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Primordial Magnetic Fields, Right Electrons, and the Abelian Anomaly

M. Joyce[1,2] and M. Shaposhnikov[1]

[1] *Theory Division, CERN, 1211 Geneve 23, Switzerland*

[2] *School of Mathematics, Trinity College, Dublin 2, Ireland*

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Abstract

In the standard model there are charges with abelian anomaly only (e.g. right-handed electron number) which are effectively conserved in the early universe until some time shortly before the electroweak scale. A state at finite chemical potential of such a charge, possibly arising due to asymmetries produced at the GUT scale, is unstable to the generation of hypercharge magnetic field. Quite large magnetic fields ($\sim 10^{22}$ gauss at $T \sim 100$ GeV with typical inhomogeneity scale $\sim \frac{10^6}{T}$) can be generated. These fields may be of cosmological interest, potentially acting as seeds for amplification to larger scale magnetic fields through non-linear mechanisms. Previously derived bounds on exotic $B - L$ violating operators may also be evaded.

It is usually assumed that the early Universe at temperatures above the electroweak scale and below, say, $10^{12} - 10^{16}$ GeV (depending on the model of inflation) consists of an (almost) equilibrium primordial plasma of elementary particles, in which any long-range fields are absent. One exception is in the context of the problem of generating galactic magnetic fields, which may require the presence of primordial seed magnetic fields which are subsequently amplified by a galactic dynamo mechanism (see, e.g., [1]). The creation of long range magnetic fields requires that conformal invariance be broken in the coupling of the electromagnetic field to gravity [2], and a number of mechanisms based on different ideas about this breaking have been proposed to date [2,3]. In this letter we argue that there may be a relation between the appearance of magnetic fields in the early Universe and two other, apparently completely unrelated, phenomena : (i) The smallness of the electron Yukawa coupling constant, and (ii) possible lepton asymmetry of the early Universe.

In short, the logic goes as follows. There are three exact conservation laws in the standard electroweak theory. The associated conserved charges can be written as $N_i = L_i - \frac{1}{3}B$, where L_i is the lepton number of ith generation and B is the baryon number. The fourth possible combination, $B + \sum_i L_i$ is not conserved because of electroweak anomalous processes, which are in thermal equilibrium in the range 100 GeV $< T <$ 10^{12} GeV [4]. Now, if $h_e = 0$, where h_e is the right electron Yukawa coupling constant, then the electroweak theory on the classical level shows up a higher symmetry, associated with the chiral rotation of the right electron field. For the small actual value of the Yukawa coupling ($h_e = 2.94 \times 10^{-6}$ in the MSM) this symmetry has an approximate character. At temperatures higher than $T_R \simeq 80$ TeV perturbative processes with right electron chirality flip are slower than the expansion rate of the Universe [5], and therefore this symmetry may be considered as an exact one on the classical level at $T > T_R$ [6]. (The importance of this symmetry for the consideration of the wash-out of the GUT baryon asymmetry by anomalous electroweak B and L non-conserving reactions was realized in refs. [7,5,8].) Suppose now that an excess of right electrons over positrons was created by some means at $T > T_R$ (e.g. by a GUT mechanism for baryogenesis). Now the right electron number current j_R^μ is violated in the minimal standard model (MSM) as described by the anomaly equation

$$\partial_\mu j^\mu_R = -\frac{g'^2 y_R^2}{64\pi^2} f_{\mu\nu}\tilde{f}^{\mu\nu}, \tag{1}$$

where $f(\tilde{f})$ are the $U_Y(1)$ hypercharge field strengths (and their duals) respectively, g' is the associated gauge coupling and $y_R = -2$ is the hypercharge of the right electron. The number of the right electrons N_R therefore changes with the Chern-Simons (CS) number of the hypercharge field configuration as $\Delta N_R = \frac{1}{2} y_R^2 \Delta N_{cs}$ with

$$N_{cs} = -\frac{g'^2}{32\pi^2}\int d^3\vec{x}\epsilon_{ijk} f_{ij} b_k, \tag{2}$$

where b_k is the hypercharge field potential.

One can now see qualitatively that there is an instability in hot matter with an excess of right electrons towards formation of hypercharge fields with CS number as follows. (The line of reasoning presented here is similar to the consideration of cold fermionic matter with anomalous charges in [9]). The energy density "sitting" in right electrons with a chemical potential μ_R is of order $\mu_R^2 T^2$, and their number density of order $\mu_R T^2$. On the other hand this fermionic number can be absorbed by a hypercharge field of order $g'^2 k b^2$, with energy of order $k^2 b^2$, where k is the momentum of the classical hypercharge field and b is its amplitude. Therefore, at $b > T/g'^2$ and $k \sim \mu_R T^2/(g'^2 b^2)$ the gauge field configuration has the same fermion number as the initial one, but smaller energy. An instability to generation of hypercharge magnetic field, which tends to "eat up" real fermions, results. It is important here that at temperatures $T > T_R$ the electroweak symmetry is "restored", and that the U(1) hypercharge magnetic field is massless at that time. (No term like $m_Y^2 b^2$ is generated in any order of perturbation theory in abelian gauge theory at high temperature [10]; the lattice study in [11] confirmed this expectation for SU(2)×U(1) EW theory beyond perturbation theory). If the hypercharge magnetic fields survive until the time of the EW phase transition ($T \sim 100$ GeV), they will give rise to ordinary magnetic fields because of electroweak mixing. In the rest of this paper we present quantitative estimates of the (hypercharge) magnetic fields which may be produced by this effect.

Let us discuss first the possible origin and the magnitude of the the required right electron number asymmetry $\delta_R = e_R/s$, where $s = \frac{2}{45}\pi^2 T^3 N_{eff}$ is the entropy density with $N_{eff} = N_b + \frac{7}{8}N_f = 106.75$ the total effective number of degrees of freedom of the MSM. In principle δ_R produced by out of equilibrium decay at the GUT scale can be as large as $\sim 10^{-2} - 10^{-4}$ (for a review see, e.g. [12]). This is quite consistent with the magnitude of the final baryon asymmetry δ_B being that observed since there is no simple general relation between the two numbers. In theories like those discussed in [5,8] with L violating processes at intermediate scales one has $\delta_B \sim \delta_R$, at least in the case that the L violating processes go out of equilibrium before the e_R violating ones come into equilibrium. In [8] the case is considered where the L violation continues for just long enough to reduce the final δ_B to the observed one from an initially larger value fixed by δ_R. And, in a simple GUT like $SU(5)$ in which the charges $N_i = L_i - \frac{1}{3}B$ are conserved, we can have $\delta_B = 0$ at the electroweak scale irrespective of the value of δ_R during the time it is effectively conserved. In the rest of this letter we will simply assume the existence of a primordial density of e_R, with its chemical potential as a free input parameter, assumed only small enough to be treated perturbatively. We also assume that no hypercharge magnetic fields existed before the right electron excess is generated.

The effect of the anomaly on the gauge field dynamics is given through the term in the effective Lagrangian

$$\delta\mathcal{L} = \frac{g'^2}{4\pi^2}\mu_R \epsilon_{ijk} f_{ij} b_k. \tag{3}$$

which is obtained by integrating out the fermions at finite chemical potential [13]. It simply describes how winding the gauge fields to give CS number changes the energy of the system because it changes the number of fermions as described by the anomaly equation. Adding this term to the ordinary Lagrangian for the gauge fields leads to the equations of motion

$$\frac{\partial \vec{H}}{\partial t} + \vec{\nabla} \times \vec{E} = 0, \tag{4}$$

$$\vec{E} = \frac{1}{\sigma}(\vec{\nabla} \times \vec{H} + \frac{4\alpha'}{\pi}\mu_R \vec{H}). \tag{5}$$

where $\alpha' = g'^2/4\pi$. These are simply Maxwell's equations with the additional term due to the anomaly ($\propto \mu_R \vec{H}$) and the assumption that the total (hypercharge) current is given by $\vec{j} = \sigma \vec{E}$, where σ is the conductivity of the plasma and $\vec{E}$ is the (hyper-)electric field. We have also dropped the term $\partial \vec{E}/\partial t$ since (as we will see below) the fields always evolve on a time-scale which is much longer than σ^{-1}. In the expanding FRW Universe with scale factor a the equations have exactly the same form in conformal time coordinates $\tau = \int a^{-1}(t)dt$, but with the replacements $\mu_R \to \mu_R a$ and $\sigma(\propto T) \to \sigma a$. The fields $\vec{E}$ and $\vec{H}$ are those given by their standard definitions in the conformal frame which will be related to the physical fields at the appropriate point below. We also have the following kinetic equation for μ_R :

$$\frac{1}{a}\frac{\partial(\mu_R a)}{\partial \tau} = -\frac{\alpha'}{\pi}\frac{783}{88}\frac{1}{a^3 T^2}\vec{E}\cdot\vec{H} - \Gamma_R(\mu_R a), \tag{6}$$

in which the first term describes the change in the chemical potential due to the anomaly ($f\tilde{f} \propto \vec{E}\cdot\vec{H}$), and the second the change due to the perturbative processes which flip electron chirality with the rate $\Gamma_R = \frac{T_R}{M_0}T$ ($M_0 = M_{pl}/1.66\sqrt{N_{eff}} \simeq 7.1 \times 10^{17}$ GeV). The numerical coefficient $\frac{783}{88}$ comes from the relationship between right electron chemical potential and right electron number asymmetry (in terms of which the anomaly is expressed)

$$\mu_R = \frac{2}{45}\pi^2 N_{eff}[\frac{783}{88}\delta_R - \frac{201}{88}\delta_1 + \frac{15}{22}(\delta_2 + \delta_3)]T,$$

which is obtained from a local thermal equilibrium calculation in the EW theory with three fermionic generations and one scalar doublet, with the conserved charges assumed to be $N_i(=\delta_i s)$ and $e_R(=\delta_R s)$.

With a Fourier mode decomposition $\vec{H}(\vec{x}) = \int d^3\vec{k}\vec{H}(\vec{k})e^{-i\vec{k}.\vec{x}}$ with $\vec{H}(\vec{k}) = h_i\vec{e}_i$ where $i = 1, 2, \vec{e}_i^{\,2} = 1, \vec{e}_i \cdot \vec{k} = 0, \vec{e}_1 \cdot \vec{e}_2 = 0$, the linear equations (4) and (5) become

$$\begin{aligned} \partial_\tau h_1 + \frac{k^2}{\sigma a}h_1 - \frac{4i\mu|k|}{\sigma}h_2 = 0 \\ \partial_\tau h_2 + \frac{k^2}{\sigma a}h_2 + \frac{4i\mu|k|}{\sigma}h_1 = 0 \end{aligned} \tag{7}$$

where $\mu \equiv \frac{g'^2}{4\pi^2}\mu_R$. The mode

$$h_2(\tau, k) = -ih_1(\tau, k) = -\frac{i}{2}(h_1(0, k) + ih_2(0, k))exp(\lambda_+(\tau)) \tag{8}$$

where

$$\lambda_{\pm}(\tau) = -\frac{k}{\sigma a}(k\tau \mp 4\int_0^\tau d\tau' \mu a) \tag{9}$$

is an unstable mode which is growing at conformal time τ if $k < 4\mu(\tau)a(\tau)$. It has the property $\vec{E}(\vec{k}) = \frac{1}{\sigma a}(-|k| + 4\mu a)\vec{H}(\vec{k})$. (The other orthogonal mode decays at any $|k|$.)

Consider now the approximation in which the chemical potential μ is a constant. The growing instability starts to develop at $T \sim T_g$ where we define T_g to be

$$8(\frac{\mu}{T})^2 \frac{1}{\sigma/T}\frac{M_o}{T_g} = 1. \tag{10}$$

(when the maximally growing mode with $k = 2\mu a$ has begun growing significantly). A necessary requirement for the instability to develop is that $T_g > T_R$, since if this is not satisfied the second term in (6) will rapidly reduce μ towards zero. Translated into a minimum value for $\delta \equiv \frac{\mu}{T}$ this requires $\delta > \delta_{crit} = 10^{-6}$ (using $\sigma \approx 68T$ [14]). For $\delta < \delta_{crit}$ no non-trivial dynamics result from the presence of such a chemical potential since the unstable modes are frozen on the relevant time-scale. If $\delta > \delta_{crit}$ the evolution of the instability for $T < T_g$ will be given by the simple growth factor above, until the time at which the growth becomes significant enough that the first term in (6) is important. To estimate when this is and what the amplitude of the field is at that time it is sufficient to calculate the CS number as a function of time. It is given (per co-moving volume) by

$$\begin{aligned} n_{cs}(\tau) &= -\frac{g'^2}{32\pi^2} < \epsilon_{ijk} f_{jk}(\tau) b_i(\tau) > \\ &\approx -\frac{g'^2}{64\pi^4}\int_0^{4\mu a} dk e^{2\frac{k(4\mu a - k)\tau}{\sigma a}} k^2 f(k), \end{aligned} \tag{11}$$

neglecting all but the growing mode. We have also taken $< b_i(\vec{k},\tau) b_j^*(\vec{l},\tau) > |_{\tau=0} = \delta^3(\vec{k}-\vec{l})\delta_{ij} < b^2(k) >_0$, assuming translational and rotational invariance of the initial perturbations, and assumed that the perturbations are thermal in origin, with the appropriate normalizations, $< b^2(k) >_0 = \frac{1}{2k(2\pi)^3} f(k)$ where $f(k) = (e^{\frac{k}{T_0}} - 1)^{-1}$ is the bosonic distribution function and T_0 the temperature at which we define $a_0 = 1$. Defining ϵ by $\frac{1}{2} y_R^2 n_{cs} \equiv \epsilon \Delta e_R a^3$, where $\Delta e_R = \frac{88}{783}\frac{\mu_R}{T} T^3$ i.e. the difference between the right electron density in the initial state and the $\mu_R = 0$ state, the linear approximation breaks down when $\epsilon \sim 1$. Evaluating the integral in (11) we find $\epsilon \approx 2 \times 10^{-6} \delta \frac{1}{\sqrt{\alpha}} e^{\alpha}$ where $\alpha = T_g/T$ with T_g as in (10). Thus for a few expansion times after the mode starts growing at temperature T_g we have $\epsilon < 1$ and the linear approximation is valid. The corresponding physical magnetic field H_{phy} can be estimated by putting $|n_{cs}| \approx \frac{g'^2}{16\pi^2} kb^2 = \frac{1}{2}\epsilon \Delta e_R a^3$ and using $kb = a^2 H_{phy}$, where $k \sim 2\mu a$ (i.e. assuming the maximal growing mode to dominate). Putting in the numbers this gives a physical magnetic field of strength $H_{phy} \approx 2 \times 10^2 \sqrt{\epsilon \delta \frac{k_{phy}}{T}} T^2$ at a physical length scale $k_{phy}^{-1} \equiv (\frac{k}{a})^{-1} \sim \frac{1}{2\delta T}$. For $\delta \sim 5 \times 10^{-6}$ we are in the linear regime until $T_R \sim 80$ TeV and at that point therefore have a magnetic field of strength $H_{phy} \sim 6 \times 10^{26}$ gauss (1 GeV^2 $= 1.95 \times 10^{20}$ gauss) at a length scale of $\sim 10^5/T$ (compared to a horizon scale of $\sim 10^{13}/T$).

How do the fields evolve for $T < T_R$? In the case that the linear (i.e. constant chemical potential) approximation is good, one expects that the growth will rapidly turn into decay as μ is damped. Within a few expansion times the growth will be undone as the maximally growing mode $k \sim 2\mu a$ now decays with exponent $-(k^2/\sigma)a\tau$. What about the case when this linear approximation breaks down? To treat this case we must analyse the full non-linear set of equations (4)-(6). We have done this numerically with the simplifying assumption that the distribution of right electron number is homogeneous in space. Then the two linear equations (4) and (5) can be solved exactly for any time dependent μ, the solution inserted in (6), and the averaging over thermal initial conditions performed. The resulting equation is

$$\frac{\partial \mu a}{\partial \tau} = (\frac{\alpha'}{\pi})^2 \frac{783}{88} \frac{1}{32\pi^2 \sigma a^2 T} \int_0^\infty dk k^2 [(k - 4\mu a) exp(2\lambda_+(\tau)) - (k + 4\mu a) exp(2\lambda_-(\tau))], \quad (12)$$

a length scale of $10^5/T$a length scale of $10^5/T$with $\lambda_\pm(\tau)$ given by (9). Our results show that the chemical potential, typical physical momentum of the magnetic field configuration and the magnetic field energy scale as

$$\frac{\mu_R}{T} \propto \frac{k_{phy}}{T} \propto \frac{H_{phy}^2}{T^4} \propto \left(\frac{T}{T_g}\right)^{\frac{1}{2}}. \quad (13)$$

in the range $T_g > T > T_R$. This behaviour can be easily understood qualitatively as follows. As the instability develops, the linear approximation breaks down and μ starts significantly decreasing. This shifts the growth of modes to longer wavelengths. This procedure continues, growth of any mode eventually turning itself off and increasing the growth coefficient of modes at larger scales. The minimum value of μ which can be reached at any given time τ (and, correspondingly, the maximum physical scale for the sourced fields) is simply that given by (10), solved for μ with T_g replaced by the temperature $T(\tau)$ i.e it is just the minimal chemical potential required to drive a growing mode at that time in the linear approximation. The parametric dependence on the temperature observed follows from the fact that the chemical potential (and maximally growing mode) trace these values. The dependence of the magnetic field energy follows from the expression we derived in the previous paragraph by setting the CS number of the configuration to cancel the total fermion number, but now taking the appropriate scaling for k_{phy} itself.

Evolving the system forward from T_R to the electroweak scale T_{ew} we see the damping of the fields for δ in the linear regime anticipated above, as the perturbative processes erase the chemical potential driving the growth. As δ increases, however, this damping becomes less efficient, and for $\delta > 2 \times 10^{-4}$ we find that the damping has not set in at all by the electroweak scale. The reason for this behaviour is also simple. For a mode which evolves in the linear regime, the growth and decay exponents are effectively the same for the modes which grow significantly. Once we enter the non-linear regime this is no longer true, since the maximally growing mode carries the integrated effect of growth until any given time i.e. it has grown with exponent $k \int \frac{\mu(\tau)}{\sigma} d\tau$ which is much greater than $\frac{k^2}{\sigma a}\tau$. Put another way, the mode has been able to grow on a scale significantly larger than the diffusion length for magnetic field at the relevant time, and it takes some time after the end of the growth for the latter scale to catch up and undo the effect of the instability. For δ in this region we also see that the typical scale of magnetic fields k and value of the chemical potential at T_{ew} do not depend on initial asymmetry, $\frac{2k_{phy}}{T} \simeq \frac{4\mu}{T} \simeq \frac{10^6}{T}$. The amplitude of magnetic field scales as $H \propto \sqrt{\delta}$ and e.g. for $\delta = 10^{-2}$ we find $H \simeq 4 \times 10^{22}$ gauss. This is as we would expect from the discussion of the scaling above.

What is the ultimate fate of these magnetic fields? Unless some other effect comes into play in the dynamics, the fields will decay. One such effect is turbulence. With the full set of MHD equations (which include the velocity of the fluid which we have neglected) there is a transition to a turbulent regime when the magnetic Reynold's number $R = \sigma L v$ is large [16]. The reason we have evolved the equations to the electroweak scale is that, if the electroweak phase transition is of first order, it serves as a source of turbulence [15]. Since we have here $\sigma \sim 10^2 T$ and magnetic fields which begin to grow on length scales L up to $\sim 2 \times 10^6/T$ we expect to enter the turbulent regime if there are bulk velocities of greater than $\sim 5 \times 10^{-9}$, which are certainly larger that the expected velocity of the bubble walls. A recent study of this phenomenon [16] suggests that the effect of this turbulence is to transfer the magnetic energy to larger length scales, thus evading the Silk argument [17]. If true, the fields generated by the mechanism under discussion may play the role of the seed galactic magnetic fields. Note that the seed fields we obtain at the electroweak scale with the mechanism we have discussed ($\sim 10^{22}$ gauss) are much larger than those generated at bubbles walls ($\sim 10^{-2}$ gauss) which were suggested as seeds for amplification through turbulence in [15]. It is also worth mentioning the particular structure of the magnetic fields appearing because of the abelian anomaly. The CS wave (8) has a non-zero value of $\vec{H} \cdot \vec{\partial} \times \vec{H}$ and thus breaks parity. Could it be that the rotation of galaxies are related to this? Study of the entire set of MHD equations with the additional anomalous terms discussed in this letter will be required to address this question.

Finally let us mention that the processes we have considered also affect the bounds on the strength of exotic interactions with $B - L$ violation derived from the requirement that GUT baryon asymmetry is not erased by sphalerons [5,8] (which is important if no baryon asymmetry is created at the electroweak scale). If the right electron asymmetry produced at the GUT scale is small enough ($\delta < 10^{-6}$), then the bounds are obviously not affected since abelian anomaly does not play any role. If, on the other hand, δ is large enough that significant CS number survives remains in the condensate until the electroweak scale, any bound on the strength of exotic interactions can be evaded. Irrespective of the effect of any $B - L$ violation until that point the remaining CS number will be converted into quarks and leptons carrying net baryon number at the electroweak phase transition. The final baryon asymmetry will depend on the initial value of δ and the exact strength of the phase transition (which will determine how the B violating processes turn off). Conversely, given detailed knowledge of the phase transition, it will be possible to place an upper bound on the initial value of δ in the very early universe, and on the strength of the magnetic fields resulting at the electroweak phase transition.

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