Structure formation with a self-tuning scalar field

Pedro G. Ferreira[1] and Michael Joyce[2]
[1] *Center for Particle Astrophysics, 301 Leconte Hall, University of California, Berkeley, CA 94720, USA.*
[2] *School of Mathematics, Trinity College, Dublin 2, Ireland.*
(July 1997)

A scalar field with an exponential potential has the particular property that it is attracted into a solution in which its energy scales as the dominant component (radiation or matter) of the Universe, contributing a fixed fraction of the total energy density. We study the growth of perturbations in a CDM dominated $\Omega = 1$ universe with this extra field, with an initial flat spectrum of adiabatic fluctuations. The observational constraints from structure formation are satisfied as well, or better, than in other models, with a contribution to the energy density from the scalar field $\Omega_\phi \sim 0.1$ which is small enough to be consistent with entry into the attractor prior to nucleosynthesis.

The simplest viable cosmology which follows from inflation, a flat universe with pressureless matter and 5% baryonic dark matter, has been unable to fit both the cosmic background radiation (CBR) fluctuations and measurements of mass fluctuations on scales of a few Megaparsecs. The paradigm of inflation is sufficiently compelling that there have been various attempts at modifying this 'standard cold dark matter' (sCDM) scenario [1]. The possibility that some part of the energy density of the Universe is in a form other than particle-like matter has been envisaged, in particular in the form of a constant energy (ΛCDM) [2] or time-dependent coherent energy density in a scalar field [3,4]. In this letter we discuss the cosmology of a model with a scalar field which has a simple exponential potential. It is distinctly different from other scalar field cosmologies, in that its energy density plays a role from very early times, rather than just at recent epochs, and resembles much more the 'mixed dark matter' (MDM) model [5] in which there is a component of matter which is collisionless during a period of the growth of structure. The required potential has the merit that it arises quite generically in particle theories involving compactifications such as supergravity or superstring theories, and has (mainly for this reason) been quite extensively discussed in the context of inflationary models.

Let us first explain the properties of an exponential potential which make it a particular and interesting case. The equations of motion in an expanding FRW universe for the homogeneous mode of a scalar field ϕ with potential $V(\phi)$ coupled to ordinary matter only through gravity are

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} + a^2V'(\phi) = \frac{1}{a^2}\frac{d}{d\tau}(a^2\dot{\phi}) + a^2V'(\phi) = 0 \quad (1)$$

$$\mathcal{H}^2 = \frac{1}{3M_p^2}(\frac{1}{2}\dot{\phi}^2 + a^2V(\phi) + a^2\rho_n) \quad (2)$$

$$\dot{\rho_n} + n\mathcal{H}\rho_n = 0 \quad (3)$$

where ρ_n is the energy density in radiation ($n = 4$) or non-relativistic matter ($n = 3$), $\mathcal{H} = \frac{\dot{a}}{a}$ is the conformal expansion rate of the universe with scale factor a, dots are derivatives w.r.t. conformal time τ, $' = \frac{d}{d\phi}$ and $M_P = 2.4 \times 10^{18}$GeV is the reduced Planck mass. Multiplying (1) by $\dot{\phi}$ and integrating, one obtains

$$\rho_\phi(a) = \rho(a_o)e^{-\int_{a_o}^{a} 6(1-\xi(a))\frac{da}{a}} \quad (4)$$

where $\rho_\phi = \frac{1}{2a^2}\dot{\phi}^2 + V(\phi)$ is the total scalar energy, and $\xi = V(\phi)/\rho_\phi$. In general therefore the energy density of a scalar field has the range of possible scaling behaviours $\rho \propto 1/a^m$ with $0 \leq m \leq 6$, and the scaling is completely determined by the ratio of its potential to its kinetic energy.

The special cosmological solutions in which we are interested here are attractor solutions of (1) - (3) for the case of an exponential potential $V(\phi) = V_o e^{-\lambda\phi/M_P}$, which were given in [3] and [6]. In these solutions the scalar field evolves so that its total energy density ρ_ϕ scales in the same way as the dominant component (i.e. $\rho_\phi \propto 1/a^n$) and contributes a *fixed* fraction of the total energy density given by

$$\Omega_\phi \equiv \frac{\rho_\phi}{\rho_\phi + \rho_n} = \frac{n}{\lambda^2} \qquad \xi \equiv \frac{V(\phi)}{\frac{1}{2a^2}\dot{\phi}^2 + V(\phi)} = 1 - \frac{n}{6} \quad (5)$$

for $\lambda > 1/\sqrt{n}$. Note that it is λ alone which determines the solution. The existence of the attractor can be understood to follow from the fulfillment of two conditions: (i) ρ_ϕ scales *faster* than $1/a^n$ if $\rho_\phi >> \rho_n$ and, (ii) scales *slower* than $1/a^n$ if $\rho_\phi << \rho_n$. These two behaviours tend to drive the two components to the attractor which lies between them. That the first condition is satisfied can be seen from solving (1) - (3) with $\rho_n = 0$ for the exponential potential. There is then a different set of attractors [8] in which

$$\xi = 1 - \frac{\lambda^2}{6} \qquad \rho_\phi \propto \frac{1}{a^{\lambda^2}} \quad (6)$$

where $\lambda < \sqrt{6}$.For $\lambda > \sqrt{6}$ there is not a single attractor, but all solutions have $\xi \to 0$ asymptotically (and, therefore, $\rho \propto 1/a^6$). The condition $\lambda > 1/\sqrt{n}$ for the attractor (5) is indeed therefore just that anticipated. The second condition can be understood qualitatively as follows. Taking, for simplicity, the case $n = 4$, (1) - (3) with $V(\phi) = 0$ give $\dot{\phi}(t) = \dot{\phi}_o(\frac{a_o}{a})^2$ and therefore

$$\phi(\tau) = \phi_o + \dot{\phi}_o \tau_o \ln \frac{\tau}{\tau_o} \qquad \rho_\phi >> \rho_\gamma \tag{7}$$

$$\phi(\tau) = \phi_o + \dot{\phi}_o \tau_o \big(1 - \frac{\tau_o}{\tau}\big) \qquad \rho_\phi << \rho_\gamma \tag{8}$$

These will also hold as approximate solutions in the case that the potential energy is sub-dominant. The first solution shows how, for a sufficiently steep exponential, the potential energy can remain small relative to the kinetic energy ($\sim 1/\tau^3$) so that the rapid scaling (associated with $\xi << 1$) can be maintained. The field in both the attractor solutions (5) and (6) has this same logarithmic time dependence. On the other hand, the second limit shows how the larger damping due to radiation domination slows down the evolution of the field giving an almost constant potential energy which will thus ultimately catch up with the kinetic energy, increasing ξ and causing the scalar energy to scale slower.

A potential which is less steep than the exponential will not satisfy the first condition [7], and a steeper potential (e.g. $\sim e^{-\phi^2/M_P^2}$) will always decay asymptotically relative to the other components. The existence of this particular attractor cosmological solution is thus quite specific to the exponential potential. Further this is in fact a potential which can arise quite generically in particle physics theories involving compactified dimensions (with internal dimensions characterized by M_P). For this reason it has been considered quite extensively in the context of inflation [8–10], since for $\lambda < \sqrt{2}$ the solutions (6) describe 'power-law' inflation (with $a \propto t^{2/\lambda^2}$ in terms of physical time $t = \int a d\tau$). Examples of specific supergravity theories in which such potentials are obtained are given in [9], and various higher dimensional theories of gravity in which they arise discussed in detail in [10] and [11].

If such a field does exist, it will enter the attractor and contribute a fraction of the energy density (fixed by λ) at some time determined by its initial energy density. Nucleosynthesis provides the earliest constraint on how large such a contribution can be. The expansion rate of the Universe at nucleosynthesis is increased over its standard model value by the same amount as ΔN_{eff} relativistic degrees of freedom, with $\Omega_\phi = \frac{3}{4}\frac{7\Delta N_{eff}/4}{10.75+7\Delta N_{eff}/4}$, where Ω_ϕ is the fraction contributed in the matter era. There is some disagreement on the precise nucleosynthesis constraint on ΔN_{eff}, but a bound of $\Delta N_{eff} = 0.9$ is given by various authors [12] or even a more conservative one of $\Delta N_{eff} = 1.5$ by others [13], which corresponds to $\Omega_\phi < 0.1 - 0.15$.

Prima facie this constraint would seem to require entry into the attractor after nucleosynthesis if the scalar field is to play any significant role cosmologically [3]. The requirement of entry after nucleosynthesis would apparently mandate the unattractive fine-tuning (typical of scalar field models) of the initial energy density in the potential to some small value. It was in fact the incor-

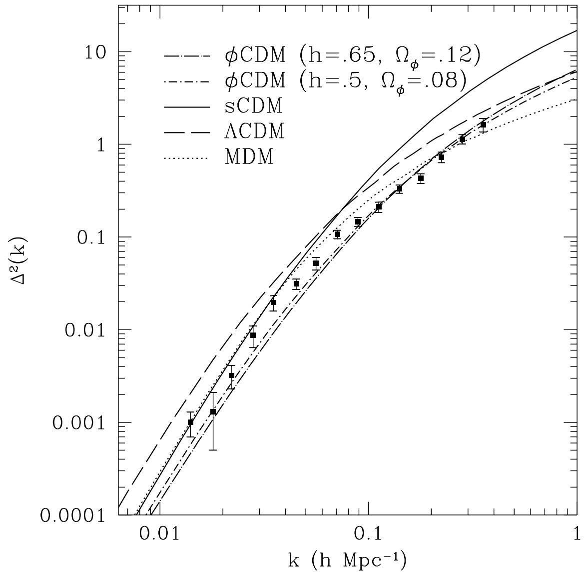

FIG. 1. Mass variance per unit $\ln k$ computed from Boltzman code for different models compared with that inferred from a compilation of galaxy surveys [19]

rectness of this second assumption which motivated the present study: If, prior to nucleosynthesis, the energy density in the exponential field with $\lambda > \sqrt{6}$ dominates over that in the radiation, there will typically be a long transient period after $\rho_\phi \sim \rho_\gamma$ during which the scalar energy is very sub-dominant (much less than its value in the attractor (5)). This is simply because the ratio $\xi \to 0$ in the kinetic energy dominated pure scalar cosmology, but is of order one in the attractor with radiation. During the time that ξ is increasing (potentially many expansion times as it cannot grow faster than a^6) the scalar field energy continues to red-shift away as $1/a^6$. Such a dominance by kinetic energy can occur in certain post-inflationary cosmologies which have considerable interest in their own right [14], [15]. It has transpired from the present work however that the first reason for disregarding this model is also incorrect, and that entry to the attractor prior to nucleosynthesis is in fact consistent - quite simply because the small contribution has a compensating long time to act.

We have carried out a detailed calculation of the evolution of perturbations in this cosmology (which we refer to as ϕCDM). We assume that the attractor is established at the beginning of our numerical simulation, deep in the radiation era, and take an initial standard inflationary scale-invariant spectrum of adiabatic perturbations. The relevant equations are the linearized coupled Einstein-Boltzmann equations given in [16], supplemented by the scalar field and its perturbations $\phi_{total} = \phi(\tau) + \varphi(\tau, \mathbf{x})$, with evolution equation

$$\ddot{\varphi} + 2\mathcal{H}\dot{\varphi} - \nabla^2\varphi + a^2 V''\varphi + \frac{1}{2}\dot{\phi}\dot{\gamma} = 0 \tag{9}$$

and additional components to the perturbed energy-momentum tensor:

$$
\begin{aligned}
a^2\delta T^0_0 &= -\dot\phi\dot\varphi - a^2V'\varphi \\
-a^2\partial_i\delta T^0_i &= \dot\phi\nabla^2\varphi \\
a^2\delta T^i_i &= 3\dot\phi\dot\varphi - 3a^2V'\varphi
\end{aligned}
\qquad (10)
$$

where γ is the trace of the metric perturbation. We vary Ω_ϕ and h (where $H_0 = h100$km/s/Mpc is the Hubble constant today), keeping the remaining cosmological parameters fixed at the values of sCDM, and find the best fit model to both CMB and large scale structure. To do this we use the COBE measurement of CMB anisotropies on large scales [17] to normalize our theory [18], estimate the theoretical mass variance per unit $\ln k$, $\Delta^2(k)$ and compare with that rendered from a collection of galaxy surveys [19]. In Figure 1 we show $\Delta^2(k)$ for two best fit ϕCDM models, for sCDM, for a ΛCDM universe with $\Omega_\Lambda = 0.6$, and for an MDM model with $\Omega_\nu = 0.2$ in the form of two massive neutrinos species. It is clear that for these values ϕCDM fares as well or better than the other models. Another useful quantity to work with is the mass fluctuations on 8h^{-1}Mpc scales, $\sigma_8^2 = \int_0^\infty \frac{dk}{k}\Delta^2(k)(\frac{3j_1(kR)}{kR})^2 \mid_{R=8}$. This can be related to masses and abundances of rich clusters and supplies us with a very tight constraint on possible cosmologies; indeed current estimates give $\sigma_8 = 0.6 \pm 0.1$ [20]. A good fit to σ_8 is

$$\sigma_8(\Omega_\phi) = e^{-8.7\Omega_\phi^{1.15}}\sigma_8^{CDM} \qquad (11)$$

where σ_8^{CDM} is the COBE normalized sCDM σ_8. Again we see that there is range of values of Ω_ϕ and H_0 which satisfy the above constraint *and* are consistent with the limits imposed by BBN. In Figure 2 we compare the C_ℓs of our models with a compilation of data points [21]. Again they are consistent with the current data.

The evolution of perturbations in the presence of the scalar field is simple to understand. On superhorizon scales there is the usual growing mode with δ_c,$\varphi \propto \tau^2$ (where δ_c is the density contrast in the CDM). This is to be expected; the superhorizon evolution is insensitive to the "chemistry" of the matter and totally dominated by gravity. On sub-horizon scales in the radiation era, the Meszaros effect comes into play giving $\delta_c \propto \ln\tau$. The specific effect of the scalar field appears on subhorizon scales in the matter era. The perturbation in the scalar field itself has the approximate solution $\varphi \propto \frac{1}{\tau^{3/2}}J_{\frac{3}{2}}(k\tau)$ (where J_ν is a Bessel function) which, when fed back into the equation for δ_c gives an altered solution for the usual growing mode $\delta_c \propto \tau^{2-\epsilon}$ where

$$\epsilon = \frac{5}{2}(1-\sqrt{1-\frac{24}{25}\Omega_\phi}) \qquad (12)$$

This solution shows explicitly how even a small contribution from the scalar field can give a significant effect, as it acts all the way through the matter era. The expected suppression of $|\delta_c|^2$ for modes larger than k_{eq} is of order $(1+z_{eq})^{-\epsilon}$, where k_{eq} is the wavenumber of the horizon size at radiation-matter equality. This last effect is reminiscent of the evolution of perturbations in a mixed dark matter (MDM) universe where one has component of matter, ρ_ν which is collisionless for a period of time during the matter era [22].

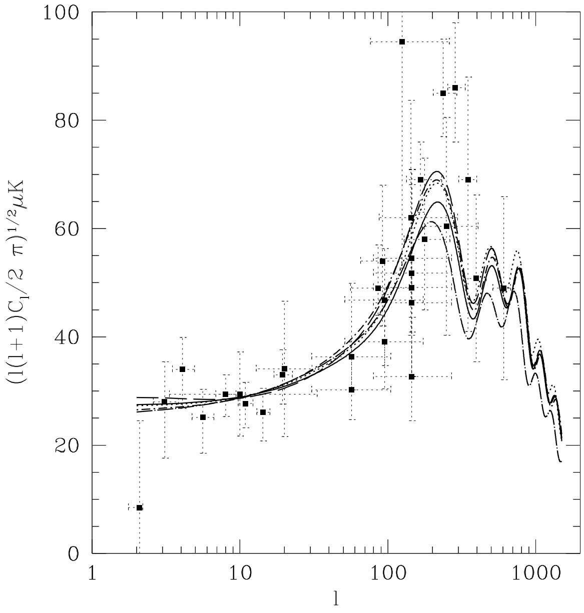

FIG. 2. Comparison of different model predictions to current experimental data. All models were COBE normalized and are labeled as in Figure 1.

It is useful to pursue a comparison between ϕCDM and MDM to identify the key differences. Firstly the scaling behaviour of the additional background energy density differs: While for ϕCDM the energy density in ϕ follows the dominant form of energy quite closely, for MDM ρ_ν changes from scaling as $1/a^4$ to scaling as $1/a^3$ when $3k_BT_\nu \simeq m_\nu$ where T_ν (m_ν) is the massive neutrino temperature (mass) and k_B is the Boltzmann constant. For a period between matter-radiation equality and this transition Ω_ν is smaller than its asymptotic value, and there is less suppression of growth in the CDM than in the case of the scalar field. A further difference is that the period of time during which perturbations are suppressed is shorter in MDM compared to ϕCDM. In both cases there is a wavenumber k_{su} which separates growing modes from damped modes. For ϕCDM this scale is roughly the horizon i.e. $k_{su} \propto \frac{1}{\tau}$, while for MDM it is the free streaming scale i.e. $k_{su} = 8a^{1/2}(m_\nu/10eV)h\mathrm{Mpc}^{-1} \propto \tau$. Clearly in the latter case any given mode of δ_c will eventually start to grow. In particular modes around k_{eq} will already have started to undergo collapse. A final important difference concerns the evolution of perturbations in the radiation era. For MDM, the perturbation in the massive neutrinos behaves much like radiation until it is well inside the horizon, and this transition is set by the Jeans scale i.e. when $k\tau \simeq 1/c_s = \sqrt{3}$. For the scalar field, on the other hand, the transition occurs for larger wavelengths, $k\tau < 1$. This means that perturbations in the CDM will stop growing earlier in ϕCDM than in MDM. The accumulated effect of these differences explains what we have observed - that, with half the energy density of MDM with two massive neutrinos, ϕCDM brings about approximately the same suppression of power on small scales. The fact that this suppression lasts until today leads to the formation of structure at higher redshifts. There is now strong observational evidence that any model of structure formation must have this feature.

Let us now turn to the effect that the scalar field has on the CMB. We shall rely on the simplified picture of [23] to understand the angular power spectrum, C_ℓ, defined as $C(\theta) = \langle \frac{\Delta T}{T}(\mathbf{n})\frac{\Delta T}{T}(\mathbf{n}')\rangle = (4\pi)^{-1}\sum(2\ell+1)C_\ell P_\ell(\cos\theta)$, with $\mathbf{n}\cdot\mathbf{n}' = \cos\theta$. For $\ell > 100$ the main features of the C_ℓs are given by the power spectrum of radiation perturbations at last scattering, δ_γ. Ignoring projection effects, one has that the structure of the peaks and troughs are given by

$$< |\delta_\gamma|^2 > \propto \cos^2(kr_s) \quad kr_s > 1 \tag{13}$$

where r_s is the sound horizon in the baryon-photon fluid, $r_s = \int_0^{\tau_*}\frac{d\tau}{3[1+R(\tau)]}$ and $R = \frac{3\rho_B}{4\rho_\gamma}$. The spatial frequency k is roughly related to the angular frequency ℓ. The fact that the properties of the C_ℓs are dominated by this quantity at $a \simeq 10^{-3}$ means that the effect of ϕ on the CMB will be much smaller than its net effect on δ_c. Adding the scalar field component brings about two effects which we can understand qualitatively. Firstly the oscillations are shifted to higher ℓs. Because of the additional energy density in the scalar field, the expansion rate will be larger and the conformal horizon will be smaller for the same red-shift in ϕCDM compared sCDM. This feeds through to give a different r_s for the same value of a, shifting the peaks as observed. The other main feature is an increase in power in the peaks. This can be understood easily using the picture outlined in [23]. The oscillations in δ_γ are driven by the evolution in the gravitational potentials, and here as in the MDM case [24] the change in the growth of metric perturbations boosts the amplitude of the peaks by a few percent.

We conclude that the cosmological model we have studied provides an interesting and distinct alternative to other models which have been proposed. It has the attractive feature that $\lambda\ (= \sqrt{3/\Omega_\phi})$, the single extra parameter compared to standard CDM, has a value which is of the order naturally expected in the many particle physics theories in which the field arises. With the launch of high resolution space based experiments, such as the Planck explorer and the MAP satellite, it should be possible to distinguish the effect on the CMB of such an exponential scalar field if it exists, or to rule out its existence and place tighter constraints on the physical theories in which these fields arise [25].

Acknowledgements: We acknowledge U. Seljak and M. Zaldarriaga for the use of their Boltzmann-Einstein solver. We thank A. Albrecht, C. Balland, M. Davis, A. Jaffe, J. Levin, A. Liddle and J. Silk for conversations. P.F. is supported by the Center for Particle Astrophysics, a NSF Science and Technology Center at U.C. Berkeley, and MJ by an Irish Government (Dept. of Education) post-doctoral fellowship.

[1] For a review see, for example, P. Steinhardt, *Cosmology at the Crossroads* in the *Proceedings of the Snowmass Workshop on Particle Astrophysics and Cosmology*, E. Kolb and R. Peccei eds. (1995).
[2] J. Ostriker and P. Steinhardt, *Nature* **377** 600 (1995).
[3] B. Ratra and J. Peebles *Phys. Rev.* **D37** 3406 (1988);
[4] K. Coble *et al.*, *Phys. Rev.* **D55** 1851 (1997).
[5] M. Davis *et al.*, *Nature* **359** 393 (1992); A. Taylor and M. Rowan Robinson, *Nature* **359** 396 (1992); J. Primack *et al.*, *Phys. Rev. Lett* **74** 2160 (1995).
[6] C. Wetterich, *Nuc. Phys.* **B252** 302 (1988) 668 and *Astron. and Astrophy.***301** 321 (1995).
[7] We have assumed the field rolls down a potential. Such scaling also pertains for a field oscillating about the minimum of a ϕ^m potential with $\frac{6m}{m+2} > n$, *cf.* M. Turner, *Phys. Rev.* **D28** 1243 (1983). In this case the second condition given is not satisfied, and there is no attractor.
[8] J. Halliwell, *Phys. Lett.* **185B** 341 (1987); J.D. Barrow, *Phys. Lett.***B187** 12 (1987).
[9] A.B. Burd and J.D. Barrow,*Nuc. Phys.* **B308** 929 (1988); J. Yokoyama and K. Maeda, *Phys. Lett.* **207B** 31 (1988).
[10] C. Wetterich, *Nuc. Phys.* **B252** 309 (1985); Q. Shafi and C. Wetterich, *Phys. Lett.* **152B** 51 (1985) and *Nuc. Phys.* **B289** 787 (1987).
[11] J. Halliwell, *Nuc. Phys.* **B266** 228 (1986); J.D. Barrow and S. Cotsakis, *Phys. Lett.***B214** 515 (1988).
[12] C. Copi, D. Schramm and M.Turner, *Phys. Rev. Lett* **75** 3981 (1995); B. Fields *et al*, *New Astronomy* **1** 77 (1996).
[13] P. Kernan and S. Sarkar, *Phys. Rev.* **D54** 3681 (1996); M. Birkel and S. Sarkar, *Astropart. Phys.* **6** 197 (1997).
[14] M. Joyce, *Phys. Rev.* **D55** 1875 (1997).
[15] B. Spokoiny, *Phys. Lett.***B315** 40 (1993).
[16] C.P. Ma and E. Bertschinger, *Astroph. Jour* **455** 7 (1995); U. Seljak and M. Zaldarriaga, *ibid* **469** 437 (1996).
[17] G. Smoot *et al.*, *Astroph. Jour.* **437**1 (1994).
[18] E. Bunn and M. White, *Astroph. Jour.* **480** 6 (1997).
[19] J. Peacock and S. Dodds, *M.N.R.A.S.* **267** 1020 (1994).
[20] S. White, G. Efstathiou and C. Frenk, *M.N.R.A.S.* **262** 1023 (1993); A. Liddle *et al.*, *M.N.R.A.S.* **281** 531 (1996).
[21] M. Tegmark and A. Hamilton, `astro-ph/9702019`; see references in `http://www.sns.ias.edu/ max/cmb/experiments.html`.
[22] J. Bond *et al.*, *Phys. Rev. Lett* **45** 1980 (1980).
[23] W. Hu, N. Sugiyama and J. Silk, *Nature* **386** 37 (1995).
[24] S. Dodelson *et al.*, *Astroph. Jour* **467** 10 (1996).
[25] G. Jungman *et al.*, *Phys. Rev* **D54** 1332 (1996); J. Magueijo and M. Hobson, `astro-ph/9610105`.