to mixed and vice-versa.

Multiplication by a whole number, such as 2, has been discussed as an operation on things: similarly, multiplication by 1/2 can be considered as an operation on things. Again we should be aware of English usage. Read '$\times(1/2)$' either as 'times one half' or 'half of.' $(1/2) \times (1/3)$ is 'a third of one half.'

Children can become accustomed to multiplying quantities by fractions. Multiplication by (1/2) involves dividing the quantity into two equal halves and selecting one: 1 lb $\times(1/2)$ is half a pound, $6 \times (1/2) = 3$, and so on. Also $3 \times (1/2)$ can be calculated, using three 1-tokens: replace each 1-token by two (1/2)-tokens, so you have six (1/2)-tokens, which can be divided into two sets of 3. So the answer is 3/2.

This is a situation where it is helpful to work with pictures. Existing textbooks are satisfactory here. Figure 9 illustrates multiplication by various multiples of 1/5.

Acetate transparencies are useful for experimenting with fractions. One can prepare squares of the same size, divided into equal vertical strips, for halves, thirds, and so on. Figure 10 shows

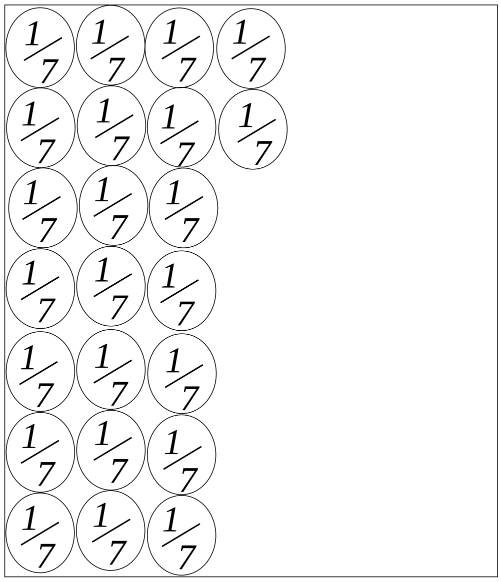


Figure 8: $\frac{23}{7} = 3\frac{2}{7}$.

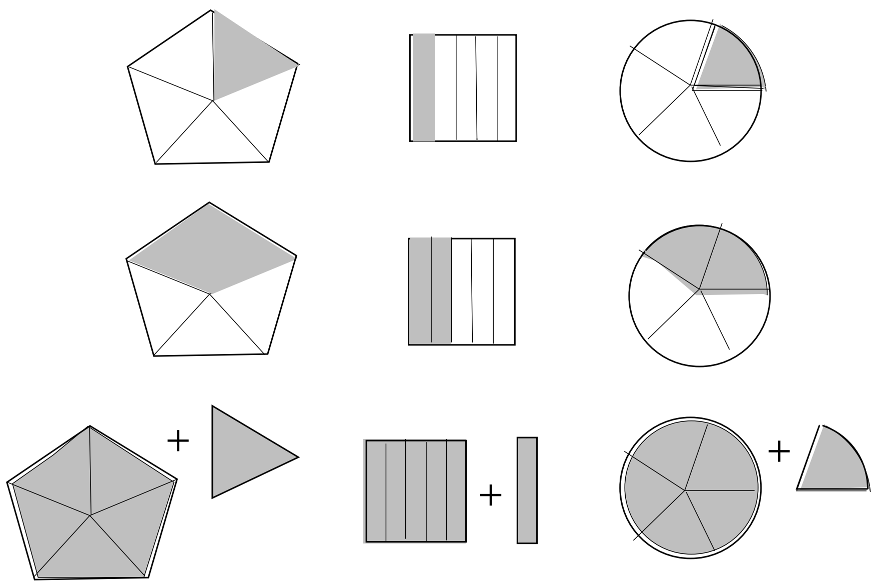


Figure 9: multiplication by 1/5, 2/5, 6/5.

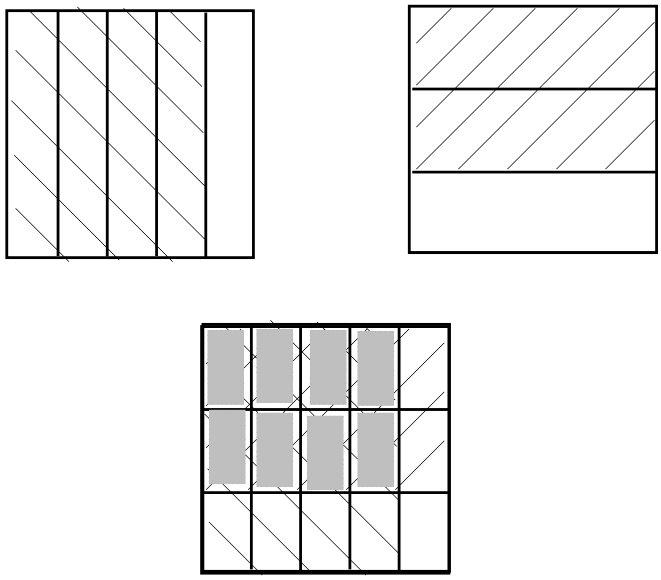


Figure 10: (2/3) of (4/5).

squares divided into fifths and thirds (they would be printed on separate slides: they can be rotated so the strips are vertical or horizontal).

The quantities 4/5 and 2/3 are illustrated by cross-hatching. By overlaying the slides, one gets $4/5 \times 2/3$, 2/3 of 4/5. The answer is 8/15. By experiments like these the child can learn the simple rule for multiplying fractions: multiply the numerators and multiply the denominators. At the same time they can learn that multiplication of fractions is commutative.

According to the Syllabus, children have difficulty in learning this .

For addition and subtraction of fractions, the 'currency model' is again useful. When adding fractions with different denominators, it is usually necessary to find another denominator to which they can both be converted. Overlaid slides can help illustrate how, to add 1/3 and 2/5, say, one needs to convert them into the same 'currency unit.' They can be expressed as multiples of 1/15, as can be seen from Figure 10. Then addition becomes easy: $5/15 + 6/15 = 11/15$.

9 Motives for changing the curriculum.

This article has been concerned with better ways of teaching the Mathematics curriculum in Primary schools. The material taught is satisfactory, but I believe it could be taught much more effectively with the aid of structural materials. The article is not concerned at all with making the Syllabus more ambitious or less ambitious.

Hwever, I do feel that there should be a single Curriculum, which every school should follow. In contrast, [3, p. 113] — admittedly a forty-year-old English publication — says

> We feel that too many factors are involved to allow of a fixed minimum content. In the first place there are great differences of environment between schools, which may be placed in industrial, rural, or suburban areas, each with its special problems, and there may have been a variety of approaches to number work in the infant

school from which the children come. The difficulties and opportunities of a very large school are quite different from those of the small rural school. Above all, since junior schools contain the whole range of ability within one generation, the pupils' mental powers must differ so widely that there should be no attempt to force children into conformity to one pattern.

All I can say is that if the educational system doesn't bother to teach all children, or to set generally attainable standards, then there is no democracy.

Of course, teachers would naturally add whatever they wish to the course, so long as they cover the basics.

The idea is to improve the *average* pupils knowledge of mathematics by making it easier to be successful. This should help make school more enjoyable for *all* children. Success in some (legitimate) activity is, of course, important to a child's general well-being. Obviously school is where it should occur. To to quote Dr. William Glassner [2, p. 5]

> a person ...will not succeed in general until he can in some way first experience success in one important part of his life.

What about those who begin badly, who come to detest school, and who resort, when possible, to truancy as an escape? Any weekday there are plenty of adolescents to be seen at large during school hours. Doubtless many of them are mitching, and doubtless many of them are up to no good.

It takes little to imagine such behaviour leading to social problems, and Dr. Glassner's book, quoted above, is most concerned with the connection. A a recent study in England has shown, not surprisingly, that prison inmates have below-average levels of literacy and academic performance in general.

In order to begin badly at mathematics, you need only make mistakes and not have the means to correct them. It is painful for a child to attempt a set of homework questions and wait till the next day to have all or most marked wrong. And almost as painful with work done in class. This can be avoided at the earliest, and probably most important, stage by teaching children to calculate with labelled tokens. It must surely be difficult to make mistakes with them. I believe that if children use such aids right from the start, and continue using them (for different purposes) until the end of primary school, the average pupil will be rewarded with success.

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