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Fisher Zeroes and Singular Behaviour of the Two Dimensional Potts Model in the Thermodynamic Limit

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Abstract

The duality transformation is applied to the Fisher zeroes near the ferromagnetic critical point in the $q > 4$ state two dimensional Potts model. A requirement that the locus of the duals of the zeroes be identical to the dual of the locus of zeroes in the thermodynamic limit (i) recovers the ratio of specific heat to internal energy discontinuity at criticality and the relationships between the discontinuities of higher cumulants and (ii) identifies duality with complex conjugation. Conjecturing that all zeroes governing ferromagnetic singular behaviour satisfy the latter requirement gives the full locus of such Fisher zeroes to be a circle. This locus, together with the density of zeroes is then shown to be sufficient to recover the singular part of the thermodynamic functions in the thermodynamic limit.

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Introduction The q–state Potts model [1], introduced in 1952 as a generalization of the Ising model [2], has become the generic model for the analytical and numerical study of first and second order phase transitions [3]. Apart from the one dimensional case [1], the only solution which exists to date is for the $q = 2$ (Ising) model in two dimensions and in the absence of an external magnetic field [4]. The partition function for the standard Potts model is $Z_L(\beta) = \sum_{\{\sigma_i\}} \exp\left(\beta \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}\right)$ where $\beta = 1/(k_B T)$, and T is the temperature. The spin σ_i at site i on a d–dimensional lattice takes values $1, 2, \ldots, q$ and the total number of sites is $V = L^d$. Despite the absence of a full solution for general q, some exact results are obtainable in $d = 2$ dimensions. The first of these is that, up to an irrelevant multiplicative constant, the form of the partition function is unchanged under a duality transformation in the thermodynamic limit [1]. In terms of the (low temperature expansion) variable $u = e^{-\beta}$, this duality transformation is $u \to \mathcal{D}(u)$, where

$$\mathcal{D}(u) = \frac{1-u}{1+(q-1)u} \quad . \tag{1}$$

The critical temperature at which the phase transition occurs is invariant under (1) and is given by [1],

$$\mathcal{D}(u_c) = u_c \quad \text{or} \quad u_c = \frac{1}{1+\sqrt{q}} \quad . \tag{2}$$

Baxter has shown that *at* the critical point the model is equivalent to a solvable homogeneous ice–type model [5, 6]. By deriving the latent heat at criticality it was shown that the phase transition in the two-dimensional model is first (second) order for $q > 4$ ($q \le 4$). In fact, for the $q > 4$ case, the exact values of the latent heat, the mean internal energy and the specific heat discontinuity (but not, for example, the mean specific heat) are known [1, 5, 6, 7]. The full form of the free energy (and derivable thermodynamic functions) of the Potts model has, however, never been calculated for general q and general T. In the words of Baxter, solving the Potts model for general temperatures is, therefore, "a very tantalising problem" [6]. The Potts model is reviewed in [7].

In this paper the problem is approached using a remarkably general and recently derived result concerning the partition function zeroes of models with a first order phase transition [8].

For finite systems the zeroes of the partition function [9] are strictly complex (non–real). As $L \to \infty$ one expects these zeroes to condense onto a smooth curve whose impact on to the real parameter axis precipitates the phase transition. Fisher [10] emphasized the application of zeroes in the complex temperature plane to the study of temperature driven phase transitions. In particular, in [10], the Kaufman solution [11] of the two dimensional Ising model was used to show that the Fisher zeroes (also called complex temperature zeroes [12]) are dense on two circles in the complex u–plane in the thermodynamic limit.

Based on similarities with the Ising case, Martin and Maillard and Rammal [13, 14] conjectured that the locus of Fisher zeroes in the $d = 2$, q–state Potts model be given by an extension of (2) to the complex plane, namely $\mathcal{D}(u) = u^*$ where u^* is the complex conjugate of u (although, on this basis alone it was "not clear where this requirement comes from" [13]). This identification yields a circle with centre $-1/(q-1)$ and radius $\sqrt{q}/(q-1)$. When $q = 2$ this recovers the so–called ferromagnetic Fisher circle of the Ising model [10]. In the Ising case, the partition function is actually a function of u^2. There, the second (so–called antiferromagnetic) Fisher circle comes from the map $u \to u^{-1}$ ($\beta \to -\beta$). Numerical investigations for small lattices at $q = 3$ and 4 [13, 14] provided evidence that the Fisher zeroes indeed lie on the circle given by the identification of duality with complex conjugation. However the numerics are highly sensitive to the boundary conditions used and the situation far from criticality remained unclear. Some progress was recently made in the non–critical region using low temperature expansions for $3 \le q \le 8$ [12].

Recently, and on the basis of numerical results on small lattices (up to $L = 7$) with $q \le 10$, it has again been conjectured that for finite lattices with self–dual boundary conditions, and for other boundary conditions in the thermodynamic limit, the zeroes in the ferromagnetic regime are on the above circle [15]. The conjecture of [15] was, in fact, proven for infinite q in [16]. This circle–conjecture is similar to another recent conjecture [17], namely that the Fisher zeroes for the q–state Potts model on a triangular lattice with pure three–site interaction in the thermodynamic limit (which is also self–dual [18]) lie on a circle and a segment of the negative real axis.

All of the above conjectures regarding the locus of Fisher zeroes rely, at least in part, on numerical approaches. In this paper, the problem is addressed analytically. A requirement that taking the thermodynamic limit and application of the duality transformation to the Fisher zeroes be commutative in the $q > 4$ case (i) recovers the ratio of specific heat discontinuity to latent heat and corresponding relationships between the discontinuities of higher cumulants and (ii) analytically identifies duality with complex conjugation. Conjecturing that all zeroes governing ferromagnetic singular behaviour satisfy the latter requirement, the locus of such Fisher zeroes is shown to indeed be a circle. This locus, together with the density of zeroes is then shown to be *sufficient* to recover the singular form of all thermodynamic functions in the thermodynamic limit.

Thermodynamic Functions For finite L, the partition function can be written as a polynomial of finite degree in u, and as such, can be expressed in terms of its complex Fisher zeroes $u_j(L)$ [9] as $Z_L(\beta) \propto \prod_{j=1}^{dV}(u - u_j(L))$. The free energy is defined by $\beta f(\beta) = -\ln Z(\beta)/V$. The internal energy is therefore

$$e(\beta) = \frac{\partial(\beta f)}{\partial \beta} = \text{cnst.} + \frac{u}{V}\sum_{j=1}^{dV}\frac{1}{u - u_j(L)} \quad . \tag{3}$$

The specific heat and the general n^{th} cumulant are respectively defined as

$$c(\beta) = -k_B\beta^2\frac{\partial^2(\beta f)}{\partial\beta^2} \quad , \qquad \gamma_n(\beta) = (-)^{n+1}\frac{\partial^n(\beta f)}{\partial\beta^n} \quad . \tag{4}$$

Using the notation

$$\Delta\gamma_n \equiv \lim_{\beta\nearrow\beta_c}\gamma_n(\beta) - \lim_{\beta\searrow\beta_c}\gamma_n(\beta) \quad , \tag{5}$$

for the discontinuity in the n^{th} cumulant at the critical temperature, the exact results [1, 5, 6, 7] (in the thermodynamic limit) are

$$\overline{e} \;\equiv\; \frac{1}{2}\left(\lim_{\beta\nearrow\beta_c} e(\beta) + \lim_{\beta\searrow\beta_c} e(\beta)\right) = -\left(1 + \frac{1}{\sqrt{q}}\right) \quad , \tag{6}$$

$$
\begin{aligned}
\Delta e &= 2\left(1+\frac{1}{\sqrt{q}}\right)\tanh\left(\frac{\Theta}{2}\right)\prod_{n=1}^{\infty}\tanh^2(n\Theta) \quad , && (7)\\
\Delta c &= k_B\beta_c^2\frac{\Delta e}{\sqrt{q}} \quad , && (8)
\end{aligned}
$$

where $\Theta = \ln\left(\sqrt{q/4}+\sqrt{q/4-1}\right)$. Further results include the general higher cumulant combination $\gamma_n(\beta_c^-) - (-)^n\gamma_n(\beta_c^+)$ determinable from duality [7, 19, 20].

Partition Function Zeroes Recently, Lee [8] has derived a general theorem for first order phase transitions in which the partition function zeroes can be expressed in terms of the discontinuities in the thermodynamic functions (for finite size as well as in the infinite volume limit). For a system with a temperature–driven phase transition, Lee's result for the Fisher zeroes is

$$
\frac{\ln q}{V\Delta e} \pm i\frac{(2j-1)\pi}{V\Delta e} = \beta_c t + \frac{t^2}{2!}\frac{\Delta c/k_B}{\Delta e} + \sum_{n=3}^{\infty}\frac{(\beta_c t)^n}{n!}\frac{\Delta\gamma_n}{\Delta e} \qquad (9)
$$

where the reduced temperature is $t = 1-\beta/\beta_c$. Inverting, we find [8]

$$
\begin{aligned}
\beta_c \mathrm{Re} t_j(L) &= A_1 I_j^2 + A_3 I_j^4 + A_5 I_j^6 + \ldots + \mathcal{O}(1/V) \quad , \\
\pm\beta_c \mathrm{Im} t_j(L) &= I_j + A_2 I_j^3 + A_4 I_j^5 + \ldots + \mathcal{O}(1/V) \quad , && (10)
\end{aligned}
$$

where $I_j = (2j-1)\pi/(V\Delta e)$ and $\mathcal{O}(1/V)$ represents terms which vanish in the infinite volume limit and where the coefficients A_n are easily calculable, the first few being [8]

$$
\begin{aligned}
A_1 &= \frac{\Delta c/k_B\beta_c^2}{2\Delta e} \quad , && (11)\\
A_2 &= -2A_1^2 + \frac{\Delta\gamma_3}{3!\Delta e} \quad , && (12)\\
A_3 &= -5A_1^3 + 5A_1\frac{\Delta\gamma_3}{3!\Delta e} - \frac{\Delta\gamma_4}{4!\Delta e} \quad , && (13)\\
A_4 &= 14A_1^4 - 21A_1^2\frac{\Delta\gamma_3}{3!\Delta e} + 3\left(\frac{\Delta\gamma_3}{3!\Delta e}\right)^2 + 6A_1\frac{\Delta\gamma_4}{4!\Delta e} - \frac{\Delta\gamma_5}{5!\Delta e} \quad , && (14)\\
A_5 &= 42A_1^5 - 84A_1^3\frac{\Delta\gamma_3}{3!\Delta e} + 28A_1\left(\frac{\Delta\gamma_3}{3!\Delta e}\right)^2 + 28A_1^2\frac{\Delta\gamma_4}{4!\Delta e} - 7\frac{\Delta\gamma_3}{3!\Delta e}\frac{\Delta\gamma_4}{4!\Delta e} - 7A_1\frac{\Delta\gamma_5}{5!\Delta e} + \frac{\Delta\gamma_6}{6!\Delta e} && (15)
\end{aligned}
$$

The Locus of Zeroes From (10) the real part of the zeroes (in the thermodynamic limit) can be expressed in terms of their imaginary parts as $\beta_c \mathrm{Re} t = \mathcal{L}(\beta_c \mathrm{Im} t)$ where

$$
\mathcal{L}(\theta) = A_1\theta^2 + (-2A_1A_2 + A_3)\theta^4 + (7A_1A_2^2 - 2A_1A_4 - 4A_2A_3 + A_5)\theta^6 + \ldots \quad . \qquad (16)
$$

The zeroes are thus seen to lie on a curve. In the complex u upper half–plane the equation of this curve is

$$
\gamma^{(+)}(\theta) = u_c e^{\mathcal{L}(\theta)+i\theta} \quad . \qquad (17)
$$

This defines the locus of zeroes in the infinite volume limit.

The Dual of the Locus of Zeroes Applying the duality transformation (1) to $\gamma^{(+)}(\theta)$ and expanding in powers of θ gives

$$\begin{aligned}
&\mathrm{Re}\mathcal{D}\left(\gamma^{(+)}(\theta)\right) = u_c\Big[1+\frac{\theta^2}{2q}\left(2\sqrt{q}-2A_1q-q\right)-\frac{\theta^4}{24q^2}\Big(24\sqrt{q}-36q+14q^{\frac{3}{2}}-q^2\\
&\quad -72A_1q+72A_1q^{\frac{3}{2}}-12A_1q^2+24A_1^2q^{\frac{3}{2}}-12A_1^2q^2-48A_1A_2q^2+24A_3q^2\Big)+\ldots\Big] \quad , \qquad (18)\\
&\mathrm{Im}\mathcal{D}\left(\gamma^{(+)}(\theta)\right) = -u_c\Big[\theta-\frac{\theta^3}{6q}\Big(6-6q^{\frac{1}{2}}+q-12A_1q^{\frac{1}{2}}+6A_1q\Big)-\frac{\theta^5}{120q^2}\Big(-120+240q^{\frac{1}{2}}\\
&\quad -150q+30q^{\frac{3}{2}}-q^2+480A_1q^{\frac{1}{2}}-720A_1q+280A_1q^{\frac{3}{2}}-20A_1q^2-360A_1^2q+360A_1^2q^{\frac{3}{2}}\\
&\quad -60A_1^2q^2+480A_1A_2q^{\frac{3}{2}}-240A_1A_2q^2-240A_3q^{\frac{3}{2}}+120A_3q^2\Big)+\ldots\Big] \quad . \qquad (19)
\end{aligned}$$

The Locus of the Duals of the Zeroes Alternatively, applying the duality transformation (1) directly to the j^{th} zero in the finite–size system and expanding again, gives

$$\begin{aligned}
\beta_c\mathrm{Re}t_j^D(L) &= A_1^DI_j^2+A_3^DI_j^4+A_5^DI_j^6+\ldots+\mathcal{O}(1/V) \quad ,\\
\mp\beta_c\mathrm{Im}t_j^D(L) &= I_j+A_2^DI_j^3+A_4^DI_j^5+\ldots+\mathcal{O}(1/V) \quad , \qquad (20)
\end{aligned}$$

where the first few coefficients A_n^D are

$$\begin{aligned}
A_1^D &= q^{-\frac{1}{2}}-A_1 \quad , & (21)\\
A_2^D &= -q^{-1}+2q^{-\frac{1}{2}}A_1+A_2 \quad , & (22)\\
A_3^D &= -\frac{1}{12}q^{-\frac{1}{2}}-q^{-\frac{3}{2}}+3q^{-1}A_1-q^{-\frac{1}{2}}A_1^2+2q^{-\frac{1}{2}}A_2-A_3 & (23)\\
A_4^D &= \frac{1}{4}q^{-1}+q^{-2}-\frac{1}{3}q^{-\frac{1}{2}}A_1-4q^{-\frac{3}{2}}A_1+3q^{-1}A_1^2-3q^{-1}A_2+2q^{-\frac{1}{2}}A_1A_2+2q^{-\frac{1}{2}}A_3+A_4 & (24)\\
A_5^D &= \frac{1}{360}q^{-\frac{1}{2}}+\frac{1}{2}q^{-\frac{3}{2}}+q^{-\frac{5}{2}}-\frac{5}{4}q^{-1}A_1-5q^{-2}A_1+6q^{-\frac{3}{2}}A_1^2+\frac{1}{2}q^{-\frac{1}{2}}A_1^2-q^{-1}A_1^3-\frac{1}{3}q^{-\frac{1}{2}}A_2\\
&\quad -4q^{-\frac{3}{2}}A_2+6q^{-1}A_1A_2+q^{-\frac{1}{2}}A_2^2+3q^{-1}A_3-2q^{-\frac{1}{2}}A_1A_3+2q^{-\frac{1}{2}}A_4-A_5 \quad . & (25)
\end{aligned}$$

From (20) the real part of the dual zeroes can be expressed in terms of their imaginary parts in the thermodynamic limit as $\beta_c\mathrm{Re}t_j^D=\mathcal{L}^D(\mathrm{Im}t_j^D)$ where

$$\mathcal{L}^D(\theta)=A_1^D\theta^2+(-2A_1^DA_2^D+A_3^D)\theta^4+(7A_1^D{A_2^D}^2-2A_1^DA_4^D-4A_2^DA_3^D-A_5^D)\theta^6+\ldots \quad . \qquad (26)$$

Therefore, the locus of the dual of the upper half–plane zeroes in the thermodynamic limit is given by

$$\gamma^{(+)^D}(\theta)=u_ce^{\mathcal{L}^D(\theta)-i\theta} \quad . \qquad (27)$$

The expansion of this locus of duals is

$$\begin{aligned}
&\mathrm{Re}\gamma^{(+)^D}(\theta)\\
&\quad = u_c\Big[1+\frac{\theta^2}{2!}\Big(-1+2A_1^D\Big)+\frac{\theta^4}{4!}\Big(1-12A_1^D-48A_1^DA_2^D+24A_3^D+12{A_1^D}^2\Big)+\ldots\Big] \quad , \qquad (28)\\
&\mathrm{Im}\gamma^{(+)^D}(\theta)\\
&\quad = -u_c\Big[\theta+\frac{\theta^3}{3!}\Big(-1+6A_1^D\Big)+\frac{\theta^5}{5!}\Big(1-20A_1^D+60{A_1^D}^2-240A_1^DA_2^D+120A_3^D\Big)+\ldots\Big] \quad . \qquad (29)
\end{aligned}$$

Identification of the Dual of the Locus with the Locus of the duals In deriving the dual of the locus of zeroes (18) and (19), the duality transformation was applied *after* the thermodynamic limit of the positions of the zeroes (i.e., their locus) was taken. In (28) and (29) the duality transformation was applied to the zeroes *before* taking the thermodynamic limit. Even in the case where the finite–L system does not have duality–preserving boundary conditions, taking the thermodynamic limit restores self–duality. The dual of the (thermodynamic limit) locus of zeroes and the (thermodynamic limit) locus of the duals of the zeroes must be identical. We demand, therefore, that

$$\mathcal{D}\left(\gamma^{(+)}\right) \equiv \gamma^{(+)^D} \tag{30}$$

order by order in the expansion in θ. Up to $\mathcal{O}(\theta^2)$ this is trivial. To $\mathcal{O}(\theta^3)$ and (separately at) $\mathcal{O}(\theta^4)$ they are identical if $A_1 = 1/(2\sqrt{q})$. From (11), this is the result (8). The identity (30) at $\mathcal{O}(\theta^5)$ and (separately at) $\mathcal{O}(\theta^6)$ gives $A_3 = A_2/\sqrt{q} - q^{-3/2}(q-3)/24$, which from (12) and (13) means that

$$\Delta\gamma_4 = \frac{6}{\sqrt{q}}\Delta\gamma_3 + \frac{q-6}{q^{3/2}}\Delta e \quad . \tag{31}$$

Higher order results are obtainable using a computer algebra system such as Maple. To orders $\mathcal{O}(\theta^7)$ and $\mathcal{O}(\theta^8)$ and (separately) to orders $\mathcal{O}(\theta^9)$ and $\mathcal{O}(\theta^{10})$ one finds

$$\frac{\Delta\gamma_6}{6!\Delta e} = \frac{5}{2q^{1/2}}\frac{\Delta\gamma_5}{5!\Delta e} + \frac{q-20}{8q^{3/2}}\frac{\Delta\gamma_3}{3!\Delta e} + \frac{1}{6!q^{1/2}} - \frac{1}{8q^{3/2}} + \frac{1}{2q^{5/2}} \quad , \tag{32}$$

and

$$\begin{aligned}\frac{\Delta\gamma_8}{8!\Delta e} &= \frac{7}{2q^{1/2}}\frac{\Delta\gamma_7}{7!\Delta e} + \left(\frac{5}{24q^1/2} - \frac{35}{4q^{3/2}}\right)\frac{\Delta\gamma_5}{5!\Delta e} + \left(\frac{3}{6!q^{1/2}} - \frac{15}{16q^{3/2}} + \frac{21}{2q^{5/2}}\right)\frac{\Delta\gamma_3}{3!\Delta e} \\ &\quad + \frac{1}{8!q^{1/2}} - \frac{23}{960q^{3/2}} + \frac{5}{8q^{5/2}} - \frac{17}{8q^{7/2}} \quad ,\end{aligned} \tag{33}$$

respectively. These results and further results for the higher cumulants at criticality are also obtainable directly from the duality transformation (31) (see [7, 19, 20]).

The Full Ferromagnetic Locus of Zeroes Putting the above equations into (21) – (25) (and their higher order equivalents) yields $A_j^D = A_j$ (this has been verified up to $j = 8$). Therefore (at least up to θ^{10}) the dual of the locus of zeroes is the complex conjugate of the original locus of zeroes. We now assume that this is the case for all θ. Then, the full ferromagnetic locus of zeroes (that part of the full locus which intersects the real temperature axis at the physical ferromagnetic critical point) is found by identifying [13, 14]

$$\mathcal{D}\left(\gamma^{(+)}(\theta)\right) = \gamma^{(+)^*}(\theta) \quad , \tag{34}$$

where $\gamma^{(+)^*}$ represents the complex conjugate of $\gamma^{(+)}$. The full ferromagnetic locus is then [13, 14]

$$\gamma(\theta) = \frac{1}{q-1}\left(-1 + \sqrt{q}e^{i\theta}\right) \quad . \tag{35}$$

This circular locus is analogous to the circle theorem of Lee and Yang [9]. In the field driven case, where one is interested in Lee–Yang zeroes in the complex $z = \exp h$ plane (h is an external magnetic field), formulae analogous to (10) – (15) apply where e, c, etc. are replaced by the corresponding derivative of the free energy with respect to h (the magnetization m, the susceptibility χ, etc.). There, the partition function is unchanged under $h \to -h$ and consequently $\Delta\gamma_l = 0$ for even l. Therfore $\mathcal{L}(\theta) = 0$ and the locus of zeroes is $z = \exp i\theta$. This is Lee's proof of the Lee–Yang theorem [8]. One observes that considering $h \to -h$ as a self–duality map and identifying it with complex conjugation yields this locus.

The Singular Parts of the Thermodynamic Functions in the Thermodynamic Limit From (9), the density of zeroes in the temperature driven case is [23]

$$g(\theta) = \lim_{V\to\infty} \frac{1}{V}\frac{dj}{d\theta} = \frac{\Delta e}{2\pi}\left(1 + \frac{1}{(q-1)\gamma(\theta)}\right)\left\{1 + \frac{\Delta c/k_B\beta_c^2}{\Delta e}\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right.$$
$$\left. + \frac{1}{2!}\frac{\Delta\gamma_3}{\Delta e}\left(\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right)^2 + \frac{1}{3!}\frac{\Delta\gamma_4}{\Delta e}\left(\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right)^3 + \dots\right\} \quad . \tag{36}$$

The internal energy is (from (3) or [21, 22])

$$e = \text{cnst.} + u\int_0^{2\pi} \frac{g(\theta)}{u-\gamma(\theta)}d\theta \quad . \tag{37}$$

Therefore, from (35), (36) and (37), the internal energy is

$$e(\beta < \beta_c) \;=\; e_0 \tag{38}$$
$$e(\beta > \beta_c) \;=\; e_0 - \Delta e\left\{1 + \frac{\Delta c/k_B\beta_c^2}{\Delta e}(\beta_c - \beta) + \frac{1}{2!}\frac{\Delta\gamma_3}{\Delta e}(\beta_c-\beta)^2 + \frac{1}{3!}\frac{\Delta\gamma_4}{\Delta e}(\beta_c-\beta)^3 + \dots\right\} \tag{39}$$

with e_0 a constant (one expects that when separate Fisher loci which don't cross the positive real temperature axis are accounted for, e_0 becomes temperature dependent). At β_c the internal energy discontinuity $e(\beta = \beta_c^-) - e(\beta = \beta_c^+) = \Delta e$ is recovered. Appropriate differentiation recovers the discontinuities in specific heat and higher cumulants. Thus the full locus (35) and the density (36) are sufficient to give the *singular* parts of the thermodynamic functions in the infinite volume limit.

Conclusions In summary, we have applied the duality transformation (1), under which the $d = 2$ q–state Potts model is invariant, to the Fisher zeroes recently found by Lee [8] for systems with a first order phase transition. The requirement that the dual of the locus of zeroes be identical to the locus of the duals of the zeroes in the thermodynamic limit (i) recovers the ratio of specific heat to internal energy discontinuity at criticality and the relations between the discontinuities of higher cumulants and (ii) identifies duality with complex conjugation.

Conjecturing that all zeroes governing ferromagnetic critical behaviour satisfy (ii) gives that this full locus is the circle (35) in the complex u–plane. The equation (35) was first conjectured by Martin and Maillard and Rammal [13, 14] on the basis of analogous Ising results [10]. The same conjecture, based on numerical results for small lattices was made in [15] and proven for infinite q in [16] (see also [17]).

The locus (35), together with the density of zeroes is sufficient to recover the *singular* parts of all thermodynamic functions in the thermodynamic limit. It is to be expected that the *regular* parts come from separate loci of zeroes which don't cross the positive real temperature axis.

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