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Three-Dimensional Black Holes and String Theory

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Abstract

The exact decay rate for emission of massless minimally coupled scalar fields by a non-extremal black hole in $2+1$ dimensions is obtained. At low energy, the decay rate into scalars with zero angular momentum is correctly reproduced within conformal field theory. The conformal field theory has both left- and right-moving sectors and their contribution to the decay rate is associated naturally with left and right temperatures of the black hole.

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1 Introduction

Recently, it has become clear that a large class of extremal and near-extremal black holes allow for a conformal field theory or effective string theory description. Extremal black holes often correspond to *BPS*-states of an underlying fundamental string theory. Agreement between the Bekenstein-Hawking entropy and the counting of string states for extremal black holes in five dimensions was first obtained in [1]. However, the correspondence does not seem to be restricted to extremal black holes. Indeed, the entropy of near-extremal black holes is often completely described by an effective string theory [2]-[4].

On another front, the decay rates of non-extremal black holes have also been examined. This involves studying the absorption of quanta by the black hole, and then allowing it to evaporate, via Hawking radiation, back to extremality. In [4]-[9], the low energy scattering cross sections and decay rates for a massless minimally coupled scalar field were computed for a large class of four- and five-dimensional black holes, and agreement was found with conformal field theory or effective string theory predictions. In each of these cases, the result relied on a particular matching of solutions, in a region near the black hole horizon and an asymptotic region far from the black hole. For certain ranges of parameters inherent to the problem, this matching agrees with a conformal field theory description.

In this paper, we study the propagation a massless minimally coupled scalar field in the background geometry of the $(2+1)$-dimensional Bañados-Teitelboim-Zanelli black hole [10]. This black hole is described by two parameters, its mass M and angular momentum J. In addition, the metric has constant negative curvature, and is thus locally isometric to anti-de Sitter space. The special feature here is that the wave equation can be solved exactly, without any approximations [11]. This allows us to determine exactly the range of energy and angular momentum of the scattered field, for which the the decay rate agrees with the conformal field theory description. We find agreement for energies small in comparison to the size of the black hole, and to the curvature scale of the spacetime; in addition, one is restricted to the zero angular momentum wave. In this region, however, agreement is found for all values of M and J, and thus the conformal field theory description is not restricted to a near-extremal limit. Apart from that, we find behaviour similar to that observed in five dimensions, namely that the conformal field theory has both left- and right-moving sectors. The corresponding decay rate is then written naturally in terms of left and right temperatures of the black hole.

Similarly to the 5-D black holes, the BTZ black hole is a solution of string theory [12, 13]. The string scattering off BTZ black holes has been considered in [14, 15] (see also [16]).

2 The BTZ Black Hole

Geometrically, three-dimensional anti-de Sitter space can be represented as the $SL(2,\mathbf{R})$ group space. Isometries are then represented by elements of the group $SL(2,\mathbf{R})\times$

$SL(2,\mathbf{R})/\mathbf{Z}_2$, where the two copies of $SL(2,\mathbf{R})$ act by left and right multiplication. The BTZ black hole is obtained as the quotient space $SL(2,\mathbf{R})/\langle(\rho_L,\rho_R)\rangle$, where $\langle(\rho_L,\rho_R)\rangle$ denotes a certain finite subgroup of $SL(2,\mathbf{R})\times SL(2,\mathbf{R})/\mathbf{Z}_2$ generated by (ρ_L,ρ_R) [17]. We choose Schwarzschild-like coordinates in which the BTZ metric reads [10, 18]

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2}dr^2 + r^2\left(d\phi + N^\phi dt\right)^2, \tag{1}$$

with lapse and shift functions and radial metric

$$N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \quad N^\phi = -\frac{J}{2r^2}, \quad (|J| \leq M\ell). \tag{2}$$

The metric (1) is singular when $r = r_\pm$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}, \tag{3}$$

i.e.,

$$M = \frac{r_+^2 + r_-^2}{\ell^2}, \quad J = \frac{2r_+r_-}{\ell}. \tag{4}$$

The $M=-1$, $J=0$ metric may be recognized as that of ordinary anti-de Sitter space; it is separated by a mass gap from the $M=0$, $J=0$ "massless black hole", whose geometry is discussed in Refs. [17] and [19]. For convenience, we recall that the Hawking temperature T_H, the area of the event horizon $\mathcal{A}_H$, and the angular velocity at the event horizon Ω_H, are given by

$$T_H = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}, \quad \mathcal{A}_H = 2\pi r_+, \quad \Omega_H = \frac{J}{2r_+^2}. \tag{5}$$

The BTZ black hole is part of a solution of low energy string theory [12, 13]. The low energy string effective action is

$$I = \int d^3x\,\sqrt{-g}e^{-2\phi}\left(\frac{4}{k} + R + 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{6}$$

where ϕ is the dilaton, $H_{\mu\nu\rho}$ is an antisymmetric Kalb-Ramond field, which in three dimensions must be proportional to the volume form $\epsilon_{\mu\nu\rho}$, and k is the cosmological constant. It was observed in [12, 13] that, in three dimensions, the ansatz

$$H_{\mu\nu\rho} = \frac{2}{\ell}\epsilon_{\mu\nu\rho}, \quad \phi = 0, \quad k = \ell^2, \tag{7}$$

reduces the equations of motion of (6) to the Einstein field equations satisfied by (1). In fact, there is a corresponding exact solution of string theory, the SL$(2,\mathbf{R})$ WZW model with an appropriately chosen central charge describes the propagation of strings. By quotienting out the discrete group $\langle(\rho_L, \rho_R)\rangle$ by means of an orbifold construction, one obtains a theory that may be shown to be an exact string theoretical representation of the BTZ black hole [12, 13].

3 The Wave Equation

It has been known for some time that the minimally coupled scalar field equation can be solved exactly in the background geometry of the BTZ black hole [11]. This will allow us to determine the scattering cross section and decay rate of the scalar field exactly. Substitution of the metric (1) into the covariant Laplacian

$$\Box = \frac{1}{\sqrt{|g|}}\partial_\mu \sqrt{|g|}\; g^{\mu\nu}\partial_\nu \tag{8}$$

leads to the scalar wave equation

$$\left(-f^{-2}\partial_t^2 + f^2\partial_r^2 + \frac{1}{r}\left(\partial_r r f^2\right)\partial_r - \frac{J}{r^2}f^{-2}\partial_t\partial_\phi - \frac{A}{r^2}f^{-2}\partial_\phi^2\right)\Psi = 0, \tag{9}$$

where

$$f^2 = \frac{1}{\ell^2 r^2}(r^2 - r_-^2)(r^2 - r_+^2), \quad A = M - \frac{r^2}{\ell^2}. \tag{10}$$

This suggests the ansatz

$$\Psi(r,t,\phi) = R(r,\omega,m)e^{-i\omega t + im\phi}, \tag{11}$$

leading to the radial equation for $R(r)$

$$\partial_r^2 R(r) \quad + \quad \left(-\frac{1}{r} + \frac{2r}{r^2 - r_-^2} + \frac{2r}{r^2 - r_+^2}\right)\partial_r R(r) + f^{-4}\left(\omega^2 - \frac{J\omega m}{r^2} + \frac{Am^2}{r^2}\right)R(r). \tag{12}$$

Changing variables to $x = r^2$, the radial equation becomes

$$(x - x_-)(x - x_+)\partial_x^2\; R(x) + [2x - x_+ - x_-]\partial_x\; R(x) + K(x)\; R(x) = 0, \tag{13}$$

where

$$K(x) = \frac{\ell^2}{4f^2}\left(\omega^2 - \frac{J\omega m}{x} + \frac{Am^2}{x}\right). \tag{14}$$

We introduce a further change of variables by defining

$$z = \frac{x - x_+}{x - x_-}. \tag{15}$$

The radial equation then becomes

$$z(1-z)\partial_z^2\, R(z) + (1-z)\partial_z\, R(z) + \left(\frac{A_1}{z} + B_1\right) R(z) = 0, \tag{16}$$

where

$$A_1 = \left(\frac{\omega - m\Omega_H}{4\pi T_H}\right)^2, \quad B_1 = -\frac{x_-}{x_+}\left(\frac{\omega - m\Omega_H \frac{x_+}{x_-}}{4\pi T_H}\right)^2. \tag{17}$$

The hypergeometric form of (16) becomes explicit upon removing the pole in the last term through the ansatz

$$R(z) = z^\alpha g(z), \quad \alpha^2 = -A_1. \tag{18}$$

We then have

$$z(1-z)\partial_z^2 g(z) + (2\alpha+1)(1-z)\partial_z g(z) + (A_1 + B_1)g(z) = 0. \tag{19}$$

In the neighbourhood of the horizon, $z=0$, two linearly independent solutions are then given by $F(a,b,c,z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c, z)$, where

$$\begin{aligned} a+b &= 2\alpha, \\ ab &= \alpha^2 - B_1, \\ c &= 1+2\alpha. \end{aligned} \tag{20}$$

Note that $c = a+b+1$.

4 The Decay Rate

We choose the solution which has ingoing flux at the horizon, namely,

$$R(z) = z^\alpha F(a,b,c,z). \tag{21}$$

To see this, we note that the conserved flux for (12) is given, up to an irrelevant normalisation, by

$$\mathcal{F} = \frac{2\pi}{i}\left(R^*\Delta\partial_r R - R\Delta\partial_r R^*\right), \tag{22}$$

where $\Delta = rf^2$. The flux can be evaluated by noting that

$$\Delta\partial_r = \frac{2\Delta_-}{\ell^2} z\partial_z, \tag{23}$$

where $\Delta_- = x_+ - x_-$. Then, using the fact that ab is real, we find the total flux (which is independent of z) to be given by

$$\mathcal{F}(0) = \frac{8\pi\Delta_-}{\ell^2}\mathrm{Im}[\alpha]\,|F(a,b,c,0)|^2 = 2\mathcal{A}_H(\omega - m\Omega_H). \tag{24}$$

In order to compute the absorption cross section, we need to divide (24) by the ingoing flux at infinity. The singularity of (12) at infinity is such that it admits one solution of the form [20]

$$u_1(y) = y^2 \sum_{n=0}^{\infty} c_n y^n, \tag{25}$$

where $y = \ell/r$. A second linearly independent solution of (12) at infinity is then given by

$$u_2(y) = \sum_{n=0}^{\infty} d_n y^n + A u_1(y)\log(y), \tag{26}$$

where A is a constant. Up to second order in y, we have

$$u_1(y) = y^2, \quad u_2(y) = 1 - -\left(\frac{\omega^2\ell^2 - m^2}{2}\right) y^2 \log(y). \tag{27}$$

The distinction between ingoing and outgoing waves is complicated by the fact that the BTZ-spacetime is not asymptotically flat. For a tachyon field, ingoing and outgoing waves have been defined in e.g. [15]. A naive extrapolation to massless fields is not sensible as the resulting ingoing and outgoing waves are given by u_1 and u_2 in (27), which both have vanishing flux. However, we can define ingoing and outgoing waves to be complex linear combinations of u_1 and u_2 which have positive and negative flux, respectively. This leads to

$$R^{in} = A_i\left(1 - i\frac{c\ell^2}{r^2}\right), \quad R^{out} = A_o\left(1 + i\frac{c\ell^2}{r^2}\right), \tag{28}$$

where c is some positive dimensionless constant, which we take to be independent of the frequency ω. We note the comparison here with the near region behaviour of the ingoing and outgoing solutions in the four-dimensional case (eqn. (2.22) in [7]). The ingoing flux is correspondingly

$$\mathcal{F}_{in} = 8\pi c|A_i|^2. \tag{29}$$

The asymptotic behaviour of (21) for large r is readily available [21], and we can then match this to (28) to determine the coefficients A_i and A_o. We find

$$
\begin{aligned}
A_i + A_o &= \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}, \\
A_i - A_o &= -\frac{\Delta_-\Gamma(a+b+1)}{ic\ell^2}\Bigg\{\frac{\log(\Delta_-/\ell^2)+\psi(a+1)+\psi(b+1)--\psi(1)-\psi(2)}{\Gamma(a)\Gamma(b)} \\
&+ \frac{\alpha}{\Gamma(a+1)\Gamma(b+1)}\Bigg\}, \qquad (30)
\end{aligned}
$$

where ψ is the digamma function. We can estimate the relative importance of the two terms in (30) as follows. Firstly, we note that

$$
ab = -\frac{\ell^2}{4\Delta_-}(\omega^2\ell^2 - m^2), \quad \alpha = i\frac{(\omega - m\Omega_H)\ell^2 r_+}{2\Delta_-}. \qquad (31)
$$

Using $\Gamma(z+1) = z\Gamma(z)$, we find

$$
\begin{aligned}
A_i - A_o &= -\frac{\Gamma(a+b+1)}{c\Gamma(a+1)\Gamma(b+1)}\Big\{\frac{(\omega - m\Omega_H)r_+}{2} + \frac{i(\omega^2\ell^2 - m^2)}{4}\Big[\log(\Delta_-/\ell^2) \\
&+ \psi(a+1)+\psi(b+1)-\psi(1)-\psi(2)\Big]\Big\}. \qquad (32)
\end{aligned}
$$

Furthermore,

$$
a = \frac{i\ell}{2(r_+ - r_-)}\,(\ell\omega - m)\,, \quad b = \frac{i\ell}{2(r_+ + r_-)}\,(\ell\omega + m)\,. \qquad (33)
$$

If $m=0$ and $\omega << \min(\frac{1}{r_+}, \frac{1}{\ell})$, the difference $A_i - A_o$ in (32) is small compared to the sum $A_i + A_o$, so that

$$
A_i \simeq \frac{1}{2}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}. \qquad (34)
$$

This approximation means that the Compton wavelength of the scattered particle is much bigger that the size of the black hole and the scale set by the curvature of the anti-de Sitter space. Note that the logarithmic term in (32) ensures that the term in braces is finite for all values of ℓ/r_+, i.e., all values of M and J. In particular, the extreme limit $J = \pm M\ell$, or equivalently $\Delta_- = 0$, is well defined. Hence, for $m=0$ and ω small in the sense defined above, the approximation (34) should be valid. For $m \neq 0$ on the other hand, there is no obvious choice of black hole parameters for which the term in braces in (32) is small, and hence (34) is modified for $m \neq 0$.

Let us consider the $m=0$ wave and assume $\omega << \min(\frac{1}{r_+}, \frac{1}{\ell})$ so that (34) is valid. Then the partial wave absorption cross section is given by

$$\sigma^{m=0} = \frac{\mathcal{F}(0)}{\mathcal{F}_{in}} = \frac{1}{\pi c}\mathcal{A}_H \omega \frac{|\Gamma(a+1)\Gamma(b+1)|^2}{|\Gamma(a+b+1)|^2}. \tag{35}$$

In order to relate the partial wave cross section to the plane wave cross section σ_{abs}, we need to divide $\sigma^{m=0}$ by ω [22]. We find

$$\sigma_{abs} = \mathcal{A}_H \frac{|\Gamma(a+1)\Gamma(b+1)|^2}{|\Gamma(a+b+1)|^2}, \tag{36}$$

where we have chosen c so that $\sigma_{abs}(\omega) \to \mathcal{A}_H$ for $\omega \to 0$ [22]. The decay rate Γ of a non-extremal black hole is then given by

$$\begin{aligned} \Gamma &= \frac{\sigma_{abs}}{e^{\frac{\omega}{T_H}} - 1} = T_H \mathcal{A}_H \omega^{-1} e^{-\frac{\omega}{2T_H}} |\Gamma(a+1)\Gamma(b+1)|^2 \\ &= 4\pi^2 \ell^2 \omega^{-1} T_L T_R e^{-\frac{\omega}{2T_H}} \left| \Gamma\left(1 + i\frac{\omega}{4\pi T_L}\right) \Gamma\left(1 + i\frac{\omega}{4\pi T_R}\right) \right|^2, \end{aligned} \tag{37}$$

where the left and right temperatures are defined by

$$T^{-1}_{L/R} = T_H^{-1}\left(1 \pm \frac{r_-}{r_+}\right), \tag{38}$$

and we have used $\mathcal{A}_H = 2\pi r_+$.

We now compare this decay rate with the conformal field theory prediction. As explained in [5, 7, 23], in the effective string theory picture such decays are described by a coupling of the spacetime scalar field to an operator with dimension 1 in the conformal field theory, both in the left- and right-moving sector. This might have been expected since the WZW-model of which the BTZ-black hole is a classical solution contains both chiralities [13]. A calculation closely analogous to that presented in [7] shows the emission rate is given by

$$\int d\sigma^+ \, e^{-i\omega(\sigma^+ - i\epsilon)} \left[\frac{2T_R}{\sinh(2\pi T_R \sigma^+)}\right]^2 \int d\sigma^- e^{-i\omega(\sigma^- - i\epsilon)} \left[\frac{2T_R}{\sinh(2\pi T_R \sigma^-)}\right]^2. \tag{39}$$

Performing the $\sigma^\pm$ integrals we then obtain, up to an undetermined numerical constant,

$$\Gamma = 4\ell^2 \omega^{-1} T_L T_R e^{-\frac{\omega}{2T_H}} \left| \Gamma\left(1 + i\frac{\omega}{4\pi T_L}\right) \Gamma\left(1 + i\frac{\omega}{4\pi T_R}\right) \right|^2, \tag{40}$$

where ℓ^2 has been included for dimensional reasons, and a factor $1/\omega$ accounts for the normalisation of the outgoing scalar [7]. Hence, the CFT-prediction (40) and the semiclassical decay rate (37) agree. Note that the above analysis is valid for all values of M and J, subject to the low energy restriction. Thus, for the three-dimensional black hole the CFT description is not restricted to the near-extreme, or near-*BPS* limit; we recall that an extreme BTZ-black hole is also a *BPS* configuration, as shown in [24].

Note also that supersymmetry has not played a role in the calculations presented here. In the four-dimensional case [7], supersymmetry appeared indirectly in determining the relevant conformal field theory. However, such an analysis fails in the 3-dimensional case. Similarly, in five dimensions, bosons and fermions must propagate on the effective string in order to recover the decay rates for even and odd angular momenta alike. Again, this fails in three dimensions because of the restriction $m=0$. On the other hand, it has been shown [24] that the BTZ-black hole admits one supersymmetry in the extremal case with $M \neq 0$, and two supersymmetries if furthermore $M=0$. In the latter case, it may be viewed as the ground state of $(1,1)$-adS supergravity.

A final remark concerns the duality between the BTZ-black hole and the black string [13]. This suggests that an analysis similar to that presented here might go through for the black string. A question that arises naturally here is how the two low energy decay rates compare. For the tachyon, the reflection coefficients have been obtained in [25] for the black string, and in [14, 15] for the case of the black hole.

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