Asymptotic expansion of the lattice scalar propagator in coordinate space

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Abstract

The asymptotic expansion of the massive scalar field propagator on a n- dimensional lattice is derived. The method used is based on the evaluation of the asymptotic expansion of the modified Bessel function $\mathrm{I}_\nu(\nu^2\beta)$ as the order ν grows to infinity.

Key words: Lattice propagator. Asymptotic methods. Bessel function $\mathrm{I}_\nu(\nu^2\beta)$.

1 Introduction

In perturbation theory, the advantage of a coordinate space description as a way of studying the divergences arising in Feynman diagrams has been discussed by many authors (*e.g.* [1,2]). In the continuum, the analytic expression for the scalar field propagator in position space is well-known. On the lattice, however, the standard representation for the scalar propagator involves integrals over Bessel functions and has proved to be very difficult to analyze in the continuum limit.

Several attempts have been made to derive a suitable expansion for the lattice scalar propagator in the limit where the lattice spacing goes to zero. In particular, we mention the analysis performed by Lüscher and Weisz for the massless propagator [3] as well as the research carried out by Burgio, Caracciolo and Pelissetto in [4]. In the first of these studies, the x- dependence of the massless propagator is asymptotically derived. In the second, the m-dependence is obtained for the propagator at $x = 0$. The propagator at any other point $x \neq 0$ is then expressed, through a set of recursion relations, in terms of the value which its massless counterpart assumes on an element of the unit hypercube (i. e, x_μ either 0 or 1). In the present paper we wish to tackle the most general case of both m and x non-vanishing.

The procedure that we adopt attacks the question directly at its core. We derive an asymptotic expansion of the modified Bessel function of the first kind which appears in the expression of the lattice propagator. The order and argument of this Bessel function go to infinity at different rates as the lattice spacing decreases to zero. We have found no tabulated asymptotic expansion for this particular case. Expansions for the modified Bessel function of the first kind, $I_\nu(x)$, are, indeed, available for the cases where either the argument x or the order ν becomes large, or for the case where both x and ν grow to infinity keeping the value of their respective ratio constant (approximantion by tangents). Unfortunately, none of the cases just mentioned characterizes the modified Bessel function at hand, and we have had to develop, as a result, a new asymptotic expansion.

We introduce now briefly the notations that will be assumed in the following sections. Throughout this paper, we shall work on a n- dimensional lattice of finite spacing a. The convention

$$x^p = \sum_{\mu=1}^{n} x_\mu^p, \quad \text{with } p \text{ an integer }, \tag{1}$$

will also be used.

The Euclidean free scalar propagators in a n-dimensional configuration space will be denoted by $\Delta^{\mathrm{C}}(x;n)$ and $\Delta^{\mathrm{L}}(x;n)$ with

$$\Delta^{\mathrm{C}}(x;n) = \int_{-\infty}^{+\infty} \frac{d^n p}{(2\pi)^n} \frac{e^{\imath px}}{m^2 + p^2} , \tag{2}$$

$$\Delta^{\mathrm{L}}(x;n) = \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \frac{d^n p}{(2\pi)^n} \frac{e^{\imath px}}{m^2 + \hat{p}^2} , \tag{3}$$

referring to the propagator evaluated in the continuum and on the lattice, respectively. Note that in eq. (3) we have introduced the short-hand notation

$$\hat{p}^2 = \frac{4}{a^2} \sum_{\mu=1}^{n} \sin^2\left(\frac{p_\mu a}{2}\right) . \tag{4}$$

2 The propagator on the lattice

It is a well- known result that the continuum scalar field propagator in a 4-dimensional configuration space can be written in terms of the modified Bessel function K_1. The technique consists in using the Schwinger's representation for the propagator in momentum space. Re-expressing, then, the integral over the four-momentum as a product of four one-dimensional integrals and completing the square in the resulting exponential, we can finally perform the integration and obtain our final formula. Adopting the notational convention given by eq. (1), we write the n-dimensional expression as follows

$$
\begin{aligned}
\Delta^{\mathrm{C}}(x;n) &= \int_{-\infty}^{+\infty} \frac{d^n p}{(2\pi)^n} \frac{e^{\imath p x}}{m^2+p^2} \\
&= \int_0^{\infty} d\alpha e^{-m^2\alpha} \prod_{\mu=1}^{n} \int_{-\infty}^{+\infty} \frac{dp_\mu}{(2\pi)} \exp\left\{-p_\mu^2\alpha + \imath p_\mu x_\mu\right\} \\
&= (2\pi)^{-n/2} \left[\frac{(x^2)^{1/2}}{m}\right]^{1-n/2} \mathrm{K}_{1-\mathrm{n}/2}\left[\mathrm{m}(\mathrm{x}^2)^{1/2}\right].
\end{aligned} \tag{5}
$$

The derivation of the standard representation for the lattice propagator in configuration space is carried out in a much similar fashion. Indeed, we have

$$
\begin{aligned}
\Delta^{\mathrm{L}}(x;n) &= \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \frac{d^n p}{(2\pi)^n} \frac{e^{\imath p x}}{m^2+\hat{p}^2} \\
&= \int_0^{\infty} d\alpha e^{-m^2\alpha} \left\{\prod_{\mu=1}^{n} e^{-\frac{2\alpha}{a^2}} \left(\frac{1}{a}\right) \int_{-\pi}^{\pi} \frac{d\vartheta_\mu}{(2\pi)} e^{\frac{2\alpha}{a^2}\cos\vartheta_\mu} \cos\left[\left(\frac{x_\mu}{a}\right)\vartheta_\mu\right]\right\} \\
&= \int_0^{\infty} d\alpha e^{-m^2\alpha} \left\{\prod_{\mu=1}^{n} e^{-\frac{2\alpha}{a^2}} \left(\frac{1}{a}\right) \mathrm{I}_{\frac{\mathrm{x}_\mu}{\mathrm{a}}}\left(\frac{2\alpha}{\mathrm{a}^2}\right)\right\}
\end{aligned} \tag{6}
$$

with $\mathrm{I}_{\frac{\mathrm{x}_\mu}{\mathrm{a}}}\left(\frac{2\alpha}{\mathrm{a}^2}\right)$ corresponding to the modified Bessel function of the first kind. Unfortunately, the integral appearing in eq. (6) cannot be trivially solved. As a consequence, we are not able in this case to express the propagator in closed form. What we really wish to do here, though, is to show that in the continuum limit, *i.e.* when $a \to 0$, $\Delta^{\mathrm{L}}(x;n)$ is given by the sum of $\Delta^{\mathrm{C}}(x;n)$ plus a series of correction terms depending on increasing powers of the lattice spacing a.

As already mentioned in the introduction, the most direct way to proceed in order to achieve our goal is trying to derive the asymptotic expansion for the modified Bessel function $\mathrm{I}_{\frac{\mathrm{x}_\mu}{\mathrm{a}}}(\frac{2\alpha}{\mathrm{a}^2})$ as $a \to 0$.

The strategy to adopt for the actual derivation of the expansion is determined by the fact that the order and the argument of the modified Bessel function become large at different rates. The standard techniques of *global* analysis (e.g, the steepest descents method) are not of much use in this case. As a consequence, we are forced to choose here a *local* analysis approach. This implies beginning our study by examining the differential equation satisfied by the modified Bessel function at hand. With the purpose of determining uniquely the asymptotic behaviour of the solution, we shall also impose in the end the condition that the series representation derived reproduce (through its leading term) the continuum result $\Delta^{\mathrm{C}}(x;n)$.
We first commence by setting

$$\beta \equiv \frac{2\alpha}{x_\mu^2} \quad and \quad \nu \equiv \frac{x_\mu}{a} \quad , \tag{7}$$

i.e. $\mathrm{I}_{\mathrm{x}_\mu/\mathrm{a}}(2\alpha/\mathrm{a}^2) \to \mathrm{I}_\nu(\nu^2\beta)$.
Thus, we wish to find an expansion for $\mathrm{I}_\nu(\nu^2\beta)$ as $\nu \to \infty$. We observe now that $\mathrm{I}_\nu(\nu^2\beta)$ satisfies a differential equation of the form: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\nu^2)y=0$. Hence, performing the change of variable

$$x \to \nu^2\beta \tag{8}$$

we obtain

$$\frac{\partial^2\mathrm{I}_\nu}{\partial\beta^2}(\nu^2\beta)+\frac{1}{\beta}\frac{\partial\mathrm{I}_\nu}{\partial\beta}(\nu^2\beta)-\nu^2\left[\frac{1}{\beta^2}+\nu^2\right]\mathrm{I}_\nu(\nu^2\beta)=0. \tag{9}$$

Eq. (9) simplifies slightly if we make the substitution

$$\mathrm{I}_\nu(\nu^2\beta)=\frac{\mathrm{C}}{\sqrt{\beta}}\mathrm{Y}(\beta,\nu). \tag{10}$$

C represents here the free parameter whose value shall be fixed later on according to the prescription made at the beginning of this paragraph.
By using eq. (10), we now get a new differential equation for $\mathrm{Y}(\beta,\nu)$, namely

$$\frac{\partial^2\mathrm{Y}(\beta,\nu)}{\partial\beta^2}-\left[\nu^4+\frac{\nu^2-\frac{1}{4}}{\beta^2}\right]\mathrm{Y}(\beta,\nu)=0. \tag{11}$$

We consider at this stage the limit $\nu \to \infty$. We now need to make some kind of assumption on the form of leading term governing the expansion of the solution to eq. (11) as ν becomes large. With this purpose, we consider a substitution originally suggested by Carlini (1817), Liouville (1837) and

Green (1837) and whose underlying idea is that the controlling factor in the asymptotic expansion of a function is usually in the form of an exponential. Hence, we assume the asymptotic relation

$$\mathrm{Y}(\beta,\nu) \sim \mathrm{e}^{\nu^2\beta+\int \mathrm{w}(\beta,\nu)\mathrm{d}\beta}. \tag{12}$$

Actually, the form which we have used here for $\mathrm{Y}(\beta,\nu)$ differs slightly from the standard Carlini's substitution since it implicitly assumes that not only the leading term, but also all the subsequent terms in the expansion are expressible as exponentials. The reason which led us to adopt eq. (12) as a viable option is essentially twofold: on one hand, it is dictated by the necessity of recovering the well-known expression for the free scalar propagator in the continuum; on the other, the idea spurred in analogy to the method adopted in Watson [5] and originally due to Meissel (1892) in order to deduce the asymptotic expansion for the Bessel function of the first kind, $\mathrm{J}_\nu(\nu \mathrm{z})$ with ν large.
Carrying out the substitution in the differential equation for $\mathrm{Y}(\beta,\nu)$, we end up with a first-order inhomogeneous differential equation for the new function $w(\beta,\nu)$, *i.e.*

$$\frac{\partial w(\beta,\nu)}{\partial\beta} + w^2(\beta,\nu) + 2\nu^2 w(\beta,\nu) - \frac{\nu^2-\frac{1}{4}}{\beta^2} \sim 0. \tag{13}$$

Our aim, at this point, is to look for a suitable series representation for the solution to this equation. With this purpose, we make the following observations. First of all, if we suppose that $2\nu^2 w(\beta,\nu) - \nu^2/\beta^2 \sim 0$ represents a good approximation to eq. (13) when $\nu \to \infty$, then it is straightforward to derive the leading behaviour of $w(\beta,\nu)$ as $(2\beta^2)^{-1}$. We note that this gives exactly the continuum result $\Delta^{\mathrm{C}}(x;n)$ once the overall solution to eq. (9) is normalized by fixing the arbitrary parameter C to be equal to $(2\pi\nu^2)^{-1/2}$. Secondly, we observe that all the non-leading terms in the expansion of $w(\beta,\nu)$ should feature only even powers of ν, since only even powers of this variable appear in eq. (13).
As a result of the remarks just made, we assume that $w(\beta,\nu)$ admits an asymptotic series representation for ν large of the type

$$w(\beta,\nu) = \sum_{n=0}^{\infty} a_n(\beta)\nu^{-2n}. \tag{14}$$

This is consistent with the approximation that we considered when we deduced the form of the leading term of the solution. Substituting now this formula into eq. (13), we get

$$2\sum_{n=-1}^{\infty} a_{n+1}(\beta)\nu^{-2n} \sim \frac{\nu^2-\frac{1}{4}}{\beta^2} - \sum_{n=0}^{\infty}\left\{\left[\sum_{m=0}^{n} \mathrm{a_m}(\beta)\mathrm{a_{n-m}}(\beta)\right] + \frac{\partial}{\partial\beta}a_n(\beta)\right\}\nu^{-2n}. \tag{15}$$

Matching, order by order, the coefficients corresponding to the same power of ν, we are now able to deduce the set of relations defining the coefficients $a_n(\beta)$. We have

$$a_0(\beta) = \frac{1}{2\beta^2} \quad ; \quad a_1(\beta) = -\frac{1}{8\beta^2} + \frac{1}{2\beta^3} - \frac{1}{8\beta^4} \quad ; \tag{16}$$

$$a_{n+1}(\beta) = -\frac{1}{2}\left\{\sum_{m=0}^{n} a_m(\beta)a_{n-m}(\beta) + \frac{\partial}{\partial\beta}a_n(\beta)\right\} \qquad n \geq 1. \tag{17}$$

The coefficients $a_n(\beta)$ can be, at this stage, computed iteratively leading to

$$\begin{aligned} w(\beta,\nu) &= \frac{1}{2\beta^2} - \left[\frac{1}{8\beta^2} - \frac{1}{2\beta^3} + \frac{1}{8\beta^4}\right]\nu^{-2} \\ &+ \left[-\frac{1}{8\beta^3} + \frac{13}{16\beta^4} - \frac{1}{2\beta^5} + \frac{1}{16\beta^6}\right]\nu^{-4} \\ &+ \left[-\frac{25}{128\beta^4} + \frac{7}{4\beta^5} - \frac{115}{64\beta^6} + \frac{1}{2\beta^7} - \frac{5}{128\beta^8}\right]\nu^{-6} + \mathcal{O}(\nu^{-8}). \end{aligned} \tag{18}$$

The asymptotic expansion of $\mathrm{I}_\nu(\nu^2\beta)$ as $\nu \to \infty$ can be now easily derived. The result reads as follows

$$\begin{aligned} \mathrm{I}_\nu(\nu^2\beta) &\sim_{\nu\to\infty} (2\pi\nu^2\beta)^{-\frac{1}{2}} \exp\left(\nu^2\beta - \frac{1}{2\beta}\right) \\ &\times \left\{1 + \left[\frac{1}{8\beta} - \frac{1}{4\beta^2} + \frac{1}{24\beta^3}\right]\nu^{-2}\right. \\ &\quad + \left[\frac{9}{128\beta^2} - \frac{29}{96\beta^3} + \frac{31}{192\beta^4} - \frac{11}{480\beta^5} + \frac{1}{1152\beta^6}\right]\nu^{-4} \\ &\quad + \left[\frac{75}{1024\beta^3} - \frac{751}{1536\beta^4} + \frac{1381}{3072\beta^5} - \frac{1513}{11520\beta^6} + \frac{4943}{322560\beta^7}\right. \\ &\quad \left.\left. - \frac{17}{23040\beta^8} + \frac{1}{82944\beta^9}\right]\nu^{-6} + \mathcal{O}(\nu^{-8})\right\}. \end{aligned} \tag{19}$$

Dividing this equation by $\mathrm{I}_\nu(\nu^2\beta)$ and plotting the result against β for a fixed (large) value of ν, we see a fluctuation of the ratio around unity. This reproduces exactly the behaviour which ought to be expected by the ratio of two functions asymptotic to each other. Furthermore, we observe that the fluctuation around 1 is a feature of the whole positive β-axis, therefore implicitly suggesting the validity of our expansion even for large values of β.

Recalling the expressions of β and ν in terms of α, x_μ and a, we are now able to derive the formula for the n- dimensional propagator up to sixth order in

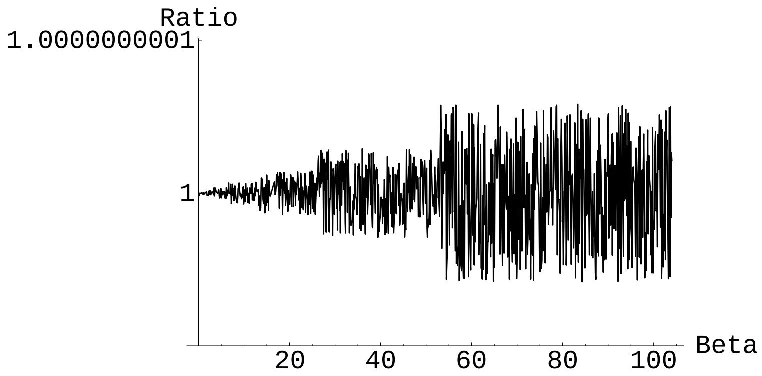

Fig. 1. Behaviour of the ratio of the asymptotic expansion of $\mathrm{I}_\nu(\nu^2\beta)$ with $\mathrm{I}_\nu(\nu^2\beta)$ as a function of β with ν equal to 100.

the lattice spacing. In fact, we can do more than that. We consider at this point a generalisation of $\Delta^{\mathrm{L}}(x;n)$ by introducing an arbitrary exponent q in the denominator of eq. (6); that is, we now look at

$$\begin{aligned}\Delta^{\mathrm{L}}(x;n;q) &= \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \frac{d^n p}{(2\pi)^n} \frac{e^{\imath px}}{\{m^2+\hat{p}^2\}^q} \\ &= \frac{1}{\Gamma(q)} \int_0^\infty d\alpha \alpha^{q-1} e^{-m^2\alpha} \left\{ \prod_{\mu=1}^{n} e^{-\frac{2\alpha}{a^2}} \left(\frac{1}{a}\right) \mathrm{I}_{\frac{\mathrm{x}_\mu}{\mathrm{a}}} \left(\frac{2\alpha}{\mathrm{a}^2}\right) \right\}. \end{aligned} \tag{20}$$

The main reason for considering this generalized quantity rests in the fact that eq. (20) represents a key element in the expression of general one-loop lattice integrals with bosonic propagators and zero external momenta [4].
Having reached a formula for the asymptotic expansion of the modified Bessel function as the lattice spacing vanishes, the study of the continuum limit of $\Delta^{\mathrm{L}}(x;n;q)$ is not technically more difficult than the analysis of the same limit for the n- dimensional lattice propagator $\Delta^{\mathrm{L}}(x;n) \equiv \Delta^{\mathrm{L}}(x;n;1)$. Indeed, we simply have now to substitute eq. (19) into eq. (20) and carry out the product over the dimensional index μ. The resulting α-integrals are all well-defined and finite. We can, therefore, proceed to their evaluation and obtain

$$\Delta^{\mathrm{L}}(x;n;q) \sim_{a\to 0} \frac{(4\pi)^{-n/2}}{\Gamma(q)} \Big\{\Delta^{\mathrm{L}}_0(x;n;q) + a^2\Delta^{\mathrm{L}}_2(x;n;q) + a^4\Delta^{\mathrm{L}}_4(x;n;q) + a^6\Delta^{\mathrm{L}}_6(x;n;q) + \mathcal{O}(a^8)\Big\}. \tag{21}$$

The full expression of each of the coefficients $\Delta^{\mathrm{L}}_{2i}(x;n;q)$ $(i = 0,1,2,3)$ is given in terms of the new function $\mathcal{P}_\rho(m;x)$ defined as

$$\mathcal{P}_\rho(m;x) = \left[\frac{2m}{(x^2)^{\frac{1}{2}}}\right]^\rho \mathrm{K}_\rho\left[\mathrm{m}\left(\mathrm{x}^2\right)^{\frac{1}{2}}\right] \tag{22}$$

with $\mathrm{K}_\rho\left[\mathrm{m}\,(\mathrm{x}^2)^{\frac{1}{2}}\right]$ representing, as usual, the modified Bessel function of the second kind and ρ a real number. Therefore, we have

$$\Delta^{\mathrm{L}}_0(x;n;q) = 2\mathcal{P}_{\frac{n}{2}-q}(m;x) \tag{23}$$

$$\Delta^{\mathrm{L}}_2(x;n;q) = \frac{n}{8}\mathcal{P}_{1+\frac{n}{2}-q}(m;x) - \frac{x^2}{8}\mathcal{P}_{2+\frac{n}{2}-q}(m;x) + \frac{x^4}{96}\mathcal{P}_{3+\frac{n}{2}-q}(m;x) \tag{24}$$

$$\begin{aligned}\Delta^{\mathrm{L}}_4(x;n;q) &= \frac{n(n+8)}{256}\mathcal{P}_{2+\frac{n}{2}-q}(m;x) - \left[\left(\frac{n}{128} + \frac{13}{192}\right)x^2\right]\mathcal{P}_{3+\frac{n}{2}-q}(m;x)\\ &+ \left[\frac{n+24}{1536}x^4 + \frac{(x^2)^2}{256}\right]\mathcal{P}_{4+\frac{n}{2}-q}(m;x)\\ &- \left[\frac{x^6}{1280} + \frac{x^2x^4}{1536}\right]\mathcal{P}_{5+\frac{n}{2}-q}(m;x) + \frac{(x^4)^2}{36864}\mathcal{P}_{6+\frac{n}{2}-q}(m;x)\end{aligned} \tag{25}$$

$$\begin{aligned}\Delta^{\mathrm{L}}_6(x;n;q) &= \left[\frac{n(n-1)(n-2)}{12288} + \frac{3n(3n+22)}{4096}\right]\mathcal{P}_{3+\frac{n}{2}-q}(m;x)\\ &- \left[\frac{(n-1)(n-2)}{4096} + \frac{85n+666}{12288}\right]x^2\mathcal{P}_{4+\frac{n}{2}-q}(m;x)\\ &+ \left[\frac{(n-1)(n-2)+59n+1102}{49152}x^4 + \frac{3n+52}{12288}(x^2)^2\right]\mathcal{P}_{5+\frac{n}{2}-q}(m;x)\\ &- \left[\frac{3n+160}{61440}x^6 + \frac{(x^2)^3}{12288} + \frac{3n+98}{73728}x^2x^4\right]\mathcal{P}_{6+\frac{n}{2}-q}(m;x)\\ &+ \left[\frac{5}{57344}x^8 + \frac{x^4(x^2)^2}{49152} + \frac{n+48}{589824}(x^4)^2 + \frac{x^2x^6}{20480}\right]\mathcal{P}_{7+\frac{n}{2}-q}(m;x)\\ &- \left[\frac{x^2(x^4)^2}{589824} + \frac{x^4x^6}{245760}\right]\mathcal{P}_{8+\frac{n}{2}-q}(m;x) + \frac{(x^4)^3}{21233664}\mathcal{P}_{9+\frac{n}{2}-q}(m;x).\end{aligned} \tag{26}$$

The expansion obtained for $\Delta^{\mathrm{L}}(x;n;q)$ clearly shows how the finite corrections introduced by formulating the theory on a lattice can be analytically expressed by a series of increasing (even) powers of the lattice spacing a with coefficients given by analytic functions of the mass and space coordinate times a modified Bessel function of the second kind of increasing order ρ.

We intend now to demonstrate how eq. (21) is in perfect agreement with both the studies performed in [3] and [4]. With this aim, we analyse $\Delta^{\mathrm{L}}(x;n;q)$ in the limit $m(x^2)^{1/2} \to 0$. Given the functional dependence of eq. (21) on $\mathcal{P}_\rho(m;x)$ and given the definition in eq. (22), this translates into considering the appropriate expansion for the Bessel function $\mathrm{K}_\rho\left[\mathrm{m}(\mathrm{x}^2)^{\frac{1}{2}}\right]$. We wish to recall at this point that the series representation of the modified Bessel function of the second kind assumes different forms depending on whether the order ρ is a real (ρ_{re}) or integer (ρ_{in}). In particular, for $m(x^2)^{1/2}$ small and $\rho = \rho_{\mathrm{re}}$ we have

$$\mathcal{P}_{\rho_{\mathrm{re}}}(m;x) \sim \frac{\pi}{2}\frac{1}{\sin\rho_{\mathrm{re}}\pi}\left\{\frac{2^{2\rho_{\mathrm{re}}}}{\Gamma(1-\rho_{\mathrm{re}})}\frac{1}{(x^2)^{\rho_{\mathrm{re}}}} - \frac{m^{2\rho_{\mathrm{re}}}}{\Gamma(1+\rho_{\mathrm{re}})}\right\} \tag{27}$$

while, for $m(x^2)^{1/2}$ still small and $\rho = \rho_{\mathrm{in}}$ $(\rho_{\mathrm{in}} \neq 0)$ [1], we find

$$\begin{aligned}\mathcal{P}_{\rho_{\mathrm{in}}}(m;x) &\sim \frac{2^{2\rho_{\mathrm{in}}-1}}{(x^2)^{\rho_{\mathrm{in}}}}\Gamma(\rho_{\mathrm{in}}) \\ &+(-1)^{\rho_{\mathrm{in}}+1}\frac{m^{2\rho_{\mathrm{in}}}}{\Gamma(1+\rho_{\mathrm{in}})}\left\{\ln\left[\frac{m(x^2)^{1/2}}{2}\right] - \frac{1}{2}\psi(1) - \frac{1}{2}\psi(1+\rho_{\mathrm{in}})\right\}\end{aligned} \tag{28}$$

with ψ denoting the ψ- function [6].
Using the relation $\Gamma(1-\rho_{\mathrm{re}})\Gamma(\rho_{\mathrm{re}}) = \pi/\sin\rho_{\mathrm{re}}\pi$, we find that $\mathcal{P}_{\rho_{\mathrm{re}}}(m;x)$ and $\mathcal{P}_{\rho_{\mathrm{in}}}(m;x)$ assume the same functional form $2^{\rho-1}\Gamma(\rho)/(x^2)^\rho$ $(\rho > 0)$ as $m \to 0$. The limits $m \to 0$ and $\rho_{\mathrm{re}} \to \rho_{\mathrm{in}}$ are, therefore, uniform. After performing the final mechanical analysis to set $n = 4$ and $q = 1$, we find that the massless limit of our continuum expansion for $\Delta^{\mathrm{L}}(x;n;q)$ reproduces exactly the expansion obtained by Lüscher and Weisz for the massless 4-dimensional propagator [3] The limit $x \to 0$ proves to be more difficult to analyse. The short-distance behaviour is, indeed, singular and the limit not uniform in this case. Note that this observation matches the analogous remark made in [1] about the behaviour of the dimensionally regularized propagator. Observe also that the logarithmic mass-behaviour described in [4] by Burgio, Caracciolo and Pelissetto is recovered, in our formulation, for integral values of ρ.

3 Conclusions

In the present work, we derived an asymptotic expansion for the modified Bessel function $\mathrm{I}_\nu(\nu^2\beta)$ as $\nu \to \infty$. The expansion obtained was of vital importance to analytically evaluate the continuum expansion of both the lattice

[1] For $\rho_{\mathrm{in}} = 0$ the correct expansion reads $\mathcal{P}_0(m;x) \sim \psi(1) - \ln\left[\frac{m(x^2)^{1/2}}{2}\right]$.

scalar propagator in a n-dimensional configuration space and its related generalised quantity $\Delta^{\mathrm{L}}(x; n; q)$. The study of the small $m(x^2)^{1/2}$-behaviour of $\Delta^{\mathrm{L}}(x; n; q)$ in the limit $a \to 0$ was shown to involve only the standard series expansion of modified Bessel functions of either real (ρ_{re}) or integer (ρ_{in}) order. The uniformity of the limits $m \to 0$ and $\rho_{\mathrm{re}} \to \rho_{\mathrm{in}}$ was observed and the Lüscher and Weisz expansion for the massless propagator in 4-dimensions recovered. The result obtained in [4] for the mass-dependence was also reproduced for integral values of ρ.

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References

[1] J. Collins, *Renormalization* (Cambridge University Press, 1984).

[2] D. Z. Freedman, K. Johnson and J. I. Latorre, Nucl. Phys. B 371 (1992) 353-414.

[3] M. Lüscher and P. Weisz, Nucl. Phys. B 445 (1995) 429-450.

[4] G. Burgio, S. Caracciolo, A. Pelissetto, Nucl. Phys. B 478 (1996) 687-722.

[5] G. N. Watson, *A treatise on the theory of Bessel functions, Second Edition* (Cambridge University Press, 1962).

[6] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products, Fourth Edition* (Academic Press, 1965).

[7] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers* (McGraw-Hill, 1978).

[8] M. Creutz, *Quarks, gluons and lattices* (Cambridge University Press, 1983).

[9] S. Wolfram, *Mathematica* (Addison-Wesley, 1988).

[10] I. Montvay and G. Munster, *Quantum fields on a lattice* (Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1994).

[11] Z. X. Wang and D. R. Guo, *Special functions* (World Scientific, 1989).

[12] A. F. Nikiforov and V. B. Uvarov, *Special functions of mathematical physics* (Birkhäuser, 1988).