A note on positivity of elementary operators

Richard M. Timoney

Abstract

We show that operators on $n \times n$ matrices which are representable in the form $T(X) = \sum_{i=1}^{\ell} a_i X b_i$ (for a_i and b_i $n \times n$ matrices) and are k-positive for $k = [\sqrt{\ell}]$ must be completely positive. As a consequence, elementary operators on a C*-algebra with minimal length ℓ which are k-positive for $k = [\sqrt{\ell}]$ must be completely positive.

For A a C*-algebra, an operator $T\colon A \to A$ is called an *elementary operator* if T can be expressed in the form $Tx = \sum_{i=1}^{\ell} a_i x b_i$ with a_i and b_i ($1 \leq i \leq \ell$) in the multiplier algebra $M(A)$ of A. (We will mainly be concerned with the case where A is the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices and then $M(A) = A$. Indeed $M(A) = A$ if A is unital.) Such representations of T may not be unique. The smallest ℓ in such representations of T is called the *minimal length* of T. If A is a prime C*-algebra ($M_n(\mathbb{C})$ is prime) and the collections $\{a_i : 1 \leq i \leq \ell\}$ and $\{b_i : 1 \leq i \leq \ell\}$ are each linearly independent, then ℓ is the minimal length of T [4]. If A is prime and such an elementary operator T sends the set $A_h = \{x \in A : x = x^*\}$ of hermitian elements of A into itself, then T is representable as

$$Tx = \sum_{i=1}^{\ell} \lambda_i a_i^* x a_i \tag{1}$$

with $a_i \in M(A)$ ($1 \leq i \leq \ell$) linearly independent and λ_i ($1 \leq i \leq \ell$) nonzero real numbers [4]. (By linear independence of the a_i, the ℓ in (1) must be minimal.)

An operator $T\colon A \to A$ which preserves the set A_h of hermitian elements is called *positive* if $Tx \geq 0$ whenever $x \in A$ is positive. T is called n-positive if the operator $T^{(n)}$ is positive, where $T^{(n)}$ acts on the $n \times n$ matrices $M_n(A)$ with entries in A and is given by $T^{(n)}(x_{ij})_{i,j} = (T(x_{ij}))_{i,j}$. The operator T is called *completely positive* if it is n-positive for all $n = 1, 2, 3, \ldots$. This terminology goes back to Stinespring [6].

For the case $A = M_n(\mathbb{C})$ all linear operators on $M_n(\mathbb{C})$ are elementary operators with length at most n^2. An example of Choi [1] says that the operator $Tx = (n-1)\mathrm{trace}(x) - x$ on $M_n(\mathbb{C})$ is $(n-1)$-positive but not n-positive. It is not hard to compute that the minimal length of this operator is n^2. In fact, T can be expressed as

$$\begin{aligned} Tx &= \sum_{i,j=1}^{n}(n-1){e_{ij}}^* x e_{ij} - I^* x I \\ &= \sum_{i\neq j}(n-1){e_{ij}}^* x e_{ij} - \sum_{i=1}^{n}((n-1)e_{ii} - I)^* x e_{ii}, \end{aligned}$$

where e_{ij} denotes the $n\times n$ matrix with 1 in the (i,j) place and zeroes elsewhere. Thus we have expressed Tx in the form $\sum_{i=1}^{n^2} a_i x b_i$ with independent $\{a_i\}$ and $\{b_i\}$ and so T has length n^2.

It is also well known that n-positive operators on $M_n(\mathbb{C})$ are completely positive. (We include a proof of this result below.)

A result of Choi [1] as generalised by Takasaki and Tomiyama [7] characterises k-positivity of operators T on $M_n(\mathbb{C})$ in terms of properties of an $n^2\times n^2$ matrix associated with T. Let $\gamma(T)$ denote the block matrix in $M_n(M_n(\mathbb{C}))$ where the (i,j) block is $T(e_{ij})$. We regard $\gamma(T)$ as an element of $M_{n^2}(\mathbb{C})$. Clearly γ establishes a linear bijection between linear operators on $M_n(\mathbb{C})$ and $M_{n^2}(\mathbb{C})$.

Proposition 1 ([1] and [7]) *An operator $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is k-positive for $1\le k\le n$ if and only if the matrix $(I_n\otimes P)\gamma(T)(I_n\otimes P)$ is positive for all rank k (hermitian) projections $P\in M_n(\mathbb{C})$.*

If $P=(p_{ij})_{ij}$, then the matrix $I_n\otimes P$ may be viewed as the block matrix in $M_n(M_n(\mathbb{C}))$ with (i,j) block equal to $p_{ij}I_n$. For positivity of hermitian elements of $M_{n^2}(\mathbb{C})$, we regard them as hermitian or quadratic forms on $\mathbb{C}^{n^2}$. We choose to identify a row vector $z=(z_1,z_2,\ldots,z_{n^2})\in\mathbb{C}^{n^2}$ with the $n\times n$ matrix $\rho(z)$ which has i^{th} row $(z_{n(i-1)+1}, z_{n(i-1)+2},\ldots,z_{ni})$. With this identification, observe that $\langle z,w\rangle = \text{trace}(\rho(z)\rho(w)^*)$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{C}^{n^2}$. In other words ρ carries the standard inner product on $\mathbb{C}^{n^2}$ to the Hilbert-Schmidt inner product on $M_n(\mathbb{C})$.

Lemma 2 *For $z\in\mathbb{C}^{n^2}$ and $P\in M_n(\mathbb{C})$, $\rho(z(I_n\otimes P)) = P^t\rho(z)$ (where P^t denotes the transpose of P). Moreover the subset*

$$\{\rho(z(I_n\otimes P)) : z\in\mathbb{C}^{n^2}, P=P^*=P^2\in M_n(\mathbb{C}), \textit{rank}(P)=k\}$$

of $M_n(\mathbb{C})$ is the set of matrices in $M_n(\mathbb{C})$ of rank at most k.

Proof. Write $z\in\mathbb{C}^{n^2}$ as $z=(z^1,z^2,\ldots,z^n)$ where each $z^i\in\mathbb{C}^n$, so that the z^i are the rows of $\rho(z)$. If $P=(p_{ij})_{ij}\in M_n(\mathbb{C})$, then $w=z(I_n\otimes P)$ has $w^i=\sum_j z^j p_{ji}$, taking $w=(w^1,w^2,\ldots,w^n)$ as for z. The rows of $P^t\rho(z)$ are again $\sum_j p_{ji}z^j$. □

Lemma 3 *Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the elementary operator $Tx = a^*xa$ (for $a \in M_n(\mathbb{C})$) and let $P \in M_n(\mathbb{C})$ be a hermitian projection. Then the hermitian form on $\mathbb{C}^{n^2} = M_n(\mathbb{C})$ associated with $S = (I_n \otimes P)\gamma(T)(I_n \otimes P)$ is*

$$zSz^* = |\textit{trace}(a\rho(z(I_n \otimes P))^*)|^2$$

Proof. First write $w = z(I_n \otimes P) \in \mathbb{C}^{n^2}$ and

$$zSz^* = z(I_n \otimes P)\gamma(T)(I_n \otimes P)z^* = w\gamma(T)w^*.$$

Suppose $a = (a_{ij})_{i,j}$. Then the blocks of $\gamma(T)$ are

$$T(e_{ij}) = \sum_{r,s,t,u} \overline{a_{sr}} e_{rs} e_{ij} a_{tu} e_{tu} = \sum_{r,u} \overline{a_{ir}} a_{ju} e_{ru} = (\overline{a_{ir}} a_{ju})_{r,u}.$$

It follows that

$$\begin{aligned} \gamma(T) &= (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{nn})^*(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{nn}) \\ &= \rho^{-1}(a)^* \rho^{-1}(a) \end{aligned}$$

is a rank one hermitian operator and

$$w\gamma(T)w^* = |\langle \rho^{-1}(a), w \rangle|^2.$$

From this it follows that

$$w\gamma(T)w^* = |\text{trace}(a\rho(w)^*)|^2.$$

This completes the proof of the Lemma. □

We can now deduce the following characterization of positivity for elementary operators.

Theorem 4 *Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an elementary operator of the form $Tx = \sum_{i=1}^{\ell} \lambda_i a_i x a_i*$ (where $\lambda_i \in \mathbb{R}$ and $a_i \in M_n(\mathbb{C})$ for each i). Suppose $1 \le k \le n$. Then T is k-positive if and only if the hermitian form $Q_T\colon M_n(\mathbb{C}) \to \mathbb{R}$ given by*

$$Q_T(z) = \sum_{i=1}^{\ell} \lambda_i |\textit{trace}(a_i z^*)|^2$$

satisfies $Q_T(z) \ge 0$ whenever $z \in M_n(\mathbb{C})$ has rank at most k.

Proof. By linearity of γ and Lemma 3 it follows that for P a rank k projection on $\mathbb{C}^n$, $(I_n \otimes P)\gamma(T)(I_n \otimes P)$ gives rise to the hermitian form $R\colon \mathbb{C}^{n^2} \to \mathbb{R}$ with

$$R(w) = \sum_{i=1}^{\ell} \lambda_i |\text{trace}(a_i \rho(w(I_n \otimes P))^*)|^2$$

and so $R(w) = Q_T(\rho(w(I_n \otimes P)))$. The result now follows from Proposition 1 and Lemma 2. □

Note that the theorem may be viewed in a geometrical light. It states that the cone of matrices which are negative with respect to Q_T cannot intersect the rank k matrices if T is k-positive.

As an example, we establish the following well-known result via this theorem: *If a linear operator $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is n-positive, then it is completely positive.* We know any T which preserves the hermitian subset of $M_n(\mathbb{C})$ can be represented in the form (1) with $\ell \leq n^2$. If any of the λ_i is negative, we claim that T cannot be n-positive. To show this, observe that if (say) $\lambda_1 < 0$, then by linear independence, we can find $z \in M_n(\mathbb{C})$ so that trace$(a_1^* z) \neq 0$ but trace$(a_i^* z) = 0$ for $2 \leq i \leq \ell$. From Theorem 4 it follows that $Q_T(z) = \lambda_1 |\text{trace}(a_1^* z)|^2 < 0$ and so T cannot be n-positive. Once we know that $\lambda_i > 0$ for all i, it is easy to see that T is completely positive.

Proposition 5 *If an elementary operator $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ ($n \geq 2$) has length at most $n^2 - 1$ and T is $(n-1)$-positive, then T is completely positive.*

Proof. For T to be $(n-1)$-positive, it must preserve Hermitian operators and so must be representable in the form (1) with a_i ($1 \leq i \leq \ell$) independent elements in $M_n(\mathbb{C})$, $\ell < n^2$ and $\lambda_i \in \mathbb{R} \setminus \{0\}$. For the sake of arriving at a contradiction, suppose $\lambda_1 < 0$.

From Theorem 4 we see that $Q_T(z) \geq 0$ whenever $z \in M_n(\mathbb{C})$ has rank at most $n - 1$ (that is, whenever z is singular).

From linear independence of $\{a_1, a_2, \ldots, a_\ell\}$ plus the fact that $\ell \leq n^2 - 1$, we can see that we can find a nonzero $x \in M_n(\mathbb{C})$ with trace$(a_i x^*) = 0$ for $1 \leq i \leq \ell$ and $y \in M_n(\mathbb{C})$ with trace$(a_1 y^*) \neq 0$ but trace$(a_i y^*) = 0$ for $2 \leq i \leq \ell$. From Theorem 4, we can see that

$$Q_T(\mu x + y) = \lambda_1 |\text{trace}(a_1 y^*)|^2 < 0$$

for all $\mu \in \mathbb{C}$. If there is any choice of $\mu \in \mathbb{C}$ where $\mu x + y$ has rank strictly less than n (that is where $\det(\mu x + y) = 0$) then we have shown that T is not $(n-1)$-positive and obtained the desired contradiction. However, there remains the case where $\mu x + y$ is always invertible.

As the set $\{z \in M_n(\mathbb{C}) : Q_T(z) < 0\}$ is open and contains y, it contains a ball around y. For any z in this ball, we have

$$Q_T(\mu x + z) = Q_T(z) < 0$$

(for all $\mu \in \mathbb{C}$) and it easy to see that there must be a choice of z where $\det(\mu x + z)$ is not a constant function of μ. (To verify this assertion, suppose that x has rank j and choose unitary matrices $U, V \in M_n(\mathbb{C})$ so that UxV is diagonal with diagonal entries $(\alpha_1, \alpha_2, \ldots, \alpha_j, 0, \ldots, 0)$. Then $\det(\mu x + z) = \det(\mu UxV - UzV)/\det(UV)$ has degree at most j in μ and the coefficient of μ^j is the determinant of the bottom right $(n-j)\times(n-j)$ block of UzV times $(1/\det(UV))\prod_{i=1}^{j}\alpha_i$. We can find z arbitrarily close to y where this coefficient is nonzero.)

We can now choose μ to be a solution of $\det(\mu x + z) = 0$ and get the required contradiction to the $(n-1)$-positivity of T. $\square$

Our main result is the following, which leads to improvements of some results of Li [3], Mathieu [5] and Hou [2].

Theorem 6 *If an elementary operator $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ has length at most $(k+1)^2 - 1$ and T is k-positive (where $1 \le k < n$), then T is completely positive.*

Proof. As before, we can assume that T is given by (1) with $\ell < (k+1)^2$, $\lambda_i \in \mathbb{R} \setminus \{0\}$ and $\{a_1, a_2, \ldots, a_\ell\}$ linearly independent in $M_n(\mathbb{C})$. It is enough to show that T is n-positive and by induction on k it is clearly sufficient to show that T is $(k+1)$-positive.

By Theorem 4, if T is not $(k+1)$-positive, then there is some $w \in M_n(\mathbb{C})$ of rank $k+1$ with $Q_T(w) < 0$. We can find unitary matrices $U, V \in M_n(\mathbb{C})$ so that $UwV = r$ is diagonal with diagonal entries $(r_1, r_2, \ldots, r_{k+1}, 0, \ldots, 0)$. Now

$$\begin{aligned} Q_T(w) &= Q_T(U^* r V^*) \\ &= \sum_i \lambda_i |\text{trace}(a_i V r^* U)|^2 \\ &= \sum_i \lambda_i |\text{trace}((U a_i V) r^*)|^2 \\ &= Q_S(r) \end{aligned}$$

for S the elementary operator $Sz = \sum_i \lambda_i b_i^* z b_i$ and $b_i = U a_i V$. Since $Q_S(z) = Q_T(U^* z V^*)$, it is clear from Theorem 4 that S is also k-positive.

We now consider the compression of S to the subspace of $M_n(\mathbb{C})$ consisting of matrices which are zero outside the top left $(k+1) \times (k+1)$ block. We can identify this subspace with $M_{k+1}(\mathbb{C})$. Let $P\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ be the projection $P(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{k+1})$ (or the associated $(k+1) \times n$ matrix). By the compression we mean the map $R\colon M_{k+1}(\mathbb{C}) \to M_{k+1}(\mathbb{C})$ given by

$Rz = P(Sz)P^*$. As the operator on $M_n(\mathbb{C}) \to M_{k+1}(\mathbb{C})$ given by $z \mapsto PzP^*$ is completely positive (and so is the inclusion operator $M_{k+1}(\mathbb{C}) \to M_n(\mathbb{C})$: $z \mapsto P^*zP$ implicit above), R is k-positive. Also $R(z) = \sum_i \lambda_i P b_i^* z b_i P^* = \sum_i \lambda_i (Pb_iP^*)^* z (Pb_iP^*)$ and so R is an elementary operator on $M_{k+1}(\mathbb{C})$ of length at most $(k+1)^2 - 1$ and k-positive.

By Proposition 5 R is $(k+1)$-positive and so Q_R is always nonnegative. But it is easy to see that

$$\begin{aligned} Q_R(PrP^*) &= \sum_i \lambda_i |\text{trace}(Pb_iP^*Pr^*P^*)|^2 \\ &= \sum_i \lambda_i |\text{trace}(b_i r^*)|^2 \\ &= Q_S(r) = Q_T(w) < 0. \end{aligned}$$

This contradiction shows that T must be $(k+1)$-positive and completes the proof. □

Our next task is to extend Theorem 6 to the infinite dimensional case where the C*-algebra is $B(H)$, the algebra of bounded operators on an infinite-dimensional Hilbert space. As $B(H)$ is prime, the representation (1) is available for elementary operators on $A = B(H)$.

Lemma 7 *Let H be a Hilbert space and $T\colon B(H) \to B(H)$ an elementary operator of the form $Tx = \sum_{i=1}^{\ell} \lambda_i a_i^* x a_i$ ($\lambda_i \in \mathbb{R}$ and $a_i \in B(H)$). If $Tx \geq 0$ whenever $x \in B(H)$ is a rank one positive operator, then T is positive.*

Proof. Every finite-rank positive element of $B(H)$ can be expressed as a finite sum of rank one positive operators and so $Tx \geq 0$ whenever $x \in B(H)$ is positive and finite rank.

For an arbitrary positive $x \in B(H)$ we have $x_P = PxP$ positive and finite rank whenever $P \in B(H)$ is a finite rank hermitian projection. The net $(x_P)_P$, indexed by the finite rank projections ordered by inclusion of the ranges, converges to x in the weak sense. That is, if $v \in H$ then $\langle x_P v, v\rangle \to \langle xv, v\rangle$. Indeed $\langle x_P v, v\rangle = \langle xPv, Pv\rangle = \langle xv, v\rangle$ as long as $v \in PH$. Now, $\langle T(x_P)v, v\rangle = \sum_{i=1}^{\ell} \lambda_i \langle x_P a_i v, a_i v\rangle \to \sum_i \lambda_i \langle x a_i v, a_i v\rangle = \langle T(x)v, v\rangle$. Thus $\langle T(x)v, v\rangle \geq 0$ since $T(x_P) \geq 0$. □

Corollary 8 *Let H be a Hilbert space and $T\colon B(H) \to B(H)$ an elementary operator of minimal length ℓ. If T is k-positive for some $k \geq 1$ with $(k+1)^2 > \ell$, then T is completely positive.*

Proof. Let T be given by (1) with $a_i \in B(H)$, $\lambda_i \in \mathbb{R}$ ($1 \leq i \leq \ell$). As the finite-dimensional case is covered by Theorem 6, we look at the case where

H is infinite dimensional. To show that T is n-positive for $n > k$, we consider $T^{(n)}\colon M_n(B(H)) = B(H^n) \to B(H^n)$ and observe that $T^{(n)}X = \sum_{i=1}^{\ell} \lambda_i (a_i^{(n)})^* X a_i^{(n)}$, where $a_i^{(n)} \in B(H^n)$ acts on H^n by a_i acting on each 'coordinate' separately. By Lemma 7, $T^{(n)}$ is positive if it maps rank one positive elements of $B(H^n)$ to positive elements. A rank one positive element of $B(H^n)$ is of the form $X(v) = \langle v, w\rangle w$ for some $w \in H^n$. Let $w = (w^1, w^2, \ldots, w^n)$ where each $w^i \in H$. Choose a finite rank projection $P \in B(H)$ so that the range of P contains all the w^i and all $a_j^* w^i$ ($1 \le i \le n$, $1 \le j \le \ell$). Then $P^{(n)} X P^{(n)} = X$ and $P^{(n)} T^{(n)}(X) P^{(n)} = \sum_{i=1}^{\ell} \lambda_i P^{(n)} (a_i^{(n)})^* X a_i^{(n)} P^{(n)} = T^{(n)}(X)$.

Now consider the operator $S\colon B(H) \to B(H)$ given by

$$Sx = \sum_{i=1}^{\ell} \lambda_i (P a_i P)^* x (P a_i P) = P(T(PxP))P,$$

which is k-positive because T is and so is the map $x \mapsto PxP$. The above calculations show that $T^{(n)}(X) = S^{(n)}(X)$. Let $K = PH$ denote the (finite-dimensional) range of P and decompose all elements of $B(H)$ as 2×2 blocks with respect to $H = K \oplus K^{\perp}$. Similarly decompose $B(H^n) = M_n(B(H))$.

Let $Q\colon H \to K$ be the othogonal projection (the same as P but regarded as having values in K). The compression of S to the (K, K)-block $B(K) = QB(H)Q^*$ is a finite-dimensional elementary operator $R\colon B(K) \to B(K)$, $Ry = QS(Q^* y Q)Q^* = \sum_{i=1}^{\ell} \lambda_i b_i^* y b_i$, where b_i is $Q a_i Q^*$. R is covered by Theorem 6. Hence it is n-positive. It follows that $S^{(n)}(X) = T^{(n)}(X)$ is positive, because it is equal to $R^{(n)}((Q^{(n)}) X (Q^{(n)})^*)$ with zeroes added symmetrically. □

Corollary 8 improves on a result of Hou [2] where the same conclusion is proved under the assumption that the length of T is at most $2k + 1$.

Corollary 9 *Let A be a C*-algebra and T an elementary operator on A of minimal length at most $(k+1)^2 - 1$. If T is k-positive, then T is completely positive.*

Proof. This follows as in [5], where the same conclusion is proved under the assumption that the length of T is at most $2k + 1$. This result of [5] is also shown in [3]. □

Acknowledgment. The author thanks Martin Mathieu for his help during this work.

References

[1] M. D. Choi, *Positive linear maps on C*-algebras*, Canadian J. Math. **24** (1972) 520-529.

[2] J.-Ch. Hou, *A characterization of positive elementary operators*, J. Operator Theory, (to appear).

[3] Jiankui Li, A remark on complete positivity of elementary operators, J. Integral Equations and Operator Theory, **28** (1997) 110-115.

[4] Martin Mathieu, *Elementary operators on prime C*-algebras. I*, Math. Ann. **284** (1989) 223-244.

[5] Martin Mathieu, *Characterising completely positive elementary operators*, Bull. London Math. Soc, (to appear).

[6] W. Forrest Stinespring, *Positive functions on C*-algebras*, Proc. Amer. Math. Soc. **6** (1955) 211–216.

[7] T. Takasaki and J. Tomiyama, *On the geometry of positive maps in matrix algebras*, Math. Zeit. **184** (1983) 101-108.

School of Mathematics
Trinity College
Dublin 2
Ireland

Email: `richardt@maths.tcd.ie`