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Normalization of the chiral condensate in the massive Schwinger model

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Abstract

Within mass perturbation theory, already the first order contribution to the chiral condensate of the massive Schwinger model is UV divergent. We discuss the problem of choosing a proper normalization and, by making use of some bosonization results, we are able to choose a normalization so that the resulting chiral condensate may be compared, e.g., with lattice data.

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1 Introduction

The massive Schwinger model, or massive QED_2 with one fermion flavour,

$$L = \bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

has been studied for some time because it resembles QCD in many respects. The investigation started more than 20 years ago with some classical papers [1] – [4] and has continued ever since (for a review see e.g. [5, 6]). Quite recently, the model has been studied in some detail within mass perturbation theory [6] – [10] as well as with light-front methods [11] – [13], on the lattice [14] – [18], and by a generalized Hartree-Fock approach on the circle [19, 20]. The multi-flavour case, too, has received some attention recently [20] – [26].

Some of the features that make the model so attractive are the presence of instanton-like gauge field configurations, and, consequently, a nontrivial vacuum structure (θ vacuum) [27, 1, 2]; further the chiral anomaly [28] and the formation of a nonzero chiral condensate $\langle\bar{\Psi}\Psi\rangle$ [27].

In the massless case ($m = 0$), the chiral condensate may be computed exactly [27], [30] – [33]. For $m \neq 0$, a mass perturbation expansion can be performed and corrections to $\langle\bar{\Psi}\Psi\rangle_{m=0}$ may be computed [7, 6]. However, already at order m^1 a UV singularity occurs that has to be regularized, and a proper normalization for $\langle\bar{\Psi}\Psi\rangle_m$ has to be chosen. It is the purpose of this article to discuss this point and to arrive at an expression for $\langle\bar{\Psi}\Psi\rangle_m$ that may be compared, e.g., to lattice computations. In the sequel, all computations are done for two-dimensional, Euclidean space-time.

2 Chiral condensate up to order m

The Euclidean, bosonized version of the theory (1) reads [1, 2, 4, 25]

$$L_{\mathrm{b}} = -N_\mu\Big[\phi(\Box - \mu^2)\phi + \frac{e^\gamma}{2\pi}\mu m\cos(\sqrt{4\pi}\phi + \theta)\Big] \tag{2}$$

where $\mu = e/\pi^{1/2}$ is the Schwinger mass, N_μ denotes normal ordering w.r.t. μ, θ is the vacuum angle, and $\gamma = 0.5772$ is the Euler constant.

The vacuum condensate $\langle\bar{\Psi}\Psi\rangle_m$ is

$$\langle\bar{\Psi}\Psi\rangle_m = -\frac{e^\gamma}{2\pi}\mu\langle\cos(\sqrt{4\pi}\phi + \theta)\rangle_m \tag{3}$$

within the bosonized version of the theory and may be evaluated by a perturbation expansion in m. The lowest order expression is the wellknown condensate of the massless model [31] – [33]

$$\langle\bar{\Psi}\Psi\rangle_{m=0} = -\frac{e^\gamma}{2\pi}\mu\cos\theta. \tag{4}$$

For a higher order computation it is useful to rewrite the interaction part of the bosonic Lagrangian like

$$L_{\mathrm{b,I}} = -\frac{e^\gamma}{2\pi}\mu m \frac{1}{2} N_\mu \Big[e^{i\theta} e^{i\sqrt{4\pi}\phi} + e^{-i\theta} e^{-i\sqrt{4\pi}\phi} \Big], \tag{5}$$

because the exponentials $\exp(\pm i\sqrt{4\pi}\phi)$ have especially simple n-point functions within the massless model, e.g. (see [6])

$$\langle e^{\sigma_1 i\sqrt{4\pi}\phi(x_1)} e^{\sigma_2 i\sqrt{4\pi}\phi(x_2)} \rangle_{m=0} = e^{\sigma_1\sigma_2 4\pi D_\mu(x_1-x_2)} \tag{6}$$

where $\sigma_1, \sigma_2 = \pm 1$, $D_\mu(x)$ is the massive scalar propagator

$$D_\mu(x) = -\frac{1}{2\pi} K_0(\mu|x|), \qquad \widetilde{D}_\mu(p) = \frac{-1}{p^2+\mu^2} \tag{7}$$

(K_0 ... McDonald function). Further, powers of $e^{i\theta}$ indicate the contributing instanton sectors ($e^{in\theta}$ ... instanton number $k=n$).

The exponentials $\exp(\pm 4\pi D_\mu(x))$ have the limit $\lim_{|x|\to\infty} \exp(\pm 4\pi D_\mu(x)) = 1$, therefore a disconnected piece has to be subtracted, and the correct propagators for the mass perturbation expansion are the functions

$$E_\pm(x,\mu) = e^{\pm 4\pi D_\mu(x)} - 1. \tag{8}$$

For the chiral condensate in order m^1 one finds easily (see [7, 6]; in [7, 6] the sign of $\langle\bar\Psi\Psi\rangle$ is reversed due to an opposite sign convention for the fermion mass m in the Lagrangian)

$$\begin{aligned} \langle\bar\Psi\Psi\rangle_m^{(1)} &= -\frac{1}{2} m\mu^2 \frac{e^{2\gamma}}{4\pi^2} \int d^2x (E_+(x,\mu)\cos 2\theta + E_-(x,\mu)) \\ &= -\frac{1}{2} m \frac{e^{2\gamma}}{4\pi^2} \int d^2x (E_+(x,1)\cos 2\theta + E_-(x,1)). \end{aligned} \tag{9}$$

Taking into account the short-distance behaviour of $K_0(z)$,

$$K_0(z) \sim -\gamma - \ln\frac{z}{2} \qquad \text{for} \qquad z \to 0 \tag{10}$$

one easily finds that the integral w.r.t. $E_+(x,1)$ is UV finite,

$$E_+ := \int d^2x E_+(x,1) = -8.9139 \tag{11}$$

whereas the integral w.r.t. $E_-(x,1)$ is logarithmically UV divergent. Actually, these findings can be understood immediately from ordinary perturbation theory (in e) for the massless Schwinger model. The $E_+(x,1)$ contribution in (9) is purely non-perturbative (in e) (it receives contributions from the instanton sectors $k=\pm 2$). On the other hand, the $E_-(x,1)$ contribution (from the $k=0$ sector)

$$\langle\bar\Psi\Psi\rangle_m^{(1),k=0} = -\frac{m}{2}\int d^2x \langle\bar\Psi(x)\Psi(x)\bar\Psi(0)\Psi(0)\rangle_{c,m=0}^{k=0} \tag{12}$$

(c ... connected part) is purely perturbative. For small distances $|x|$ the integrand in (12) behaves like

$$\langle\bar{\Psi}(x)\Psi(x)\bar{\Psi}(0)\Psi(0)\rangle_{c,m=0}^{k=0} = \mu^2 \frac{e^{2\gamma}}{4\pi^2} e^{2K_0(\mu|x|)} - 1 \sim \frac{1}{2\pi^2 x^2}, \tag{13}$$

which is just the lowest order contribution of the perturbative expansion (in e) within the massless Schwinger model

$$\langle\bar{\Psi}(x)\Psi(x)\bar{\Psi}(0)\Psi(0)\rangle_{m=0}^{k=0} = \mathrm{tr} G_0(x) G_0(-x) + o(e^2) = \frac{1}{2\pi x^2} + o(e^2) \tag{14}$$

where $G_0(x) = i\gamma_\mu x^\mu/(2\pi x^2)$ is the free, massless fermion propagator. Higher order contributions may be ignored in the $|x| \to 0$ limit (asymptotic freedom). [These higher order terms consist of all possible insertions of massive photon lines into the two massless fermion propagators $G_0(x)$, and turn out to exponentiate, leading to formula (6) for $\sigma_1\sigma_2 = -1$ (the photon acquires the Schwinger mass μ via the Schwinger mechanism, i.e., the summation of all vacuum polarization insertions, see [29, 28, 34]).]

In [7] we regulated the chiral condensate by just isolating this free fermion singularity via a partial integration

$$\begin{aligned}
\langle\bar{\Psi}\Psi\rangle_m^{(1),k=0} &= -\frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}\int d^2x (e^{2K_0(|x|)} - 1) \\
&= -\frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}\lim_{\epsilon\to 0} 2\pi \int_\epsilon^\infty \frac{dr}{r}(e^{2K_0(r)+2\ln r} - r^2) \\
&= -m\pi\frac{e^{2\gamma}}{4\pi^2}\Big(\lim_{\epsilon\to 0}\Big[\ln r (e^{2K_0(r)+2\ln r} - r^2)\Big]_\epsilon^\infty \\
&+ \int_0^\infty dr \ln r [2(K_1(r) - \frac{1}{r})e^{2K_0(r)+2\ln r} + r]\Big) \\
&= \frac{m}{\pi}\lim_{\epsilon\to 0}\ln\epsilon - \frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}E_-
\end{aligned} \tag{15}$$

$$E_- := 2\pi\int_0^\infty dr \ln r [2(K_1(r) - \frac{1}{r})e^{2K_0(r)+2\ln r} + r] = 9.7384 \tag{16}$$

($K_0' = -K_1$) where we performed the limit where it is safe. Normalizing the chiral condensate by just omitting the $\ln\epsilon$ term is plausible from the viewpoint of mass perturbation theory, because the latter relies on the (exact solution of the) massless Schwinger model, and omitting the $\ln\epsilon$ term just amounts to omitting the non-interacting contribution of the massless Schwinger model.

However, this normalization is not appropriate for a comparison with, e.g., lattice data (as was pointed out in [20]). Instead, one has to choose the normalization of ordinary perturbation theory,

$$\langle\bar{\Psi}\Psi\rangle'_m = \langle\bar{\Psi}\Psi\rangle_m - \langle\bar{\Psi}\Psi\rangle_m^{e=0}. \tag{17}$$

Here one may wonder whether this normalization may be chosen within the context of mass perturbation theory. We will find that this is possible due to the specific two-dimensional feature of bosonization, as we want to discuss now.

To obtain $\langle\bar\Psi\Psi\rangle_m^{e=0}$ in the bosonic language, we would just like to redo our computation for the $\mu\to 0$ limit of the bosonic Lagrangian $L_{\rm b}$, (2). However, as both coupling constants and the normal ordering in $L_{\rm b}$ depend on μ, we should first do a renormal-ordering. Using the normal-ordering relation (see [35])

$$N_m e^{\pm i\beta\phi(x)} = (\frac{\mu}{m})^{\frac{\beta^2}{4\pi}} N_\mu e^{\pm i\beta\phi(x)} \tag{18}$$

we find (up to an irrelevant additive constant)

$$L_{\rm b} = -N_m\Big[\phi(\Box-\mu^2)\phi + \frac{e^\gamma}{2\pi}m^2\cos(\sqrt{4\pi}\phi+\theta)\Big] \tag{19}$$

where now the limit $\mu\to 0$ can be performed. Further, for $\mu = 0$ the vacuum angle θ may be compensated by a shift of the field ϕ and can, therefore, be set equal to zero (i.e., there are no instanton sectors when there is no gauge interaction), and one gets

$$L_{\rm b}^{e=0} = -N_m\Big[\phi\Box\phi + \frac{e^\gamma}{2\pi}m^2\cos\sqrt{4\pi}\phi\Big]. \tag{20}$$

Within the bosonic approach we now would be left with the task of performing a perturbation expansion for a massless scalar field, which is IR divergent. But at this point bosonization results may be used. The Lagrangian $L_{\rm b}^{e=0}$ is a version of the sine-Gordon model, which is known to be the QFT analog of the massive Thirring model [35]. More precisely, after a volume cutoff is introduced, the perturbative expansion of the sine-Gordon model with the cosine term as interaction Lagrangian is equivalent to a perturbative expansion of the massive Thirring model with the fermion mass term as interaction term. Specifically, when the coefficient β in $\cos\beta\phi$ is $\beta=\sqrt{4\pi}$, the bosonic theory (20) is equivalent to the “Thirring model” with zero coupling, i.e., the model with one free, massive fermion. The equivalence of the two perturbation expansions mentioned above implies that the fermionic Lagrangian has to be normal-ordered w.r.t. zero fermion mass.

The essential point is, of course, that the VEV $\langle\bar\Psi\Psi\rangle_m^{e=0}$ can be computed exactly in the fermionic formulation, without the need to actually perform a (IR-divergent) mass perturbation expansion. The correct normal-ordering prescription just means that the regularized expression for $\langle\bar\Psi\Psi\rangle_m^{e=0}$ has to vanish in the $m\to 0$ limit. Explicitly we find from ordinary Feynman rules

$$\langle\bar\Psi\Psi\rangle_m^{e=0} = \int\frac{d^2p}{4\pi^2}{\rm tr}\frac{-i}{\not p - im} = \int\frac{d^2p}{4\pi^2}\frac{2m}{p^2-1} = -\frac{m}{4\pi^3}\int d^2pd^2xe^{ipx}K_0(|x|). \tag{21}$$

Once the integral is regularized, this expression indeed vanishes for $m\to 0$.

Now we have to subtract this expression from the $k=0$ contribution to the chiral condensate in order m, (9). For a unified regularization prescription of both terms we slightly shift the argument of $K_0(|x|)$, $K_0(|x|) \to K_0(|x|+\epsilon)$, in both expressions and obtain

$$\begin{aligned}
\langle\bar\Psi\Psi\rangle_m^{\prime(1),k=0} &= \langle\bar\Psi\Psi\rangle_m^{(1),k=0} - \langle\bar\Psi\Psi\rangle_m^{e=0} \\
= -\frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}2\pi\int_0^\infty drr(e^{2K_0(r+\epsilon)}-1) &+ \frac{m}{\pi}\int d^2x\delta(x)K_0(x+\epsilon) \\
= -m\pi\frac{e^{2\gamma}}{4\pi^2}\int_\epsilon^\infty dr(r-\epsilon)(e^{2K_0(r)}-1) &+ \frac{m}{\pi}K_0(\epsilon) \\
= -m\pi\frac{e^{2\gamma}}{4\pi^2}\int_\epsilon^\infty dr(\frac{1}{r}-\frac{\epsilon}{r^2})(e^{2K_0(r)+2\ln r}-r^2) &+ \frac{m}{\pi}K_0(\epsilon) \\
= \frac{m}{\pi}\ln\epsilon - \frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}E_- &+ \frac{m}{\pi}(-\ln\epsilon-\gamma+\ln 2) \\
+m\pi\frac{e^{2\gamma}}{4\pi^2}\epsilon\Big[\frac{-1}{r}(e^{2K_0(r)+2\ln r}-r^2)\Big]_\epsilon^\infty &+ o(\epsilon) \\
= \frac{m}{\pi}(-\gamma+\ln 2) - \frac{m}{2}\frac{e^{2\gamma}}{4\pi^2}E_- + m\pi\frac{e^{2\gamma}}{4\pi^2}e^{-2\gamma+2\ln 2} &+ o(\epsilon)
\end{aligned} \tag{22}$$

where we performed a partial integration in both integrals and used the result (15) for the first integral. The remainder of the second integral that we did not display explicitly is of order ϵ. Putting everything together, we obtain for the chiral condensate up to order m

$$\langle\bar\Psi\Psi\rangle_m' = -\frac{e^\gamma}{2\pi}\mu\cos\theta - m\Big[\frac{e^{2\gamma}}{8\pi^2}(E_+\cos 2\theta + E_-) - \frac{1}{\pi}(1+\ln 2-\gamma)\Big] \tag{23}$$

$$= -0.283\,\mu\cos\theta + 0.358\,m\cos 2\theta - 0.036\,m \tag{24}$$

where we inserted all the numbers in the last line. For $\theta = 0$ we obtain

$$\langle\bar\Psi\Psi\rangle_{m,\theta=0}' = -0.283\,\mu + 0.322\,m \tag{25}$$

which may be compared to the lattice results of [18] and to the generalized Hartree-Fock computation of [20], see Fig. 1. The results can be seen to agree modestly for sufficiently small fermion mass m.

3 Discussion

With the help of bosonization arguments we succeeded in finding a normalization of the chiral condensate, $\langle\bar\Psi\Psi\rangle_m' = \langle\bar\Psi\Psi\rangle_m - \langle\bar\Psi\Psi\rangle_m^{e=0}$, that implies a matching between mass perturbation theory and ordinary, electric charge perturbation theory. This normalization is the appropriate one for a comparison with lattice data.

Observe that we have two consistency checks for our bosonization prescription of $\langle\bar\Psi\Psi\rangle_m^{e=0}$. Firstly, the precise cancellation of the singularity shows that we have properly matched the "coupling constants" of the bosonic and fermionic prescriptions of $\langle\bar\Psi\Psi\rangle_m^{e=0}$. Secondly, as already mentioned, the regularized expression for $\langle\bar\Psi\Psi\rangle_m^{e=0}$ has to vanish

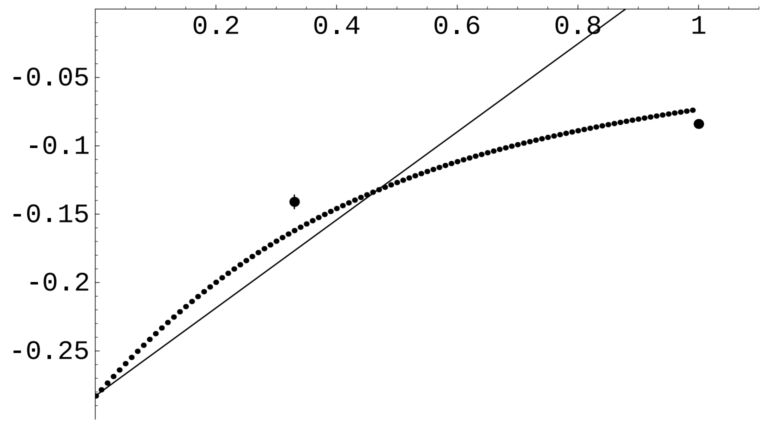

Figure 1: The figure shows fermion mass m/μ (x-axis) vs. chiral condensate $\langle\bar{\Psi}\Psi\rangle'_m/\mu$ (y-axis), both in units of μ. The straight line is the mass perturbation result (25); the dotted curve is the generalized Hartree-Fock result of [20]; the two points with error bars are the lattice results $(0.33, -0.141)$, $(1.00, -0.084)$ of [18]

in the $m \to 0$ limit because of the bosonization rules, which it does indeed. Actually, $\langle\bar{\Psi}\Psi\rangle_m^{e=0}$ turns out to be exactly of order m, (21), as it must be for dimensional reasons.

This fact may lead to some speculation about UV-finiteness. The point is that the normalization $\langle\bar{\Psi}\Psi\rangle'_m = \langle\bar{\Psi}\Psi\rangle_m - \langle\bar{\Psi}\Psi\rangle_m^{e=0}$ is the normalization of ordinary (in e) perturbation theory. Further, within ordinary perturbation theory, the massive Schwinger model is super-renormalizable (i.e., finite after normal-ordering) [1, 2]. This leads to the conjecture that all the higher order contributions to $\langle\bar{\Psi}\Psi\rangle_m$ within mass perturbation theory are UV finite. It is, however, a rather difficult (and not yet solved) problem to prove (or disprove) this UV finiteness directly within mass perturbation theory.

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