June 28, 1998

Renormal-order improvement of the Schwinger mass

C. Adam*
School of Mathematics, Trinity College, Dublin 2

Abstract

The massive Schwinger model may be analysed by a perturbation expansion in the fermion mass. However, the results of this mass perturbation theory are sensible only for sufficiently small fermion mass. By performing a renormal-ordering, we arrive at a chiral perturbation expansion where the expansion parameter remains small even for large fermion mass. We use this renormal-ordered chiral perturbation theory for a computation of the Schwinger mass and compare our results with lattice computations.

*)email address: adam@maths.tcd.ie, adam@pap.univie.ac.at

1 Introduction

The Schwinger model [1, 2], or two-dimensional QED with one fermion flavour, is one of the most widely studied quantum field theoretic models. It has the attractive feature of being rather simple, while, nevertheless, it has many properties similar to those of more complicated gauge theories. Among them are the chiral anomaly [3]–[7], the formation of a chiral condensate and the non-trivial vacuum structure (θ-vacuum) [2, 5], [8]–[12]. In addition, the fundamental fermion is confined, and only mesonic states occur in the physical particle spectrum [1, 2].

The Lagrangian density of the model is

$$L = \bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

When the fermion is massless ($m = 0$), the model may be solved exactly, and it turns out to be equivalent to the theory of one free, massive boson field ("Schwinger boson") with Schwinger mass $\mu = e/\sqrt{\pi}$.

For the massive ($m \neq 0$) Schwinger model, the Schwinger boson turns into an interacting particle, and its mass acquires corrections. The investigation of the massive Schwinger model started more than 20 years ago with a series of now classical papers [13]–[17] and has continued since then (see [5, 18] for a review). Quite recently, the corrections to the Schwinger mass, as well as the formation of higher bound states and the chiral condensate in the massive Schwinger model, have been studied by a variety of methods. Among these methods are mass perturbation theory [18]–[22], light-front quantization [23]–[28], lattice computations [29]–[34], and a generalized Hartree-Fock approach [35]–[39]. The multi-flavour case has been discussed, e.g., by [37]–[42].

In this article we want to compute the Schwinger mass of the massive Schwinger model by a method that is similar to mass perturbation theory. By performing a renormal-ordering, we shall find a new expansion parameter (instead of the fermion mass m) that remains rather small even for large m. Therefore, the resulting chiral perturbation expansion will produce a Schwinger mass that tends to the result of mass perturbation theory for small fermion mass m, but remains reasonable even for large m. Finally, we shall compare our results to some lattice data.

We use the bosonized, Euclidean version of the massive Schwinger model in the sequel, and our conventions for the massive scalar propagator are

$$D_\mu(x) = -\frac{1}{2\pi}K_0(\mu|x|), \qquad \widetilde{D}_\mu(p) = \frac{-1}{p^2 + \mu^2}, \tag{2}$$

where K_0 is the McDonald function.

2 Renormal-ordered chiral perturbation theory

Our starting point is the Euclidean, bosonized Lagrangian density of the massive Schwinger model [13, 14, 17],

$$L = -N_\mu \Big[\frac{1}{2}\Phi(\Box - \mu^2)\Phi + \frac{e^\gamma}{2\pi}\mu m \cos(\sqrt{4\pi}\Phi + \theta)\Big]. \tag{3}$$

Here $\mu = e/\pi^{1/2}$ is the Schwinger mass of the massless ($m = 0$) Schwinger model, θ is the vacuum angle and $\gamma = 0.5772$ is the Euler constant. Further, N_μ denotes normal-ordering w.r.t. μ.

By making use of the well-known normal-ordering relation

$$N_{\bar\mu} e^{\pm i\beta\Phi(x)} = (\frac{\mu}{\bar\mu})^{\frac{\beta^2}{4\pi}} N_\mu e^{\pm i\beta\Phi(x)} \tag{4}$$

(see [43]), we may rewrite the Lagrangian density (3) for arbitrary normal-ordering mass $\bar\mu$ like

$$L = -N_{\bar\mu} \Big[\frac{1}{2}\Phi(\Box - \mu^2)\Phi + \frac{e^\gamma}{2\pi}\bar\mu m \cos(\sqrt{4\pi}\Phi + \theta)\Big] \tag{5}$$

up to an irrelevant constant. Next, let us observe that the interaction term $\sim \cos\sqrt{4\pi}\Phi$ contains a Φ^2 piece, too. We now want to shift this Φ^2 piece from the interaction term to the mass term in (5), and we want to choose the normal-ordering mass $\bar\mu$ equal to the total mass of the resulting mass term. We find the equation

$$\bar\mu^2 = \mu^2 + 2e^\gamma \cos\theta\, \bar\mu m \tag{6}$$

with the solution

$$\bar\mu = e^\gamma \cos\theta\, m + \sqrt{\mu^2 + e^{2\gamma}\cos^2\theta\, m^2}. \tag{7}$$

Our new Lagrangian density reads

$$\begin{aligned} L &= -N_{\bar\mu}\Big[\frac{1}{2}\Phi(\Box - \bar\mu^2)\Phi + \lambda(\frac{1}{2}(e^{i\theta}e^{i\sqrt{4\pi}\Phi} + e^{-i\theta}e^{-i\sqrt{4\pi}\Phi}) + 2\pi\cos\theta\,\Phi^2 - \cos\theta)\Big] \\ &=: L_0 + L_{\mathrm{I}} \end{aligned} \tag{8}$$

$$\lambda = \frac{e^\gamma}{2\pi} m\bar\mu, \tag{9}$$

where we have subtracted a constant ($\cos\theta$) in the interaction term in order to avoid infinite vacuum energy contributions to the perturbation series. Further, we rewrote the sine and cosine of the field Φ in terms of exponentials, because the latter ones have simpler Wick contractions (see (12) below). By this simple rewriting of the Lagrangian density we have achieved a major advantage. A perturbative expansion of the original version (3) of the Lagrangian is an expansion in powers of m and will, obviously, only lead to sensible results for sufficiently small m. On the other hand, for the new version (8) of the Lagrangian, the (dimensionless) expansion parameter of the perturbation expansion is

$$\frac{\lambda}{\bar\mu^2} = \frac{e^\gamma m}{2\pi\bar\mu} = \frac{e^\gamma m}{2\pi(e^\gamma\cos\theta\, m + \sqrt{\mu^2 + e^{2\gamma}\cos^2\theta\, m^2})}, \tag{10}$$

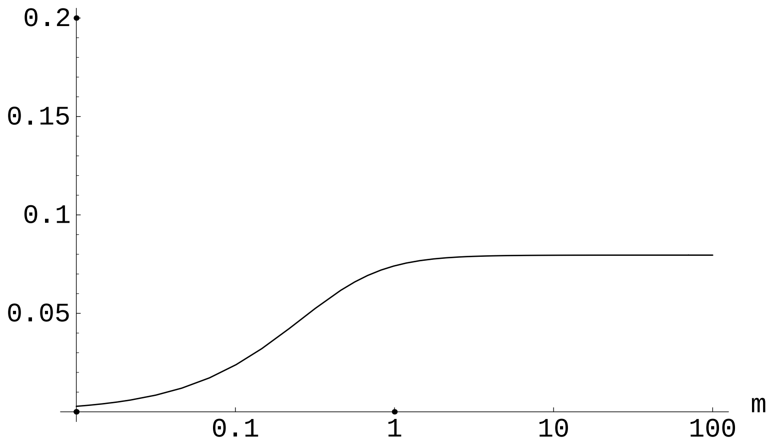

Figure 1: The figure shows how the expansion parameter $\frac{\lambda}{\bar{\mu}^2}$ (y-axis) varies with the fermion mass m. We use units $\mu = 1$.

which is rather small for all m provided that the vacuum angle θ is not too large. E.g., for $\theta = 0$ we find

$$0 = (\frac{\lambda}{\bar{\mu}^2})(m = 0, \theta = 0) < (\frac{\lambda}{\bar{\mu}^2})(m, \theta = 0) < (\frac{\lambda}{\bar{\mu}^2})(m = \infty, \theta = 0) = \frac{1}{4\pi} \tag{11}$$

(see Fig. 1).

Therefore, the perturbative expansion for the theory defined by (8) leads to finite results for all m and, as the expansion parameter is quite small, one may hope that these results are at least qualitatively correct even for large m. The interaction Lagrangian in (8) contains a Φ^2 piece in addition to the exponentials, therefore the resulting chiral perturbation theory is slightly more complicated than the usual mass perturbation expansion, because it contains Wick contractions among both polynomials and exponentials in Φ. All these may be inferred from the general formula (see [18, 41, 42])

$$\langle \prod_{l=1}^{n} e^{i\sigma_l \beta \Phi(x_l)} \rangle_0 [\rho] = e^{\sum_{i<j} \sigma_i \sigma_j \beta^2 D_{\bar{\mu}}(x_i - x_j)} \cdot$$

$$\cdot e^{-\frac{1}{2} \int dy_1 dy_2 \rho(y_1) D_{\bar{\mu}}(y_1 - y_2) \rho(y_2) + i\beta \sum_{k=1}^{n} \sigma_k \int dy \rho(y) D_{\bar{\mu}}(y - x_k)}, \tag{12}$$

where $\sigma_l = \pm 1$ and $\rho(x)$ is a source term for the $\Phi(x)$ field (i.e., $\Phi(x_i)$ are inserted into the VEV (12) by taking functional derivatives $\delta/\delta\rho(x_i)$). The subscript 0 means $\lambda = 0$, and $\beta = \sqrt{4\pi}$ in our case. Further, the exponentials on the l.h.s. are normal-ordered w.r.t. the mass of the Φ field, $\bar{\mu}$.

The Lagrangian density (8), together with the Wick contraction formula (12), will now serve as a starting point for the perturbative computation of the two-point function $\langle \Phi(y_1) \Phi(y_2) \rangle_\lambda$ and for the approximate determination of the Schwinger mass.

3 Two-point function and Schwinger mass

Before actually starting the computation, let us observe that already the zeroth order Schwinger mass $\bar{\mu}$ gives a certain prediction for large m if $\theta = 0$ (numerical estimates for large m will always be done for $\theta = 0$, because for $\theta \neq 0$ the expansion parameter $\lambda/\bar{\mu}^2$ is larger in the large m region, making thereby numerical estimates less reliable). We find

$$\lim_{m\to\infty} \bar{\mu}(m,\theta=0) = 2e^{\gamma}m \simeq 3.56m. \tag{13}$$

The true value is $2m$, because in the limit of vanishing (electromagnetic) interaction the Schwinger boson Φ is just composed of two free fermions, each of mass m. Therefore, this lowest order estimate is nearly 80% off the exact value (which is not too bad for a lowest order estimate). Further, when Taylor-expanded in m, $\bar{\mu}$ reproduces the first order correction of mass perturbation theory (M_0 ... Schwinger mass in zeroth order)

$$M_0 := \bar{\mu} = \mu + e^{\gamma}\cos\theta\, m + o(\frac{m^2}{\mu^2}). \tag{14}$$

This feature should generalize to higher orders: The two perturbation expansions (in m for the Lagrangian (3); in λ for the Lagrangian (8)) are just different formulations of the same theory; further, the parameter λ (or equivalently $\lambda/\bar{\mu}^2$) has a well-defined Taylor expansion in m, starting with a linear (m^1) term. Therefore, the order n result of the perturbation series in λ, when expanded in m up to n-th order, should coincide with the n-th order mass perturbation theory result. We conclude that there is no λ^1 contribution to the λ perturbation series for the Schwinger mass.

The perturbation expansion for the two-point function is

$$\langle\Phi(y_1)\Phi(y_2)\rangle_\lambda = \frac{1}{Z}\langle\Phi(y_1)\Phi(y_2)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\prod_{l=1}^{n}\int dx_l L_{\rm I}(x_l)\rangle_0, \tag{15}$$

but we prefer not to include the $1/Z$ normalization factor explicitly, and we will remove disconnected pieces by hand. Therefore we want to compute

$$\langle\Phi(y_1)\Phi(y_2)\rangle_\lambda = \langle\Phi(y_1)\Phi(y_2)\rangle_0 + \langle\Phi(y_1)\Phi(y_2)\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\prod_{l=1}^{n}\int dx_l L_{\rm I}(x_l)\rangle_0. \tag{16}$$

First we observe that the λ^1 contribution is indeed zero. This may be checked by explicit computation, but it can be understood immediately from the following observation. When only one $L_{\rm I}$ is inserted into the two-point function (16), then for the exponentials $\exp \pm i\sqrt{4\pi}\Phi(x)$ in $L_{\rm I}(x)$ only the Φ^2 part is effective, because a higher number of boson lines emerging from the vertex $L_{\rm I}$ cannot be Wick-contracted to anything. But precisely this Φ^2 piece is subtracted in $L_{\rm I}$, (8), therefore the net contribution is zero.

Up to second order, we obtain

$$\langle\Phi(y_1)\Phi(y_2)\rangle_\lambda = -D_{\bar{\mu}}(y_1-y_2) + \frac{\lambda^2}{8}\int dx_1 dx_2 \langle\Phi(y_1)\Phi(y_2)\cdot$$

$$\cdot(e^{i\theta}e^{i\sqrt{4\pi}\Phi(x_1)} + e^{-i\theta}e^{-i\sqrt{4\pi}\Phi(x_1)} + 4\pi\cos\theta\,\Phi^2(x_1) - 2\cos\theta)\cdot$$
$$\cdot(e^{i\theta}e^{i\sqrt{4\pi}\Phi(x_2)} + e^{-i\theta}e^{-i\sqrt{4\pi}\Phi(x_2)} + 4\pi\cos\theta\,\Phi^2(x_2) - 2\cos\theta), \tag{17}$$

where we now have to perform all the Wick contractions with the help of formula (12). Using the symmetry $D_{\bar\mu}(x) = D_{\bar\mu}(-x)$ and the relabelling symmetry $\int dx_1 dx_2 f(x_1,x_2) = \int dx_1 dx_2 f(x_2,x_1)$ and omitting the disconnected pieces, we arrive at (c ...connected)

$$\langle\Phi(y_1)\Phi(y_2)\rangle_\lambda^{\mathrm{c}} = -D_{\bar\mu}(y_1-y_2) + 2\pi\lambda^2\int dx_1dx_2\cdot$$
$$\cdot\Big[-\cos 2\theta e^{4\pi D_{\bar\mu}(x_1-x_2)}(D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2) + D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1))$$
$$-e^{-4\pi D_{\bar\mu}(x_1-x_2)}(-D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2) + D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1))$$
$$+(\cos 2\theta+1)(D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1) + 2\pi D_{\bar\mu}(y_1-y_2)D^2_{\bar\mu}(x_1-x_2)$$
$$+8\pi D_{\bar\mu}(x_1-x_2)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2) + 8\pi^2 D^2_{\bar\mu}(x_1-x_2)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1))$$
$$+(\cos 2\theta+1)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1)$$
$$-4\pi(\cos 2\theta+1)D_{\bar\mu}(x_1-x_2)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2)$$
$$-(\cos 2\theta+1)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1)\Big], \tag{18}$$

where in the λ^2 part the first two terms (with the $\exp\pm 4\pi D_{\bar\mu}(x_1-x_2)$ factors) are from the $\langle\Phi\Phi\exp\Phi\exp\Phi\rangle$ contractions, the next term is from the $\langle\Phi\Phi\Phi^2\exp\Phi\rangle$ contractions, and the three last terms are from the $\langle\Phi\Phi\exp\Phi\rangle$, $\langle\Phi\Phi\Phi^2\Phi^2\rangle$ and $\langle\Phi\Phi\Phi^2\rangle$ contractions, respectively. It is important now that in expression (18) the exponentials $\exp\pm 4\pi D_{\bar\mu}(x_1-x_2) = 1 \pm 4\pi D_{\bar\mu}(x_1-x_2) + \ldots$ that are integrated separately get their constant piece 1 cancelled by some other contributions, and those exponentials that are convoluted with propagators get their first order piece $\pm 4\pi D_{\bar\mu}(x_1-x_2)$ cancelled (their constant piece may be omitted by hand, because it is a disconnected term). Therefore, we arrive at

$$\langle\Phi(y_1)\Phi(y_2)\rangle_\lambda^{\mathrm{c}} = -D_{\bar\mu}(y_1-y_2) - 2\pi\lambda^2\cdot$$
$$\cdot\Big[\cos 2\theta(E_+(\bar\mu)\int dx D_{\bar\mu}(y_1-x)D_{\bar\mu}(y_2-x) + \int dx_1dx_2 E^{(1)}_+(x_1-x_2,\bar\mu)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2))$$
$$+E_-(\bar\mu)\int dx D_{\bar\mu}(y_1-x)D_{\bar\mu}(y_2-x) - \int dx_1dx_2 E^{(1)}_-(x_1-x_2,\bar\mu)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_2)) \tag{19}$$
$$-8\pi^2(\cos 2\theta+1)\int dx_1dx_2 D^2_{\bar\mu}(x_1-x_2)D_{\bar\mu}(y_1-x_1)D_{\bar\mu}(y_2-x_1)\Big], \tag{20}$$

where $E_\pm(\bar\mu), E^{(1)}_\pm(x,\bar\mu)$ are defined like

$$E_\pm(x,\bar\mu) = e^{\pm 4\pi D_{\bar\mu}(x)} - 1, \quad E_\pm(\bar\mu) = \int d^2x E_\pm(x,\bar\mu)$$
$$E^{(1)}_\pm(x,\bar\mu) = e^{\pm 4\pi D_{\bar\mu}(x)} - 1 \mp 4\pi D_{\bar\mu}(x). \tag{21}$$

The last term in the above expression for $\langle\Phi(y_1)\Phi(y_2)\rangle^{\rm c}_\lambda$, which we singled out by its own number (20), will be significant later on.

The Fourier transform of the two-point function reads

$$\langle\widetilde{\Phi\Phi}\rangle^{\rm c}_\lambda(p) = \frac{1}{p^2+\bar\mu^2} - \frac{2\pi\lambda^2}{(p^2+\bar\mu^2)^2}\Big[\cos 2\theta(E_+(\bar\mu) + \widetilde{E}^{(1)}_+(p,\bar\mu))$$

$$+E_-(\bar\mu) - \widetilde{E}^{(1)}_-(p,\bar\mu) - 8\pi^2(\cos 2\theta+1)\widetilde{D^2_{\bar\mu}}(0)\Big] \tag{22}$$

$$\simeq \frac{1}{p^2+\bar\mu^2+2\pi\lambda^2(\cos 2\theta(E_+(\bar\mu)+\widetilde{E}^{(1)}_+(p,\bar\mu)) + E_-(\bar\mu) - \widetilde{E}^{(1)}_-(p,\bar\mu) - \frac{2\pi}{\bar\mu^2}(\cos 2\theta+1)}, \tag{23}$$

where we inverted the two-point function perturbatively and used

$$\widetilde{D^2_{\bar\mu}}(0) = \frac{1}{4\pi\bar\mu^2} \tag{24}$$

($\widetilde{D^2_{\bar\mu}}(p)$ is just a two-boson blob).

The Schwinger mass is given by the zero of the denominator in (23), which leads to the equation (after rescaling, e.g., $\widetilde{E}^{(1)}_\pm(p,\bar\mu) = \frac{1}{\bar\mu^2}\widetilde{E}^{(1)}_\pm(\frac{p}{\bar\mu},1)$, etc.)

$$-\frac{p^2}{\bar\mu^2} = 1 + \frac{2\pi\lambda^2}{\bar\mu^4}\Big[\cos 2\theta(E_+(1)+\widetilde{E}^{(1)}_+(\frac{p}{\bar\mu},1)) + E_-(1) - \widetilde{E}^{(1)}_-(\frac{p}{\bar\mu},1) - 2\pi(\cos 2\theta+1)\Big] \tag{25}$$

with the second order solution (M_2 ... second order Schwinger mass)

$$M_2^2 := -p^2|_{\lambda^2} = \bar\mu^2 + \frac{2\pi\lambda^2}{\bar\mu^2}\Big[\cos 2\theta(E_+(1)+\widetilde{E}^{(1)}_+(i,1))$$

$$+E_-(1) - \widetilde{E}^{(1)}_-(i,1) - 2\pi(\cos 2\theta+1)\Big]$$

$$= \bar\mu^2 + \frac{2\pi\lambda^2}{\bar\mu^2}(2\pi\cos 2\theta A_+ + 2\pi A_- - 2\pi(\cos 2\theta+1)), \tag{26}$$

where

$$A_\pm = \int_0^\infty dr r\Big[e^{-2K_0(r)} - 1 + I_0(r)(\pm e^{\mp 2K_0(r)} \mp 1 + 2K_0(r))\Big] \tag{27}$$

$$A_+ = -0.6599, \qquad A_- = 1.7277 \tag{28}$$

see [20, 18] (we changed notation from A, B in [20, 18] to A_+, A_- here). Re-expressed in μ, m we find

$$M_2^2 = \mu^2 + 2e^{2\gamma}\cos^2\theta\, m^2 + 2e^\gamma\cos\theta\, m\sqrt{\mu^2+e^{2\gamma}\cos^2\theta\, m^2}$$

$$+e^{2\gamma}m^2(A_+\cos 2\theta + A_-) - e^{2\gamma}m^2(\cos 2\theta+1)$$

$$= \mu^2 + 2e^\gamma\cos\theta\, m\sqrt{\mu^2+e^{2\gamma}\cos^2\theta\, m^2} + e^{2\gamma}m^2(A_+\cos 2\theta + A_-). \tag{29}$$

Observe that one piece of the second order (in λ) contribution, stemming from the term that we singled out in (20), has precisely cancelled the $o(m^2)$ piece of the lowest order (in λ) contribution, $\bar{\mu}^2$. This cancellation ensures that M^2, when Taylor expanded in m up to second order, coincides with the second order result of mass perturbation theory (MPT),

$$(M_2^{\rm MPT})^2 = \mu^2 + 2e^\gamma \cos\theta\, \mu m + e^{2\gamma} m^2 (A_+ \cos 2\theta + A_-), \tag{30}$$

see [20, 18].

Numerically, when evaluated at $\theta = 0$ and for large m,

$$\lim_{m\to\infty} M_2(m, \theta = 0) = m e^\gamma (2 + A_+ + A_-)^{1/2} \simeq 3.12m, \tag{31}$$

we find that $M_2(m = \infty, \theta = 0)$ is still about 55% off the true value. Therefore, a higher order computation would be highly desirable. However, already the third order computation is very complicated. Still, we are able to compute a part of the third order contribution with considerably less effort. To achieve this aim, we use the requirement that the perturbation series in λ, when Taylor expanded in m, has to coincide with the mass perturbation series up to the given order. The lowest order contribution (in λ) to M^2, $\bar{\mu}^2$, contains a third order (in m) piece,

$$\bar{\mu}^2|_{m^3} = e^{3\gamma} \cos^3\theta\, \frac{m^3}{\mu}, \tag{32}$$

and such a term cannot be produced by ordinary mass perturbation theory. Therefore, the third order (in λ) contribution must contain a term that, when Taylor expanded in m, precisely cancels the term (32). The contribution that does the job reads

$$\delta_1 M_3^2 = -(2\pi)^3 \cos^3\theta\, \frac{\lambda^3}{\bar{\mu}^4} = -e^{3\gamma} \cos^3\theta\, \frac{m^3}{\bar{\mu}}. \tag{33}$$

A further contribution of order λ^3 may be identified by the same argument via a closer inspection of ordinary mass perturbation theory in order m^3. For this, observe that one contribution to our second order result (26) stemmed from the evaluation of $\widetilde{E}_\pm^{(1)}(\frac{p}{\bar{\mu}}, 1)$ at $(p/\bar{\mu})^2 = -1$. For second order mass perturbation theory an identical term occurs, but this time evaluated at $(p/\mu)^2 = -1$. In addition, in mass perturbation theory there is a first order contribution $\delta(M_1^{\rm MPT})^2 = 2e^\gamma \cos\theta \mu m$, therefore there must be a third order contribution (in mass perturbation theory!)

$$\delta(M_1^{\rm MPT})^2 \frac{\partial}{\partial(-p^2)} (\cos 2\theta\, \widetilde{E}_+^{(1)}(p, 1) - \widetilde{E}_-^{(1)}(p, 1))|_{p^2=-1} \tag{34}$$

or, including all coefficients,

$$-(2e^\gamma \cos\theta \mu m) \frac{e^{2\gamma} m^2}{2\pi\mu^2} \frac{\partial}{\partial p^2} (\cos 2\theta\, \widetilde{E}_+^{(1)}(p, 1) - \widetilde{E}_-^{(1)}(p, 1))|_{p^2=-1}$$

$$= -\frac{e^{3\gamma}\cos\theta\, m^3}{\pi\mu}\frac{\partial}{\partial p^2}\int dx e^{ipx}(\cos 2\theta\, E_+^{(1)}(x,1) - E_-^{(1)}(x,1))|_{p^2=-1}$$

$$= 2e^{3\gamma}\cos\theta\,\frac{m^3}{\mu}(\cos 2\theta B_+ - B_-) \tag{35}$$

$$B_\pm = \int drr^3 I_0(r) E_\pm^{(1)}(r,1) \tag{36}$$

$$B_+ = 1.449, \qquad B_- = 1.912. \tag{37}$$

This third order contribution must be matched by a corresponding third order term in λ perturbation expansion (observe that this term in λ perturbation theory cannot be produced in the same way, because there is no first order correction δM_1 in λ perturbation theory; most likely, the term is produced by the $\langle\Phi\Phi\Phi^2\exp\Phi\exp\Phi\rangle$ contributions). This corresponding term reads

$$\delta_2 M_3^2 = 2(2\pi)^3\cos\theta\,\frac{\lambda^3}{\bar\mu^4}(\cos 2\theta\, B_+ - B_-) = 2\cos\theta\, e^{3\gamma}\frac{m^3}{\bar\mu}(\cos 2\theta\, B_+ - B_-). \tag{38}$$

Therefore, our new, partially third order corrected expression for the Schwinger mass is

$$M^2_{2\frac{1}{2}} = M_2^2 + \delta_1 M_3^2 + \delta_2 M_3^2. \tag{39}$$

Numerically, for $\theta = 0$, $M_{2\frac{1}{2}}$ has the large m limit

$$\lim_{m\to\infty} M_{2\frac{1}{2}}(m,\theta=0) = 2.58m, \tag{40}$$

which is nearly 30% above the exact value $2m$.

Of course, there are still some third order contributions missing, but as they are due to higher-dimensional convolution integrals of exponentially decaying functions, we may hope that their contribution is not too large.

We display the results of our renormal-ordered chiral perturbation theory (for $\theta = 0$) in Fig. 2, and compare them with the lattice data of [29, 30] in Fig. 3. Fig. 3 shows that our result $M_{2\frac{1}{2}}$ agrees with the lattice data very well up to $m \sim 1$; for larger m the deviation increases, reaching the above-mentioned 30% for $m \to \infty$.

For an overview on additional data for the Schwinger mass (e.g., from light-front computations) we refer to [25] (especially their Fig. 2 and Table 16), and to [26].

4 Summary

By a simple renormal-ordering of the bosonic Lagrangian of the massive Schwinger model we were able to find a new perturbation expansion parameter, $\frac{\lambda}{\bar\mu^2}$ (see (10), Fig. 1), that remains rather small even for large fermion mass m. The resulting chiral perturbation theory is similar to mass perturbation theory [20, 18], although it involves somewhat more tedious computations. We used this renormal-ordered chiral perturbation theory to compute the Schwinger mass M, including all second order (in $\frac{\lambda}{\bar{\mu}^2}$) and some third order contributions. The resulting expression for the Schwinger mass M reduces to the result from mass perturbation theory ([20]) when Taylor-expanded in the fermion mass m. For large m our result describes the known data for the Schwinger mass (e.g. the lattice data from [29]) qualitatively correct, although there remains some deviation (e.g. our result $M_{2\frac{1}{2}}$ differs by nearly 30% from the known value $2m$ at the far end $m \to \infty$).

Actually, we may conclude from Fig. 3 that high order (in $\frac{\lambda}{\bar{\mu}^2}$) perturbative contributions will be relevant for $m \to \infty$. The reason for this is that the expansion parameter $\frac{\lambda}{\bar{\mu}^2}$ is nearly constant for $m \in [1, \infty]$, see Fig. 1. Therefore, low order contributions to M will shift the curves for $M_{2\frac{1}{2}}$ (or M_2) in Fig. 3 by a nearly constant amount for $m \in [1, \infty]$. However, the curves in Fig. 3 are closer to the lattice data at $m = 1$ than at $m = \infty$, and they can only approach the lattice data everywhere when high orders in $\frac{\lambda}{\bar{\mu}^2}$ give a non-negligible contribution.

Finally, it should be mentioned that it could be interesting to apply this renormal-ordered chiral perturbation theory to the computation of other physical quantities of the massive Schwinger model (e.g., higher bound states, chiral condensate). In addition, it might be possible to use a version of this renormal-ordered chiral perturbation theory for the massive multi-flavour Schwinger model, where ordinary mass perturbation theory fails due to infrared divergencies [41, 42].

Acknowledgement

The author thanks the members of the Department of Mathematics at Trinity College, where this work was performed, for their hospitality. Further thanks are due to T. Heinzl and C. Stern for the data of [29] and [30] in Fig. 3.

References

[1] J. Schwinger, Phys. Rev. 128 (1962) 2425

[2] J. Lowenstein, J. Swieca, Ann. Phys. 68 (1971) 172

[3] S. Adler, Phys. Rev. 177 (1969) 2426

[4] J. Bell, R. Jackiw, Nuovo Cim. A60 (1969) 47

[5] E. Abdalla, M. Abdalla, K. D. Rothe, “2 dimensional Quantum Field Theory”, World Scientific, Singapore 1991

[6] C. Adam, R. A. Bertlmann, P. Hofer, Riv. Nuovo Cim. 16 (1993) 1

[7] R. Bertlmann, “Anomalies in Quantum Field Theory”, Clarendon Press, Oxford 1996

[8] J. Hetrick, Y. Hosotani, Phys. Rev. D38 (1988) 2621

[9] C. Jayewardena, Helv. Phys. Acta 61 (1988) 636

[10] I. Sachs, A. Wipf, Helv. Phys. Acta 65 (1992) 653

[11] A. Smilga, Phys. Lett. B278 (1992) 371

[12] C. Adam, Z. Phys. C63 (1994) 169

[13] S. Coleman, R. Jackiw, L. Susskind, Ann. Phys. 93 (1975) 267

[14] S. Coleman, Ann. Phys. 101 (1976) 239

[15] A. Casher, J. Kogut, L. Susskind, Phys. Rev. D10 (1974) 732

[16] J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 3594

[17] J. Fröhlich, E. Seiler, Helv. Phys. Acta 49 (1976) 889

[18] C. Adam, Ann. Phys. 259 (1997) 1

[19] C. Adam, Phys. Lett. B363 (1995) 79

[20] C. Adam, Phys. Lett. B382 (1996) 383

[21] C. Adam, Phys. Lett. B382 (1996) 111

[22] C. Adam, Phys. Rev. D55 (1997) 6299

[23] T. Eller, H. Pauli, Phys. Rev. D35 (1987) 1493

[24] T. Heinzl, Phys. Lett. B388 (1996) 129

[25] K. Harada, T. Heinzl, C. Stern, Phys. Rev. D57 (1998) 2460

[26] J. Vary, T. Fields, H. Pirner, Phys. Rev. D53 (1996) 7231

[27] S. Elser, A. Kallionatis, Phys. Lett. B375 (1996) 285

[28] M. Burkardt, hep-th 9712032

[29] D. Crewther, C. Hamer, Nucl. Phys. B170 (1980) 353

[30] S. R. Carson, R. D. Kenway, Ann. Phys. 166 (1986) 364

[31] H. Gausterer, C. B. Lang, Phys. Lett. B341 (1994) 46

[32] H. Gausterer, C. B. Lang, Nucl.Phys. B455 (1995) 785

[33] C. Gattringer, Phys. Rev. D53 (1996) 5090

[34] P. de Forcrand, J. Hetrick, T. Takaishi, A. van der Sijs, Nucl Phys. Proc. Suppl. 63 (1998) 679

[35] Y. Hosotani, R. Rodriguez, Phys. Lett. B389 (1996) 121

[36] Y. Hosotani, R. Rodriguez, hep-th 9804205

[37] J. Hetrick, Y. Hosotani, S. Iso, Phys. Lett. B350 (1995) 92

[38] J. Hetrick, Y. Hosotani, S. Iso, Phys. Rev. D53 (1996) 7255

[39] Y. Hosotani, R. Rodriguez, Phys. Lett. B375 (1996) 273

[40] A. Smilga, Phys. Rev. D55 (1997) 443

[41] C. Gattringer, HEP-TH 9503137

[42] C. Gattringer, Ann. Phys. 250 (1996) 389

[43] S. Coleman, Phys. Rev. D11 (1975) 2088

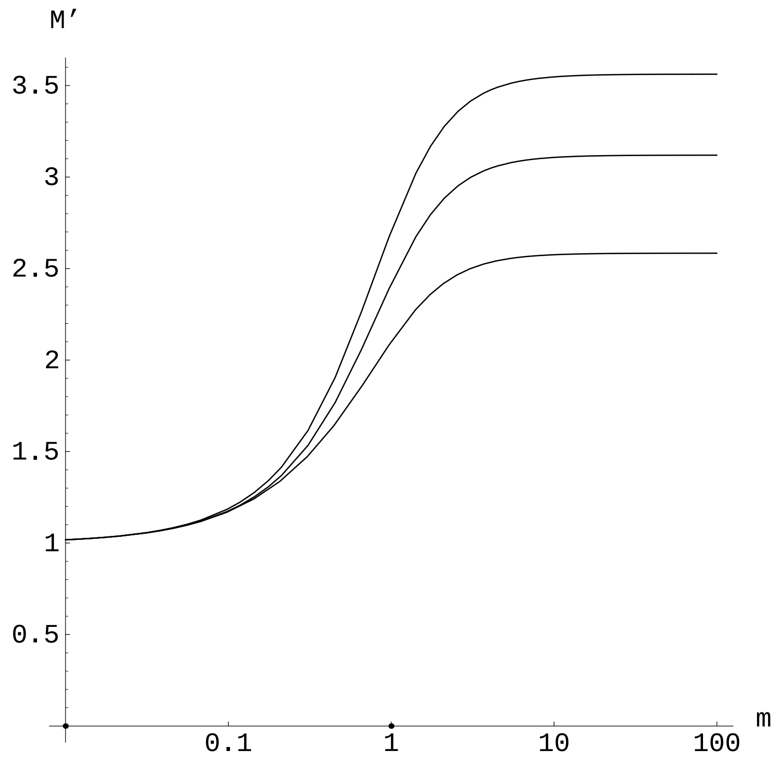

Figure 2: The figure shows successive perturbative results for the Schwinger mass. $\bar{\mu}$ is the upper curve, M_2 is the curve in the middle and $M_{2\frac{1}{2}}$ is the lowest curve. We use the rescaled Schwinger mass $M' = M/(1+m^2)^{1/2}$, like in [25, 26], and choose units $\mu = 1$.

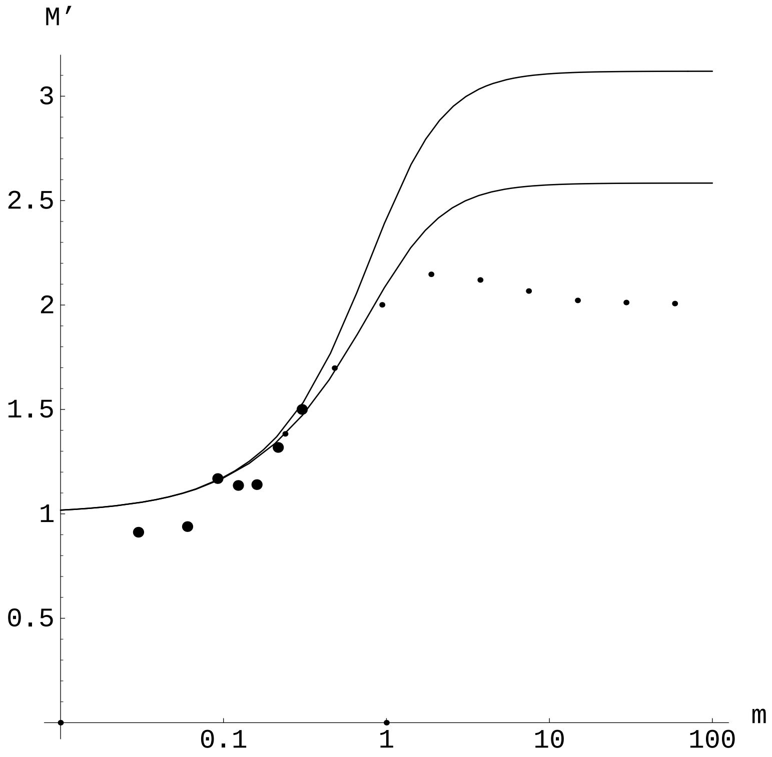

Figure 3: The figure compares our results M_2, $M_{2\frac{1}{2}}$ to the lattice data of [29] (small dots, for larger m) and [30] (fat dots, for small m, with rather large errors). We again use the rescaled Schwinger mass $M' = M/(1+m^2)^{1/2}$ and units $\mu = 1$.