Completeness of some axioms of Lukasiewicz's: an exercise in problem-solving

Colm Ó Dúnlaing*
Mathematics
Trinity College, Dublin 2, Ireland

June 30, 1997

Abstract

Among many systems of axioms for Propositional Logic, one due to Lukasiewicz is rather hard to prove complete, although it has appeared as a textbook exercise. This note retraces a roundabout route leading to a proof of its completeness. In view of this, the note is much longer than would be needed just to present the proof.

1 Propositional Logic and Lukasiewicz's system

(1.1) PROPOSITIONAL LOGIC (ZERO-ORDER LOGIC) is concerned with reasoning at is simplest. This amounts to a study of *truth-functions*, namely, functions $f\colon \{0,1\}^n \to \{0,1\}$, where 0 and 1 are regarded as *truth-values*, with 0 meaning 'false' and 1 'true.'

We begin, therefore, with a (countably infinite) set $X_1, X_2, \ldots$ of *boolean variables* (or 'atomic formulae') which can take the values 0 and 1. More complex formulae can be constructed from them using the *logical connectives* $\neg, \&, \vee, \Rightarrow, \iff$, etcetera, for 'not, and, or, implies, if and only if,' etcetera. All of these connectives, even '$\Rightarrow$,' serve only to construct new formulae out of old. The truth-function for implication assigns the value 'true' to $A \Rightarrow B$ in all cases except where A is true and B false. This has the effect that if A and $A \Rightarrow B$ are both true, then B is also true.

The following table gives the truth-functions for the Propositional connectives.

A	B	$(\neg B)$	$(A \& B)$	$(A \vee B)$	$(A \Rightarrow B)$	$(A \iff B)$
1	1	0	1	1	1	1
1	0	1	0	1	0	0
0	1		0	1	1	0
0	0		0	0	1	1

*e-mail odunlain@maths.tcd.ie

(1.2) Boolean formulae are constructed using the variables, connectives, and parentheses. So if A, B are boolean formulae then so are $(\neg A)$, $(A \& B)$, $(A \vee B)$, $(A \Rightarrow B)$, and $(A \iff B)$. The parentheses remove ambiguity. We shall omit them where the context makes the meaning clear, with the general rule that negation binds more tightly than the other connectives and the top-level parentheses are always omitted.

There is a lot of redundancy among the propositional connectives. For example, $\neg A \vee B$ and $A \Rightarrow B$ are equivalent — that is, they have the same truth-tables. In fact, from now on we shall use only the connectives '$\neg$' and '$\Rightarrow$.'

(1.3) Of particular interest are the formulae which are true under all interpretations, called *tautologies*. For example, the following formulae are tautologies, no matter what formulae A, B, and C represent. This is easily verified by assigning, in turn, all eight combinations of truth-values to A, B, and C, and computing the values of the compound formulae, which are true in all cases.

(M1) $\quad A \Rightarrow (B \Rightarrow A)$

(M2) $\quad (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

(M3) $\quad (\neg A \Rightarrow \neg B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow A)$

(1.4) A *proof system* for Propositional Logic consists of a set of formulae designated as the *axioms* of the sytem plus a set of rules called *rules of inference.*

(1.5) For example, the following proof system, which will be called 'System M' (M for Mendelson), is given in [1]. Its axioms are the above tautologies (M1–M3), and its only inference rule is Modus Ponens,

$$\text{From } A \text{ and } A \Rightarrow B \text{ infer } B.$$

Or, using the notation $\vdash_{\mathrm{M}}$ explained in 1.6,

$$A,\ A \Rightarrow B \vdash_{\mathrm{M}} B$$

It is to be emphasised that the three axiom schemes M1–M3 of system M contain an infinite number of axioms, since A, B and C can be arbitrary formulae built from the boolean variables using $\neg, \Rightarrow$, and parentheses.

(1.6) Let S be a proof-system and Γ a set of formulae, possibly empty. A *deduction from* Γ *within* S is an annotated sequence $B_1, \dots B_n$ of formulae, together with justifications for each formula B_i, namely that either (i) $B_i \in \Gamma$, (ii) B_i is an axiom of S, or (iii) B_i is deduced from earlier formulae B_j using some rule of inference of S.

We write $\Gamma \vdash_S B$ if there exists a deduction from Γ containing the formula B, and say B is *deducible from* Γ (in S). If Γ is empty we say that Γ is provable in S or simply a *theorem.*

(1.7) The intent of a proof system S for Propositional logic is that its theorems should be exactly those formulae which are true under all interpretations, i.e., the tautologies. When all theorems of S are tautologies, S is called *sound* (or consistent), and when all tautologies are theorems of S, S is called *complete.*

In the case of System M above, all the axioms are tautologies, and MP (modus ponens) applied to tautologies yields tautologies, so System M is sound. It can be shown with more difficulty that M is complete:

(1.8) Theorem *(See [1].) System M is consistent and complete.* ∎

(1.9) The system under consideration in this note is due to Lukasiewicz, and will be denoted 'System Lk.' Like System M, its only rule of inference is MP. It has three schemes of axioms,

(Lk1) $(\neg A \Rightarrow A) \Rightarrow A$,

(Lk2) $A \Rightarrow (\neg A \Rightarrow B)$, and

(Lk3) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$.

(1.10) The axioms are easily seen to be tautologies, so the system is consistent. It is not easy to show completeness. In fact, merely by combining the axioms in various ways only one result seems to emerge:

(1.11) Lemma $\vdash_{\mathrm{Lk}} A \Rightarrow A$.

Proof. We prove this (for any formula A) by exhibiting a proof within the system Lk, as follows:
(i) $A \Rightarrow (\neg A \Rightarrow A)$ (Lk2).
(ii) $(A \Rightarrow (\neg A \Rightarrow A)) \Rightarrow (((\neg A \Rightarrow A) \Rightarrow A) \Rightarrow (A \Rightarrow A))$ (Lk3).
(iii) $((\neg A \Rightarrow A) \Rightarrow A) \Rightarrow (A \Rightarrow A)$ (i, ii, MP);
(iv) $(\neg A \Rightarrow A) \Rightarrow A$ (Lk1).
(v) $A \Rightarrow A$ (iv, iii, MP). ∎

2 The Deduction Principle and 'System I'

(2.1) What we may call the *Deduction Principle* for a given proof system S is as follows: given formulae A and B and a list (or set) Γ of formulae, if $\Gamma, A \vdash_S B$ then $\Gamma \vdash_S A \Rightarrow B$.[1] It holds for the above System M and is accordingly called the *Deduction Theorem* for that system. In order to prove System Lk complete, we begin by *assuming* the Deduction Principle. As we shall see, this greatly simplfies the proof of completeness.

Let us write $\Gamma \vdash_{\mathrm{LkD}} B$ to mean that B can be deduced from Γ using the system Lk plus the Deduction Principle.

(2.2) Theorem *System Lk + (deduction principle) is complete.*

[1] Γ, A means simply $\Gamma \cup \{A\}$.

Proof. Since System M is complete, we need only show that the axioms of System M can be proved in System Lk using the Deduction Principle.

For Axiom M1 (i.e., axioms of type M1), $A \Rightarrow (B \Rightarrow A)$,
(i) $A, B \vdash_{\mathrm{Lk}} A$ (obviously).
(ii) $A \vdash_{\mathrm{Lk}} (B \Rightarrow A)$ (i, D.P.)
(iii) $\vdash_{\mathrm{LkD}} A \Rightarrow (B \Rightarrow A)$ (ii, D.P.)

Axiom M2, $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$: We take $A, (A \Rightarrow B), (A \Rightarrow (B \Rightarrow C))$ as hypotheses and apply the Deduction Principle repeatedly.
(i) $A \Rightarrow (B \Rightarrow C)$ (Hypothesis)
(ii) $A \Rightarrow B$ (Hypothesis)
(iii) A (Hypothesis).
(iv) $B \Rightarrow C$ (iii, i, MP)
(v) B (iii, ii, MP)
(vi) C (v, iv, MP)
So: (i,ii,iii) $\vdash_{\mathrm{Lk}} C$; so, by the D.P., (i,ii) $\vdash_{\mathrm{LkD}} A \Rightarrow C$; Repeating the argument twice we get

$$\vdash_{\mathrm{LkD}} (\mathrm{i}) \Rightarrow ((\mathrm{ii}) \Rightarrow (A \Rightarrow C)),$$

as desired.

Axiom M3: $(\neg Y \Rightarrow \neg X) \Rightarrow ((\neg Y \Rightarrow X) \Rightarrow Y)$.
First we claim $\neg Y \Rightarrow \neg X, \neg Y \Rightarrow X, \neg Y \vdash_{\mathrm{Lk}} Y$.
(i) $\neg Y \Rightarrow \neg X$ (Given)
(ii) $\neg Y \Rightarrow X$ (Given)
(iii) $\neg Y$ (Given)
(iv) $\neg X$ (iii, i, MP)
(v) X (iii, ii, MP)
(vi) $X \Rightarrow (\neg X \Rightarrow Y)$ (Lk2)
(vii) $\neg X \Rightarrow Y$ (v, vi, MP)
(viii) Y (iv, vii, MP) — as claimed.
Hence using the D.P.,
(ix) $\neg Y \Rightarrow \neg X, \neg Y \Rightarrow X \vdash_{\mathrm{LkD}} \neg Y \Rightarrow Y$
Continuing,
(x) $(\neg Y \Rightarrow Y) \Rightarrow Y$ (Lk1)
(ix) Y (ix, x, MP)
So $\neg Y \Rightarrow \neg X, \neg Y \Rightarrow X \vdash_{\mathrm{LkD}} Y$, and

$$\vdash_{\mathrm{LkD}} (\neg Y \Rightarrow \neg X) \Rightarrow ((\neg Y \Rightarrow X) \Rightarrow Y),$$

as asserted. This concludes the proof. ∎

(2.3) Augmenting System Lk with the Deduction Principle, therefore, results in a complete system. So to prove System Lk complete it would suffice to prove the Deduction Theorem for the system. This is too difficult to attempt directly: we begin by adding two axiom schemes which make the Deduction Theorem easy to prove.

Thus 'System I' denotes Lk augmented with the following two schemes of axioms, which are easily shown to be tautologies:

(I4) $\quad A \Rightarrow (B \Rightarrow A)$ (identical to M1)

(I5) $\quad (A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$.

(2.4) Lemma *System I is consistent and complete.*

Proof. Consistency is as for the previous systems. To show completeness, it is enough to prove the Deduction Theorem for the system. We need to prove, therefore, that if Γ is a set of formulae and A a formula and

$$\Gamma, A \vdash_{\mathrm{I}} B$$

then

$$\Gamma \vdash_{\mathrm{I}} A \Rightarrow B.$$

This is proved by induction on the length —n, say— of the shortest deduction of B from Γ, A (i.e., $\Gamma \cup \{A\}$). If $n = 1$ then B is justified in one of the following three ways: (a) it is identical to A; (b) it belongs to Γ; or (c) it is an axiom of System Lk. In case (a), $\vdash_{\mathrm{I}} A \Rightarrow B$ by Lemma 1.11. In cases (b) and (c), $\Gamma \vdash_{\mathrm{I}} B$. So:
(i) B (assumed proven).
(ii) $B \Rightarrow (A \Rightarrow B)$ (I4).
(iii) $A \Rightarrow B$ (i, ii, MP).

This disposes of the base case. For the induction, assume the result holds for all deductions of length $< n$, and a shortest deduction of length n is given for B. B must be the last formula in the deduction, and it must be deduced by Modus Ponens from two formulae C and $C \Rightarrow B$ occurring earlier in the proof. The induction hypothesis applies to these formulae, so we can assume the following:
(i) $A \Rightarrow C$ (deducible from Γ)
(ii) $A \Rightarrow (C \Rightarrow B)$ (deducible from Γ)

Continuing,
(iii) $(A \Rightarrow C) \Rightarrow ((C \Rightarrow B) \Rightarrow (A \Rightarrow B))$ (Lk3)
(iv) $(C \Rightarrow B) \Rightarrow (A \Rightarrow B)$ (i, iii, MP)
(v) $(A \Rightarrow (C \Rightarrow B)) \Rightarrow (((C \Rightarrow B) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow (A \Rightarrow B)))$ (Lk3)
(vi) $((C \Rightarrow B) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow (A \Rightarrow B))$ (ii, v, MP)
(vii) $A \Rightarrow (A \Rightarrow B)$ (iv, vi, MP)
(viii) $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (I5)
(ix) $A \Rightarrow B$ (vii, viii, MP, as required).

This concludes the proof. ∎

3 Derived Rules and 'System II'

(3.1) It is frequently useful to extend a proof system by adding new rules of inference which are redundant but lead to shorter or clearer proofs. For example, we could express the transitiity of implication by a derived rule, which we call 'transitivity:'

$$A \Rightarrow B,\ B \Rightarrow C \vdash_{\mathrm{Lk}} A \Rightarrow C.$$

It is obvious that the inference replaces a combination of Axiom Lk3 with two applications of Modus Ponens, so it is redundant, but convenient.

(3.2) Another useful rule, which abbreviating a combination of Axiom Lk3 with MP, is

$$A \Rightarrow B \vdash_{\mathrm{Lk}} ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

which we could call 'backward propagation' (the idea being that the implication $B \Rightarrow C$ propagates backward to $A \Rightarrow C$).

(3.3) A variant of this rule would be

$$B \Rightarrow C \vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C),$$

which we can call 'forward propagation.' It is not obvious how to justify the forward propagation rule within System Lk.

(3.4) However, attempts to prove the Deduction Theorem for Lk led to some experimentation with these rules, and suggested that together they would enable a proof of the Deduction Theorem. Supplying a symmetric variant of rule Lk3 would enable forward propagation to be used freely. We chose to make a stronger assumption, by adding a 'symmetrising axiom' for implication. This led to the following extension, System II, of System Lk, whose first three axiom schemes are as for System Lk, and which has the following fourth axiom scheme (of tautologies)

(II4) $\quad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)).$

Let us call the new system 'System II.' It is an extension of System Lk; it is consistent for the usual reasons. We shall show it is complete by proving all formulae of type I4 and I5; so every theorem of System I is a theorem of system II.

(3.5) System II admits the following derived rule, which we again call 'forward propagation':

$$B \Rightarrow C \vdash_{\mathrm{II}} (A_1 \Rightarrow (A_2 \Rightarrow \ldots (A_n \Rightarrow B)\ldots)) \quad \Rightarrow \quad (A_1 \Rightarrow (A_2 \Rightarrow \ldots (A_n \Rightarrow C)\ldots)).$$

This is proved by induction on n. For $n = 1$,

(i) $B \Rightarrow C$ (given)

(ii) $(A_1 \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A_1 \Rightarrow C))$ (Lk3)

(iii) $(X \Rightarrow (Y \Rightarrow Z)) \Rightarrow (Y \Rightarrow (X \Rightarrow Z))$, where X is $A_1 \Rightarrow B$, Y is $B \Rightarrow C$, and Z is $A_1 \Rightarrow C$ (II4).

(iv) $(B \Rightarrow C) \Rightarrow ((A_1 \Rightarrow B) \Rightarrow (A_1 \Rightarrow C))$ (ii, iii, MP).

(v) $(A_1 \Rightarrow B) \Rightarrow (A_1 \Rightarrow C)$ (i, iv, MP, as desired).

The induction is quite trivial:

$$B \Rightarrow C \quad \vdash_{\text{II}} \quad (A_2 \ldots (A_n \Rightarrow B) \ldots) \Rightarrow (A_2 \ldots (A_n \Rightarrow B) \ldots)),$$

(inductive hypothesis), and

$$\begin{gathered}(A_2 \ldots (A_n \Rightarrow B) \ldots) \Rightarrow (A_2 \ldots (A_n \Rightarrow B) \ldots)) \vdash_{\text{II}} \\ (A_1 \Rightarrow (A_2 \ldots (A_n \Rightarrow B) \ldots) \Rightarrow (A_1 \Rightarrow (A_2 \ldots (A_n \Rightarrow B) \ldots))),\end{gathered}$$

so the induction follows.

(3.6) Another derived rule we call 'permutation:' given a list $A_1 \ldots A_n$ of formulae and a list $X_1 \ldots X_n$ being a permutation of hte first list, and a formula B,

$$A_1 \Rightarrow (A_2 \ldots (A_n \Rightarrow B) \ldots) \quad \vdash_{\text{II}} \quad X_1 \Rightarrow (X_2 \ldots (X_n \Rightarrow B) \ldots).$$

To justify this one first proves it when the second list is derived from the first by an adjacent interchange:

$$X_1, \ldots X_n \equiv A_1, \ldots A_{i-1}, A_{i+1}, A_i, A_{i+2} \ldots A_n.$$

(i) $(A_i \Rightarrow (A_{i+1} \Rightarrow X)) \Rightarrow (A_{i+1} \Rightarrow (A_i \Rightarrow X))$, where X is $(A_{i+1} \Rightarrow (\ldots (A_n \Rightarrow B))$ (II4)

(ii) $(A_1 \Rightarrow (\ldots (A_i \Rightarrow (A_{i+1} \Rightarrow X)) \ldots)) \Rightarrow (A_1 \Rightarrow (\ldots (A_{i+1} \Rightarrow (A_i \Rightarrow X)) \ldots))$ (i, forward propagation).

(iii) $A_1 \Rightarrow (\ldots (A_i \Rightarrow (A_{i+1} \Rightarrow X)))$ (Given)

(iv) $A_1 \Rightarrow (\ldots (A_{i+1} \Rightarrow (A_i \Rightarrow X)))$ (ii, iii, MP)

The general rule follows easily by induction, since any permutation can be realised as a sequence of adjacent interchanges.

We can prove completeness of System II with a sequence of lemmas.

(3.7) Lemma $\vdash_{\text{II}} \neg A \Rightarrow (A \Rightarrow B)$.

Proof.

(i) $A \Rightarrow (\neg A \Rightarrow B)$ (Lk2)

(ii) $\neg A \Rightarrow (A \Rightarrow B)$ (i, permutation).

(3.8) Lemma $\vdash_{\text{II}} \neg\neg A \Rightarrow A$

Proof.

(i) $(\neg A \Rightarrow A) \Rightarrow A$ (Lk1)

(ii) $(\neg\neg A \Rightarrow (\neg A \Rightarrow A)) \Rightarrow (\neg\neg A \Rightarrow A)$ (i, forward propagation)

(iii) $\neg\neg A \Rightarrow (\neg A \Rightarrow A)$ (3.7)

(iv) $\neg\neg A \Rightarrow A$ (ii, iii, MP). ∎

(3.9) Lemma $\vdash {\rm II}(A \Rightarrow \neg A) \Rightarrow \neg A)$

Proof.
(i) $\neg\neg A \Rightarrow A$ (3.8)
(ii) $(A \Rightarrow \neg A) \Rightarrow (\neg\neg A \Rightarrow \neg A)$ (i, backward propagation)
(iii) $(\neg\neg A \Rightarrow \neg A) \Rightarrow \neg A$ (Lk1)
(iv) $(A \Rightarrow \neg A) \Rightarrow \neg A$ (ii, iii, transitivity). $\blacksquare$

(3.10) Lemma $\vdash {\rm II}A \Rightarrow \neg\neg A$

Proof.
(i) $A \Rightarrow (\neg A \Rightarrow \neg\neg A)$ (Lk2)
(ii) $(\neg A \Rightarrow \neg\neg A) \Rightarrow \neg\neg A$ (3.9)
(iii) $A \Rightarrow \neg\neg A$ (i,ii, transitivity). $\blacksquare$

(3.11) Lemma $\vdash {\rm II}A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

Proof.
(i) $B \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))$ (Lk2)
(ii) $(A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$ (i, forward propagation)
(iii) $A \Rightarrow (\neg B \Rightarrow ((A \Rightarrow B) \Rightarrow \neg(A \Rightarrow B)))$ (ii. permutation)
(iv) $(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow B)) \Rightarrow \neg(A \Rightarrow B)$ (3.9)
(v) $(A \Rightarrow (\neg B \Rightarrow ((A \Rightarrow B) \Rightarrow \neg(A \Rightarrow B)))) \Rightarrow (A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$ (iv, forward propagation)
(vi) $A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))$ (iii, v, MP) $\blacksquare$

(3.12) Lemma $\vdash {\rm II}(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

Proof.
(i) $\neg(A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$ (3.7)
(ii) $(A \Rightarrow (\neg(A \Rightarrow B))) \Rightarrow (A \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A))$ (i, forward propagation)
(iii) $A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))$ (3.11)
(iv) $\neg B \Rightarrow (A \Rightarrow \neg(A \Rightarrow B))$ (iii, permutation)
(v) $\neg B \Rightarrow (A \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A))$ (iv, ii, transitivity)
(vi) $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow (A \Rightarrow \neg A))$ (v, permutation)
(vii) $(A \Rightarrow \neg A) \Rightarrow \neg A$ (3.9)
(viii) $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow (A \Rightarrow \neg A))) \Rightarrow ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$ (vii, forward propagation)
(ix) $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ (vi, viii, MP). $\blacksquare$

(3.13) Lemma $\vdash {\rm II}(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

Proof.
(i) $(\neg B \Rightarrow \neg A) \Rightarrow (\neg\neg A \Rightarrow \neg\neg B)$ (3.12)
(ii) $\neg\neg B \Rightarrow B$ (3.8)
(iii) $(\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg\neg A \Rightarrow B)$ (ii, forward propagation)
(iv) $A \Rightarrow \neg\neg A$ (3.10)
(v) $(\neg\neg A \Rightarrow B) \Rightarrow (A \Rightarrow B)$ (iv, backward propagation)
(vi) $(\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (A \Rightarrow B)$ (iii, v, transitivity)
(vii) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (i, vi, transitivity). ∎

(3.14) Lemma $\vdash_{\text{II}}$ I4

Proof.
(i) $A \Rightarrow (\neg A \Rightarrow \neg B)$ (Lk2)
(ii) $(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$ (3.13)
(iii) $A \Rightarrow (B \Rightarrow A)$ (i, ii, transitivity). ∎

(3.15) Lemma $\vdash_{\text{II}}$ I5

Proof.
(i) $(A \Rightarrow \neg A) \Rightarrow \neg A$ (3.9)
(ii) $(\neg B \Rightarrow (A \Rightarrow \neg A)) \Rightarrow (\neg B \Rightarrow \neg A)$ (i, forward propagation)
(iii) $(A \Rightarrow (\neg B \Rightarrow \neg A)) \Rightarrow (\neg B \Rightarrow (A \Rightarrow \neg A))$ (II4)
(iv) $(A \Rightarrow (\neg B \Rightarrow \neg A)) \Rightarrow (\neg B \Rightarrow \neg A)$ (iii, ii, transitivity)
(v) $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ (3.12)
(vi) $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow (\neg B \Rightarrow \neg A))$ (v, forward propagation)
(vii) $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (\neg B \Rightarrow \neg A)$ (vi, iv, transitivity)
(viii) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (3.13)
(ix) $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (vii, viii, transitivity). ∎

We have just shown that all instances of axioms I4 and I5 are theorems of System II. Since System I is complete, it follows that

(3.16) Corollary *System II is complete.* ∎

4 'System III'

(4.1) We next study a system related to System II in which the 'symmetrising axiom' II4 is discarded but 'symmetric variants' of the first three axioms are added. Hence System III will be System Lk augmented with the following three schemes of tautologies:

(III4) $(A \Rightarrow \neg A) \Rightarrow \neg A$

(III5) $\neg A \Rightarrow (A \Rightarrow B)$

(III6) $(B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

Tautologies of type III4 and III5 were proved explicitly for System II (3.9 and 3.7), and III6 permits forward propagation to be used as an inference rule, so these axioms should prove useful. We prove completeness of the system the same way as with System II.

Note again that backward and forward propagation (3.2, 3.5) are permissible in this system.

(4.2) Lemma $\vdash_{\text{III}} A \Rightarrow \neg\neg A$

Proof.
(i) $A \Rightarrow (\neg A \Rightarrow \neg\neg A)$ (Lk2)
(ii) $(\neg A \Rightarrow \neg\neg A) \Rightarrow \neg\neg A$ (III4)
(iii) $A \Rightarrow \neg\neg A$ (i, ii, transitivity). ∎

(4.3) Lemma $\vdash_{\text{III}} \neg\neg A \Rightarrow A$

Proof.
(i) $\neg\neg A \Rightarrow (\neg A \Rightarrow A)$ (III5)
(ii) $(\neg A \Rightarrow A) \Rightarrow A$ (Lk1)
(iii) $\neg\neg A \Rightarrow A$ (i, ii, transitivity). ∎

(4.4) Lemma $\vdash_{\text{III}} (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

Proof.
(i) $(\neg B \Rightarrow B) \Rightarrow B$ (Lk1)
(ii) $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B)) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)$ (i, forward propagation)
(iii) $A \Rightarrow (\neg A \Rightarrow B)$ (Lk2)
(iv) $((\neg A \Rightarrow B) \Rightarrow B) \Rightarrow (A \Rightarrow B)$ (iii, backward propagation)
(v) $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (ii, iv, transitivity)
(vi) $(\neg B \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B))$ (Lk3)
(vii) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (vi, v, transitivity). ∎

(4.5) Lemma $\vdash_{\text{III}}$ I4 *(Proof same as in 3.14).* ∎

(4.6) Up to now we have approached completeness proofs by proving a series of tautologies in the respective systems. However, we have not generally used the tautologies on their own, but as part of new inference rules. This time we verify an inference rule rather than a tautology. The inference rule is used in [1] in proving completeness of System M. The phrase 'Morton's Fork' is a good mnemonic for this rule.[2]

[2] According to an encyclopedia, Cardinal John Norton (c1420–1500) was an advisor to King Henry VII of England. Cardinal Morton suggested that tax-collectors could extort huge taxes from those who spent freely, because they were obviously rich, and from those who lived frugally, because their thrift kept them rich. Hence 'Morton's Fork.'

(4.7) Lemma $X \Rightarrow Y,\ \neg X \Rightarrow Y \vdash_{\mathrm{III}} Y$.

Proof.
(i) $\neg X \Rightarrow Y$ (given)
(ii) $Y \Rightarrow \neg\neg Y$ (4.2)
(iii) $\neg X \Rightarrow \neg\neg Y$ (i, ii, transitivity)
(iv) $(\neg X \Rightarrow \neg\neg Y) \Rightarrow (\neg Y \Rightarrow X)$ (4.4)
(v) $\neg Y \Rightarrow X$ (iii, iv, MP)
(vi) $X \Rightarrow Y$ (given)
(vii) $\neg Y \Rightarrow Y$ (v, vi, transitivity)
(viii) $(\neg Y \Rightarrow Y) \Rightarrow Y$ (Lk1)
(ix) Y (vii, viii, MP). ∎

(4.8) Lemma $A \Rightarrow (A \Rightarrow B) \vdash_{\mathrm{III}} A \Rightarrow B$

Proof.
(i) $A \Rightarrow (A \Rightarrow B)$ (given)
(ii) $\neg A \Rightarrow (A \Rightarrow B)$ (III5)
(iii) $A \Rightarrow B$ (4.7). ∎

(4.9) Corollary *System* III *is complete.*

Proof. It is enough to prove the Deduction Theorem for System III. We have proved that axioms I4 can be derived within System III. We have not shown this for axiom scheme I5. However, in proving the Deduction Theorem for System I (Lemma 2.4), we used Axioms I5 only as part of the inference mechanism suggested by Lemma 4.8. Hence the same proof works for System III. ∎

5 'System IV'

(5.1) The completeness proofs given for systems II and III seem to have depended heavily on tautologies of the following type (or some varieant of it).

$$(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$$

Proof of these tautologies for System III used only the Lukasiewicz axioms and 'forward propagation' which depends on Axiom III6. As mentioned in 3.3, it is not obvious that this mechanism is valid in System Lk. However, if once again we weaken our demand for a tautology $X \Rightarrow Y$, and require only an inference rule $X \vdash Y$, there is some hope of avoiding axioms III6. And indeed this approach works. Notice that the following Lemma holds for System Lk; probably the first nontrivial theorem we can prove within the system.

(5.2) Lemma $\neg B \Rightarrow \neg A \vdash_{\mathrm{Lk}} A \Rightarrow B$

Proof.
(i) $\neg B \Rightarrow \neg A$ (given)
(ii) $(\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B)$ (i, backward propagation)
(iii) $A \Rightarrow (\neg A \Rightarrow B)$ (Lk2)
(iv) $A \Rightarrow (\neg B \Rightarrow B)$ (iii, ii, transitivity)
(v) $(\neg B \Rightarrow B) \Rightarrow B$ (Lk1)
(vi) $A \Rightarrow B$ (v, iv, transitivity). ∎

(5.3) Corollary $A \vdash_{\mathrm{Lk}} B \Rightarrow A$

Proof.
(i) A (given)
(ii) $A \Rightarrow (\neg A \Rightarrow \neg B)$ (Lk2)
(iii) $\neg A \Rightarrow \neg B$ (i, ii, MP)
(iv) $B \Rightarrow A$ (5.2). ∎

(5.4) If we could prove $A \Rightarrow \neg\neg A$ and $\neg\neg A \Rightarrow A$, we could deduce several useful corrolaries from Lemma 5.2, such as $\neg A \Rightarrow B \vdash \neg B \Rightarrow A$. A little investigation shows that $A \Rightarrow \neg\neg A$ is enough to permit arguments like those made for System III. Accordingly, we introduce our final extension of System Lk: System IV, which augments System Lk by just one scheme of tautologies:

(IV4) $\quad A \Rightarrow \neg\neg A.$

'Morton's Fork' is valid for System IV:

(5.5) Lemma $X \Rightarrow Y,\ \neg X \Rightarrow Y \vdash_{\mathrm{IV}} Y$

Proof.
(i) $\neg X \Rightarrow Y$ (given)
(ii) $Y \Rightarrow \neg\neg Y$ (IV4)
(iii) $\neg X \Rightarrow \neg\neg Y$ (i, ii, transitivity)
(iv) $\neg Y \Rightarrow X$ (5.2)
(v) $X \Rightarrow Y$ (given)
(vi) $\neg Y \Rightarrow Y$ (iv, v, transitivity)
(vii) $(\neg Y \Rightarrow Y) \Rightarrow Y$ (Lk1)
(viii) Y (vi, vii, MP). ∎

(5.6) Lemma $\vdash_{\mathrm{IV}} \neg A \Rightarrow (A \Rightarrow B)$

Proof.
(i) $\neg A \Rightarrow (\neg\neg A \Rightarrow B)$ (Lk2)
(ii) $A \Rightarrow \neg\neg A$ (IV4)
(iii) $(\neg\neg A \Rightarrow B) \Rightarrow (A \Rightarrow B)$ (ii, backward propagation)
(iv) $\neg A \Rightarrow (A \Rightarrow B)$ (i, iii, transitivity). ∎

(5.7) Lemma $A \Rightarrow (A \Rightarrow B) \vdash_{\rm IV} A \Rightarrow B$.

Proof.
(i) $A \Rightarrow (A \Rightarrow B)$ (given)
(ii) $\neg A \Rightarrow (A \Rightarrow B)$ (5.6)
(iii) $A \Rightarrow B$ (i, ii, 5.5). ∎

(5.8) Corollary *System IV is complete.*

Proof. This time we retrace a proof of the Deduction Theorem for System I, replacing instances of axioms I4 and I5 by the inference mechanisms in Lemmas 5.3 and 5.7. We obtain a proof of the Deduction Theorem in System IV, which is therefore complete. ∎

6 Completeness of Lukasiewicz's system Lk

(6.1) 'System IV' only adds one feeble-looking axiom scheme,

$$A \Rightarrow \neg\neg A,$$

to System Lk to guarantee completeness. So we are encouraged to seek a proof of $A \Rightarrow \neg\neg A$ within the system Lk.

(6.2) Reasonably enough, one could replace the above scheme by the scheme

$$\neg\neg A \Rightarrow A.$$

This is because $\neg\neg\neg A \Rightarrow \neg A \vdash_{\rm Lk} A \Rightarrow \neg\neg A$ by Lemma 5.2. However, it still seems difficult to prove either tautology.

(6.3) So we return to the possibility of proving Lemma 4.4 within System Lk, i.e., to prove

$$\vdash_{\rm Lk} (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B).$$

The difficulty is that the proof uses forward propagation, which requires Axiom scheme III6. However, it turns out that we can use something a little weaker. The following inference is merely an instance of MP:

$$X,\ X \Rightarrow (Y \Rightarrow Z)\ \vdash Y \Rightarrow Z.$$

In System II, combining the above infernce with 'permutation' yields the inference rule of Lemma 6.4 below. It turns out that we can prove it in System Lk, and this will be useful.

(6.4) Lemma $Y, X \Rightarrow (Y \Rightarrow Z) \vdash_{\rm Lk} X \Rightarrow Z$

Proof.
(i) Y (given)
(ii) $\neg Z \Rightarrow Y$ (i, 5.3)
(iii) $(Y \Rightarrow Z) \Rightarrow (\neg Z \Rightarrow Z)$ (ii, backward propagation)
(iv) $(\neg Z \Rightarrow Z) \Rightarrow Z$ (Lk1)
(v) $(Y \Rightarrow Z) \Rightarrow Z$ (iii, iv, transitivity)
(vi) $X \Rightarrow (Y \Rightarrow Z)$ (given)
(vii) $X \Rightarrow Z$ (vi, v, transitivity). ∎
Now we can prove Lemma 4.4 for System Lk.

(6.5) Lemma $\vdash_{\mathrm{Lk}} (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$.

Proof.
(i) $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B)) \Rightarrow (((\neg B \Rightarrow B) \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ (Lk3)
(ii) $(\neg B \Rightarrow B) \Rightarrow B$ (Lk1)
(iii) $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B)) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)$ (ii, i, 6.4)
(iv) $(\neg B \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow B))$ (Lk3)
(v) $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ (iv, iii, transitivity)
(vi) $A \Rightarrow (\neg A \Rightarrow B)$ (Lk2)
(vii) $((\neg A \Rightarrow B) \Rightarrow B) \Rightarrow (A \Rightarrow B)$ (vi, backward propagation)
(viii) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (v, vii, transitivity). ∎

(6.6) Lemma $\vdash_{\mathrm{Lk}} A \Rightarrow (B \Rightarrow A)$

Proof.
(i) $A \Rightarrow (\neg A \Rightarrow \neg B)$ (Lk2)
(ii) $(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$ (6.5)
(iii) $A \Rightarrow (B \Rightarrow A)$ (i, ii, transitivity). ∎
We are finally able to prove completeness of Lukasiewicz's system:

(6.7) Lemma $\vdash_{\mathrm{Lk}} \neg\neg A \Rightarrow A$

Proof.
(i) $\neg\neg A \Rightarrow (\neg A \Rightarrow \neg\neg A)$ (6.6)
(ii) $(\neg A \Rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow A)$ (6.5)
(iii) $\neg\neg A \Rightarrow (\neg A \Rightarrow A)$ (i, ii, transitivity)
(iv) $(\neg A \Rightarrow A) \Rightarrow A$ (Lk1)
(v) $\neg\neg A \Rightarrow A$ (iii, iv, transitivity). ∎
From the remarks at the beginning of this section (6.2), the above result is all we needed. So:

(6.8) Corollary *System* Lk *is complete.* ∎

7 Concluding remarks

Working backwards from Corollary 6.8, a full completeness proof for System Lk could be extracted from the foregoing development and presented in a few pages (granted that 'System M' is complete (1.8). This paper contains a great deal that is not directly relevant to such a proof.

The completeness problem may have been a textbook exercise, but the author, at least, found it extremely difficult. Indeed, it would have been completely impossible had the fact of its completeness not been already known!

Many years ago, I considered assigning this problem as homework. However, after staring at it for many hours and proving nothing more exciting than $A \Rightarrow A$ (1.10), I thought better of it. Reconsidering it a few years ago, I thought of *assuming* the Deduction Theorem and seeing how far that would get us. Completeness then became relatively trivial. This led somehow to 'System II,' Lukasiewicz's system with one extra 'permuting' axiom, and completeness of this system proved to be a feasible class-assignment (presented as a series of lemmas, of course). From System II to III was an easy step. By this time I had accumulated enough lemmas to study the proofs and see what could be dispensed with, or what could be weakened, and so on. Hence 'System IV,' which seemed to be a very modest strengthening of Lukasiewicz'z system. So by first adding strong assumptions and progressively weakening them I had found a route out of the initial impasse (1.10).

In several places progress was made by identifying where more was being proved than was needed. This was always where an axiom was only used for a derived rule. For example, 'System II' had the 'permutation' axioms

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$$

while finally we were able to prove a related derived rule

$$B,\ A \Rightarrow (B \Rightarrow C) \quad \vdash_{\mathrm{Lk}} \quad A \Rightarrow C.$$

In only one place was this 'weakening' process reversed: where the inference rule (5.2)

$$\neg B \Rightarrow \neg A \quad \vdash_{\mathrm{Lk}} \quad A \Rightarrow B$$

was strengthened to the tautologies (6.5)

$$\vdash_{\mathrm{Lk}} (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B).$$

Once this had been proved, the rest was fairly easy.

8 references

1. E. Mendelson (1987). *Introduction to mathematical logic,* 3rd Edition. Wadsworth.