Pisier's operator Hilbert space

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Abstract

The object of this seminar is to explain the definition of Pisier's operator Hilbert space, as is done in the beginning of his monograph [7]. Pisier's definition and proof of the main universal properties take only the first few pages of the monograph. However, one needs to be comfortable with thinking in terms of certain of the concepts in the theory of operator spaces before one can digest these few pages. We begin with an overview of these aspects of the theory.

Contents

1 Operator Spaces

All Banach spaces and Hilbert spaces we consider will be over $\mathbb{C}$.

Operator spaces can be looked at from two points of view, rather like one can approach C*-algebras either concretely as being *-closed subalgebras of the bounded operators on some Hilbert space or, abstractly, as a Banach algebra with an antilinear involution and satisfying certain properties.

1.1 Concrete operator spaces

Very briefly, operator sapces are closed linear subspaces of the space $\mathcal{B}(H)$ of bounded operators on some Hilbert space H.

To get the right idea, one should really consider operator spaces as a category where the appropriate mappings are the completely bounded linear mappings between operator spaces.

Associated with an operator space $X \subset \mathcal{B}(H)$, we have embeddings of the spaces $M_n(X)$ of $n \times n$ matrices with entries in X into $M_n(\mathcal{B}(H))$. We regard $M_n(\mathcal{B}(H))$ as being the bounded operators $\mathcal{B}(H^n)$ on the Hilbert space $H^n = H \oplus H \oplus H \cdots \oplus H$ (n times) using the standard idea of how matrices of operators act:

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_{1i} v_i \\ \vdots \\ \sum_{i=1}^n x_{ni} v_i \end{pmatrix}$$

If $T : X \to Y$ is a linear operator then we define $T^{(n)} : M_n(X) \to M_n(Y)$ by applying T to each entry of matrices with entries in X.

$$T^{(n)} \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} = \begin{pmatrix} T(x_{11}) & T(x_{12}) & \dots & T(x_{1n}) \\ T(x_{21}) & & \dots & T(x_{2n}) \\ \vdots & & \ddots & \vdots \\ T(x_{n1}) & T(x_{n2}) & \dots & T(x_{nn}) \end{pmatrix}$$

In the case of operator spaces $X \subset \mathcal{B}(H)$ and $Y \subset \mathcal{B}(K)$, we have norms on the matrix spaces $M_n(X)$ and $M_n(Y)$ as explained above. Thus we can compute the norm of $T^{(n)}$.

Definition 1 *The completely bounded norm of a linear operator $T : X \to Y$ between operator spaces X and Y is*

$$\|T\|_{cb} = \sup_{n \in \mathbb{N}} \|T^{(n)}\|$$

Definition 2 *A complete isometry between operator spaces X and Y is an invertible linear operator $T : X \to Y$ with*

$$\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$$

Thus we really consider operator spaces as equivalence classes of closed subspaces $X \subset \mathcal{B}(H)$ of the bounded operators on Hilbert spaces H (where H is not fixed). We consider two such subspaces to be equivalent if there is a complete isometry between them.

Put another way, we consider the category of operator spaces and completely bounded linear operators.

1.2 Abstract operator spaces

Abstractly an operator space consists of a Banach space X together with norms $\|\cdot\|_n$ on all $M_n(X)$ (= the $n \times n$ matrices with entries in X) which satisfy both

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} = \max\left(\|x\|_n, \|y\|_m\right) \tag{1}$$

$$\|\alpha x \beta\|_n \leq \|\alpha\| \, \|x\|_n \, \|\beta\| \tag{2}$$

for $x \in M_n(X)$, $y \in M_m(Y)$ and $\alpha, \beta \in M_n(\mathbb{C})$ scalar matrices. The norms of these scalar matrices are their norms as operators on an n-dimensional Hilbert space ($\mathbb{C}^n$ with the Euclidean inner product).

We have seen above that every concrete operator space $X \subset \mathcal{B}(H)$ gets norms on the matrix spaces $M_n(X) \subset \mathcal{B}(H^n)$ and the fact that these norms will satisfy the two properties (1) and (2) is quite easy to verify. Thus every concrete operator space is an operator space.

The converse, that every abstract operator space X can be realised as a concrete operator space via an isometric embedding $T : X \to \mathcal{B}(H)$ which preserves the matrix norms (more precisely so that $T^{(n)} : M_n(X) \to M_n(\mathcal{B}(H)) \equiv \mathcal{B}(H^n)$ is an isometric embedding for each n) is a deeper result. This is due to Ruan [8] but a more elegant proof can be found in [4].

2 Examples of operator spaces

1. Every C*-algebra A is an operator space, since A can be embedded as a norm closed *-subalgebra of some $\mathcal{B}(H)$.

   What is more significant is that this gives rise to a unique set of matrix norms on A. Any C*-embedding of A in a $\mathcal{B}(H)$ produces a C*-embedding of $M_n(A)$ in $\mathcal{B}(H^n)$ and the uniqueness of the matrix norms follows from the uniqueness of the C* norms on the C*-algebras $M_n(A)$.

2. Every Banach space X can be regarded as an operator space. This is because every such X can be embedded in a $\mathcal{B}(H)$. In fact we can do that in a canonical way.

   Taking the closed unit ball B_{X^*} of the dual of X (which is a compact Hausdorff space in the weak*-topology), we have an isometric embedding of X in the C*-algebra $C(B_{X^*})$ (via $x \in X$ maps to the function $x(x^*) = x^*(x)$). Hence we can use the matrix norms on the C*-algebra to get matrix norms on $M_n(X)$.

The operator space constructed in this way is denoted MIN(X). The notation seems to originate in [2] and it is shown in [2] that the MIN(X) matrix norms on the $M_n(X)$ are the smallest possible (among all choices of matrix norms compatible with the original norm on X).

3. There is also a canonical largest choice of matrix norms on a given Banach space X. It is denoted by MAX(X) in [2], where the definition is the simplest imaginable one to get this property. On$M_n(\mathrm{MAX}(X))$ we define the norm by

$$\|x\|_n = \sup \{ \|T^{(n)}(x)\|_n : T : X \to \mathcal{B}(H) \text{ an isometric embedding} \}$$

3 The dual of an operator space is an operator space

This is something that is rather counter-intuitive and indeed was not realised during some of the earlier work. This result is only true when stated appropriately. If X is an operator space, we can take the matrix norms on X^* so that $M_n(X^*)$ has a dual norm to $M_n(X)$ (via the pairing that $(x^*_{ij})(x_{kl}) = \sum_{ij} x^*_{ij}(x_{ij})$) but this results in matrix norms for X^* that satisfy a variant of (1), where max is replaced by a sum.

Operator spaces as we have defined them are sometimes referred to as L^∞-matricially normed spaces and those satisfying similar conditions, but with the max in (1) replaced by sum, are known as L^1-matricially normed spaces. This seems a natural thing to do, but leads one outside the category of operator spaces.

The operator spaces dual of an operator space was defined by Blecher in [1]. The idea is to regard elements of $M_n(X^*)$ as defining matrix-valued linear operators

$$(x^*_{ij}) : X \to M_n(\mathbb{C}) : x \mapsto (x^*_{ij}(x))$$

and then to define the matrix norm on $M_n(X^*)$ as the completely bounded norm of that operator.

$$\|(x^*_{ij})\|_n = \|(x^*_{ij})\|_{\mathrm{cb}}$$

The fact that this gives an (abstract) operator space structure to X^* is then not complicated to check. One thing that needs to be checked out is that the norm on $M_1(X^*)$ is the usual norm on the dual.

Lemma 3 *If $x^* \in X^*$, then $\|x^*\|_{cb} = \|x^*\|$.*

Proof. To compute the CB norm of x^*, we need to consider the extensions of x^* to matrices, that is

$$(x^*)^{(m)} : M_m(X) \to M_m(\mathbb{C}) : (x_{ij}) \mapsto (x^*(x_{ij}))$$

We can compute the norm of the scalar matrix on the right by applying it to a unit vector $\alpha = (\alpha_i) \in \mathbb{C}^m$ and taking the inner product of that with another unit vector $\beta = (\beta_i)$. We get

$$\left\| (x^*)^{(m)}(x_{ij}) \right\| = \sup_{\alpha,\beta} \left| \sum_{ij} \alpha_i x^*(x_{ij}) \bar{\beta}_j \right| = \sup_{\alpha,\beta} \left| x^* \left(\sum_{ij} \alpha_i x_{ij} \bar{\beta}_j \right) \right|$$

and from (2) (which can easily be generalised to rectangular matrices α and β using (1)), we can conclude that this is at most

$$\|x^*\| \, \|(x_{ij})\|_m$$

From this the lemma follows immediately.

4 Examples of duals of operator spaces

1. One way to get an operator space structure on a Hilbert space H is to embed H into $\mathcal{B}(H)$ as the operators that have nonzero entries only in the "first row". For $\mathbb{C}^n$, this means embedding it into $M_n(\mathbb{C})$ as the subspace

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The operator space structure on $\mathbb{C}^n$ which arises from this embedding will be denoted R_n. (R for row, of course).

We can do a similar construction on the first column and we get an operator space which we call C_n.

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ \alpha_2 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \alpha_n & 0 & \dots & 0 \end{pmatrix}$$

As Banach spaces (or Hilbert spaces) there is an obvious transpose map which is an isometry from R_n to C_n. However this is not a complete isometry. In the case of R_2, it is not hard to check that the extension of the transpose map to a map $M_2(R_2) \to M_2(C_2)$ maps

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The original matrix has norm $\sqrt{2}$ while the image has norm 1.

In fact, the operator spaces R_n and C_n are actually not completely isometric at all, but each one is the dual of the other.

2. It is shown in [1] that the dual of MIN(X) is MAX(X^*) (for any Banach space X) and also that the dual of MAX(X) is MIN(X^*).

   We can apply this with $X = H$ a Hilbert space.

   Again it is the case that MIN(H) and MAX(H) are not completely isometric.

4.1 The complex conjugate of a space

Before proceeding, we need to clarify the usual perception that a Hilbert space is isomorphic to its own dual. The canonical identification comes from the Riesz representation theorem, which allows every $x^* \in H^*$ to be represented as

$$x^*(x) = \langle x, x^* \rangle$$

for a unique $x^* \in H$. However this identification $x^* \in H^* \mapsto x^* \in H$ produces an antilinear isometry. Thus the correct statement is that we can canonically identify the dual H^* with the complex conjugate space $\overline{H}$, which is identical to H as a set, has the same addition operation as H, but has multiplication by scalars given by $(\lambda, y) \mapsto \bar{\lambda}x$ and inner product $(x, y) \mapsto \overline{\langle x, y \rangle} = \langle y, x \rangle$.

A convenient way of keeping track of the difference between $\overline{H}$ and H is to use $\bar{x}$ for x when x is to be considered as an element of $\overline{H}$. Then our conjugate operations will take care of themselves via the notation.$\lambda\bar{x} = \overline{\bar{\lambda}x}$ and $\langle \bar{x}, \bar{y} \rangle = \overline{\langle y, x \rangle}$.

5 The operator Hilbert Space

Theorem 4 (Pisier) *For any index set I, there exists an operator space $OH(I)$ such that*

1. *$OH(I)$ is isometric to $\ell^2(I)$ as a Banach space*

2. *The canonical identification of a Hilbert space and its conjugate dual*

   $$OH(I)^* \equiv \overline{OH(I)}$$

   is a complete isometry.

In fact Pisier [7] proves rather more than this. He shows that $OH(I)$ satisfies a universal property that implies that it is uniquely characterised up to complete isometry by these two properties.

A rather abstract approach to describing $OH(I)$ can be based on a theory of complex interpolation for operator spaces, which is also developed by Pisier in [7]. Complex interpolation is commonly used as a way of deducing boundedness of an operator on (say) L^p for $1 < p < 2$ from boundedness for $p = 1$ and $p = 2$. The general theory allows one to create intermediate spaces $(L^1, L^2)_\theta$ for $0 < \theta < 1$ (on which there is an intermediate version of the operator that remains bounded on the intermediate space). The final point is to show that one can identify the intermediate space with L^p where $1/p = (1-\theta)(1/1) + \theta(1/2)$.

Pisier's extension of this to operator spaces X and Y is to apply complex interploation not only to the spaces themselves, but also to the matrix spaces $M_n(X)$ and $M_n(Y)$ (to produce the matrix norms for the intermediate spaces $(X, Y)_\theta$). With this theory, the operator Hilbert space can be described as the midpoint space between the row operator space R_I and the column operator space C_I of "dimension" I. It can also be characterised as the midpoint between $\mathrm{MIN}(\ell^2(I))$ and $\mathrm{MAX}(\ell^2(I))$.

There is a somewhat more concrete approach which is based on the following inequality due to Haagerup [5], a sort of generalised Cauchy-Schwarz inequality.

Proposition 5 (Haagerup) *Let H and K be Hilbert spaces, $a_i \in \mathcal{B}(H)$ and $b_i \in \mathcal{B}(K)$ for $i = 0, 1, 2, \dots, n$. Then*

$$\left\| \sum_0^n a_i \otimes \bar{b}_i \right\|_{\mathcal{B}(H \otimes \bar{K})} \leq \left\| \sum_0^n a_i \otimes \bar{a}_i \right\|_{\mathcal{B}(H \otimes \bar{H})}^{1/2} \left\| \sum_0^n b_i \otimes \bar{b}_i \right\|_{\mathcal{B}(K \otimes \bar{K})}^{1/2}$$

Before explaining the proof of this inequality, we need to delve into the notation a bit. The tensor product of the Hilbert spaces H and K is most easily explained in terms of othornormal bases e_n for H and f_n for K. Then the tensor product $H \otimes K$ has an othornormal basis $e_n \otimes f_m$.

The tensor product $H \otimes \bar{K} = H \otimes K^*$ can be identified in a natural way with the class $S^2(K, H)$ of Hilbert-Schmidt operators from K to H by defining the action

$$(x \otimes \bar{y})(z) = x\langle z, y\rangle \quad \text{for } x \in H,\, y, z \in K$$

Next the tensor product of the operators $a_i \otimes \bar{b}_i$ is the linear operator which acts on the space $H \otimes \bar{K}$ by the rule

$$(a_i \otimes \bar{b}_i)(x \otimes \bar{y}) = a_i(x) \otimes \overline{b_i(y)}$$

For our purposes, we need to be clear on how this tensor product acts when we identify the space $H \otimes \bar{K}$ with Hilbert-Schmidt operators.

Lemma 6 *Let $a \in B(H)$, $b \in B(K)$. Under the canonical identification of $H \otimes \bar{K}$ with $S_2(K, H)$, the operator $a \otimes \bar{b} \in \mathcal{B}(H \otimes \bar{K})$ corresponds to the operator*

$$a \otimes \bar{b} : S_2(K, H) \to S_2(K, H) : u \mapsto aub^*$$

Proof of the lemma: $S_2(K,H)$ is spanned by operators of the form $u = x \otimes \bar{y}$ with $x \in H$, $y \in K$. For such a u, and $z \in K$, we have

$$\begin{aligned} ((a \otimes \bar{b})(u))(z) &= ((a \otimes \bar{b})(x \otimes \bar{y}))(z) \\ &= (a(x) \otimes \overline{b(y)})(z) \\ &= a(x)\langle z, b(y)\rangle = a(x)\langle b^* z, y\rangle \\ &= a(x\langle b^* z, y\rangle) \text{ using linearity of } a \\ &= a((x \otimes \bar{y})(b^* z)) = (aub^*)(z) \end{aligned}$$

Proof of the proposition: Using the lemma, we can translate the operator norm on the left of the inequality to an operator norm of operators on $S_2(K,H)$ and we compute this by applying the operator to a unit vector $u \in S_2(K,H)$ and taking the inner product of the result with another unit vector $v \in S_2(K,H)$.

$$\begin{aligned} \left\| \sum_0^n a_i \otimes \bar{b}_i \right\|_{\mathcal{B}(H \otimes \bar{K})} &= \left\| \sum_0^n a_i \otimes \bar{b}_i \right\|_{\mathcal{B}(S_2(K,H))} \\ &= \sup_{\|u\|=\|v\|=1} \left| \left\langle \left(\sum_0^n a_i \otimes \bar{b}_i \right)(u), v \right\rangle \right| \\ &= \sup_{u,v} \left| \sum_0^n \langle a_i u b_i^*, v\rangle \right| \\ &= \sup_{u,v} \left| \operatorname{tr} \sum_0^n a_i u b_i^* v^* \right| \end{aligned}$$

where we are now using the fact that the inner product on $S_2(K,H)$ is given via the trace, $\langle u, v\rangle = \operatorname{tr} uv^*$.

Next we use the fact that unit operators $u, v \in S_2(K,H)$ can be factored $u = u_1 u_2$ and $v = v_1 v_2$ with $u_1, v_1 \in S_4(H)$, $u_2, v_2 \in S_4(K,H)$ unit norm operators in the Schatten 4 class. We can then symmetrize our traces above

$$\operatorname{tr} a_i u b_i^* v^* = \operatorname{tr} a_i u_1 u_2 b_i^* v_2^* v_1^* = \operatorname{tr} v_1^* a_i u_1 (v_2 b_i u_2^*)^* = \langle v_1^* a_i u_1, v_2 b_i u_2^* \rangle$$

Thus we have

$$\left\| \sum_0^n a_i \otimes \bar{b}_i \right\|_{B(H \otimes \bar{K})} = \sup_{u_1, v_1, u_2, v_2} \left| \sum_0^n \langle v_1^* a_i u_1, v_2 b_i u_2^* \rangle \right| \tag{3}$$

The idea to prove the inequality is to use the ordinary Cauchy-Schwarz inequality for the Hilbert space $S_2(H) \oplus S_2(H) \oplus \cdots \oplus S_2(H)$ ($n+1$ times) to deduce

$$\left\| \sum_0^n a_i \otimes \bar{b}_i \right\|^2_{B(H \otimes \bar{K})} \leq \sup \left(\sum_0^n \langle v_1^* a_i u_1, v_1^* a_i u_1 \rangle \right) \left(\sum_0^n \langle v_2 b_i u_2^*, v_2 b_i u_2^* \rangle \right)$$

Here we are relying on the fact that $v_1^* a_i u_1, v_2 b_i u_2^* \in S_2(H)$, which is true because there are two factors in S_4 in each case. We can now use the argument leading to (3) to bound each of the terms on the right and get the inequality claimed in the proposition.

Corollary 7 *For a fixed infinite dimensional Hilbert space K (let us take $K = \ell^2$ to make things definite), let $\mathbb{S}$ denote the set of all finitely nonzero sequences $b_0, b_1, b_2, \dots$ from $\mathcal{B}(K)$ which satisfy*

$$\left\| \sum_0^\infty b_i \otimes \bar{b}_i \right\|_{\mathcal{B}(K \otimes \bar{K})} \leq 1$$

Then for any other Hilbert space H and any $a_0, a_1, \dots, a_n \in \mathcal{B}(H)$, we have

$$\left\| \sum_0^n a_i \otimes \bar{a}_i \right\|_{\mathcal{B}(H \otimes \bar{H})} = \sup_{b \in \mathbb{S}} \left\| \sum_0^\infty a_i \otimes \bar{b}_i \right\|_{\mathcal{B}(H \otimes \bar{K})}$$

(Here $a_i = 0$ for $i > n$.)

Proof: Using Proposition 5, we can see immediately that the supremeum is no more than the norm on the left.

It is quite easy to show that equality holds in the case $H = K$. Take

$$b_i = a_i \left\| \sum_0^n a_i \otimes \bar{a}_i \right\|_{\mathcal{B}(H \otimes \bar{H})}^{-1/2}$$

In the case $H \subset K$, we can still do this by extending the a_i to K (make them zero on $H^\perp$). This deals with all separable H, as K is infinite dimensional. For the general case, we can restrict the a_i to a separable subspace of H whcih is big enough to realise the norm of the operator $\sum_0^n a_i \otimes \bar{a}_i$ on $H \otimes \bar{H}$.

Proof of the theorem (construction of the space). We prove it only for the case $I = \mathbb{N}$.

The idea is to use the Corollary. From the corollary we have the set $\mathbb{S}$ of sequences of operators on some K and we can use this to compute norms.

For each $b = (b_0, b_1, \dots) \in \mathbb{S}$, define $K_b = K$ and $T_n^b : \bar{K}_b \to \bar{K}_b$ by $T_n^b = \bar{b}_n$. (Recall that $b_n \in \mathcal{B}(K)$ and so this makes perfect sense.) Now we put all these together to make a huge Hilbert space

$$\mathfrak{H} = \bigoplus_{b \in \mathbb{S}} \bar{K}_b$$

and "diagonal" operators $T_n \in \mathcal{B}(\mathfrak{H})$ given by

$$T_n((\bar{x}_b)_{b \in \mathbb{S}}) = (T_n^b(\bar{x}_b))_{b \in \mathbb{S}} = (\overline{b_n(x_b)})_{b \in \mathbb{S}}$$

Now we compute the operator norm on the span of the T_n

$$\begin{aligned}\left\|\sum_0^n \alpha_i T_i\right\| &= \sup_{b\in\mathbb{S}}\left\|\sum_0^n \alpha_i T_i^b\right\| \\ &= \sup_{b\in\mathbb{S}}\left\|\sum_0^n \alpha_i \bar{b}_i\right\|\end{aligned}$$

Treating the scalars α_i as linear operators of multiplication by α_i on the one-dimensional Hilbert space $H = \mathbb{C}$ and applying Corollary 7, we find that this supremum is

$$\left\|\sum_0^n \alpha_i T_i\right\| = \left(\sum_0^n |\alpha_i|^2\right)^{1/2}$$

We now define $OH(\mathbb{N})$ to be the closed span of the operators T_i. We can see now that it is a Hilbert space with the operator norm and it is an operator space because it is a subspace of $\mathcal{B}(\mathfrak{H})$.

It remains to show that $OH(\mathbb{N})$ is completely isometric to its dual. The argument just used for 1-dimensional operators applies to arbitrary operators and shows the following.

Lemma 8 *Let H be a Hilbert spaces and $a_0, a_2, \ldots, a_n \in \mathcal{B}(H)$. Then*

$$\left\|\sum_0^n a_i \otimes T_i\right\|_{\mathcal{B}(H\otimes\mathfrak{H})} = \left\|\sum_0^n a_i \otimes \bar{a}_i\right\|_{\mathcal{B}(H\otimes\bar{H})}^{1/2}$$

Before proceeding, we need to recall the following "well-known" ways of computing completely bounded norms.

Proposition 9 *Let $X \subset \mathcal{B}(H)$ be an operator space and $u : X \to \mathcal{B}(K)$ linear. (Here H and K are Hilbert spaces.) Then*

$$\|u\|_{cb} = \|Id_{\mathcal{B}(\ell^2)} \otimes u\|_{\mathcal{B}(\ell^2)\otimes X\to\mathcal{B}(\ell^2\otimes K)}$$

Here, as elsewhere, we are taking the minimal or spatial tensor product on $\mathcal{B}(\ell^2)\otimes X$ and on $\mathcal{B}(\ell^2)\otimes\mathcal{B}(K) = \mathcal{B}(\ell^2\otimes K)$. (Thus the norm on $\mathcal{B}(\ell^2)\otimes X$ is related to the operator spaces structure of $X \subset \mathcal{B}(H)$ and is the norm induced from $\mathcal{B}(\ell^2)\otimes X \subset \mathcal{B}(\ell^2\otimes H)$. The paper [2] deals with tensor products of operator spaces and this kind of tensor product is called the spatial or injective tensor product in [6]. When dealing with this tensor product, we are implicitly using the fact that the tensor product $\mathcal{B}(\ell^2)\otimes X$ is definable in terms of the abstract operator space structure of X, or in other words is unchanged if we take

another embedding of $X \subset \mathcal{B}(H_1)$ which results in a complete isometry of X.) The result seems to originate in [3].
Proof. Assume u is completely bounded. Consider projections $p_n : \ell^2 \to \ell^2$ onto the first n coordinates of ℓ^2. The operator $\mathrm{Id}_{\mathcal{B}(\ell^2)} \otimes u$ makes sense at least on the algebraic tensor product of $\mathcal{B}(\ell^2)$ and X. Let's call that operator S. Then, for finite sequences (a_i) in $\mathcal{B}(\ell^2)$ and (b_i) in X, we have

$$\begin{aligned}\|(p_n \otimes Id)S(\sum a_i \otimes b_i)(p_n \otimes Id)\| &= \|\sum p_n a_i p_n \otimes u(b_i)\| \\ &\leq \|u^{(n)}\| \, \|\sum p_n a_i p_n \otimes b_i\| \\ &\leq \|u\|_{\mathrm{cb}} \, \|\sum a_i \otimes b_i\|\end{aligned}$$

where we are using the fact that we may regard the $p_n a_i p_n$ as elements of $M_n(\mathbb{C})$ (and Id means the identity operator on $\mathcal{B}(K)$). Since the projections p_n converge in the strong operator topology to the identity, we can conclude that

$$\|S(\sum a_i \otimes b_i)\| \leq \|u\|_{\mathrm{cb}} \, \|\sum a_i \otimes b_i\|$$

and so that S extends boundedly to an operator of norm at most $\|u\|_{\mathrm{cb}}$.

For the converse, we just observe again that the restriction of $Id_{\mathcal{B}(\ell^2)} \otimes u$ to $p_n \mathcal{B}(\ell^2) p_n \otimes X$ is essentially $u^{(n)} : M_n(X) \to M_n(\mathcal{B}(K))$. Thus $\|u^{(n)}\| \leq \|Id_{\mathcal{B}(\ell^2)} \otimes u\|$ for all n.

Proposition 10 *Let $X \subset \mathcal{B}(H)$ be any operator space, (x_n) a sequence in X which spans a dense subspace of X and (ξ_n) a biorthogonal sequence to (x_n) in X^* which also spans a dense subspace of X^*. (Biorthogonal means $\xi_n(x_m) = 0$ if $n \neq m$ and $\xi_n(x_n) = 1$.)*

Let (a_n) be a finite sequence of operators on another Hilbert space K, and define $u : X \to \mathcal{B}(K)$ by $u(x) = \sum \xi_n(x) a_n$. Then

$$\|u\|_{cb} = \left\|\sum \xi_n \otimes a_n\right\|_{X^* \otimes \mathcal{B}(K)}$$

and

$$\|u\|_{cb} = \sup\left\{\left\|\sum b_n \otimes a_n\right\|_{\mathcal{B}(\ell^2 \otimes K)} : (b_n) \in \mathcal{B}(\ell^2)^{(\mathbb{N})}, \left\|\sum b_n \otimes x_n\right\|_{\mathcal{B}(\ell^2 \otimes H)} \leq 1\right\}$$

Here we are using the abbreviation $\mathcal{B}(\ell^2)^{(\mathbb{N})}$ for finitely nonzero sequences from $\mathcal{B}(\ell^2)$. Also we need to take the minimal or spatial norm on $X^* \otimes \mathcal{B}(K)$, as explained above. (Embed X^* as an operator space in some $\mathcal{B}(L)$ to get the right $X^* \otimes \mathcal{B}(K) \subset \mathcal{B}(L \otimes K)$.)
Proof of the proposition. According to Proposition 9,

$$\|u\|_{\mathrm{cb}} = \|\mathrm{Id}_{\mathcal{B}(\ell^2)} \otimes u\|_{\mathcal{B}(\ell^2) \otimes X \to \mathcal{B}(\ell^2 \otimes K)}$$

Since the finite sums $\sum b_n \otimes x_n$ are dense in $\mathcal{B}(\ell^2) \otimes X$ and $(Id_{\mathcal{B}(\ell^2)} \otimes u)(\sum_0^n b_i \otimes x_i) = \sum_1^n b_i \otimes a_i$, the second formula in the proposition comes directly from the definition of the operator norm.

The first formula follows from the first and the definition of $X^* \otimes \mathcal{B}(K)$, the definition of the operator space structure on X and the fact that we can compute norms in spatial tensor products like $X^* \otimes \mathcal{B}(K)$ via compressions to finite dimensional subspaces of K. (This last fact is used in [2].) If $p : K \to K$ is a finite rank orthogonal projection on K, then the map $a \mapsto pap : \mathcal{B}(K) \to \mathcal{B}(K)$ is a complete contraction on $\mathcal{B}(K)$ and its range may be identified with $\mathcal{B}(pK)$ and so with $M_{\text{rank }(p)}(\mathbb{C})$. Moreover

$$\|\sum \xi_n \otimes a_n\|_{X^* \otimes \mathcal{B}(K)} = \sup_p \|\sum \xi_n \otimes pa_n p\|_{X^* \otimes \mathcal{B}(\mathbb{C}^{\text{rank }(p)})}$$

where the supremum is over all finite rank orthogonal projections p on K. (If K is finite dimensional, all this is not necessary.)

We consider then the case when $K = \mathbb{C}^p$ (as we can then deal with the general case by the above reduction). Thus $a_n = (a_{nij}) \in M_p(\mathbb{C})$ and we can identify $X^* \otimes M_p(\mathbb{C})$ isometrically with $M_p(X^*)$ so that $\sum \xi_n \otimes a_n \mapsto (\sum_n a_{nij} \xi_n)$. (To see this consider $X^* \subset \mathcal{B}(L)$ as an operator space and $\mathcal{B}(L) \otimes M_p(\mathbb{C}) = \mathcal{B}(L \otimes \mathbb{C}^p)$.) From the definition of the operator space structure on X^*, we have $M_p(X^*) = CB(X, M_p(\mathbb{C}))$ isometrically and this then makes $\sum \xi_n \otimes a_n$ correspond to the $M_p(\mathbb{C})$-valued map $(\sum_n a_{nij} \xi_n(\cdot))$ on X. To calculate the completely bounded norm of that, we promote it to an operator on $M_q(X)$ with values in $M_q(M_p(\mathbb{C})) = M_{pq}(\mathbb{C})$, the operator that takes $(x_{kl}) \in M_q(X)$ to $(\sum_n a_{nij} \xi_n(x_{kl}))_{ik,jl} \in M_{pq}(\mathbb{C})$. The supremum of this over all choices of q and over all $(x_{kl}) \in M_q(X)$ with $\|(x_{kl})\|_q \leq 1$ gives us $\|\sum \xi_n \otimes a_n\|_{X^* \otimes \mathcal{B}(\mathbb{C}^p)}$.

We can make a further finite dimensional approximation and take the supremum over $x_{kl} \in \text{span }\{x_1, x_2, \ldots, x_N\}$. For such x_{kl} we have $x_{kl} = \sum_r b_{rkl} x_r$ for scalars b_{rkl} and we are considering the supremum of the norms in $M_{pq}(\mathbb{C})$ of

$$\left(\sum_n a_{nij} \xi_n(x_{kl})\right)_{ik,jl} = \left(\sum_n a_{nij} b_{nkl}\right)_{ik,jl}$$

subject to the restriction that $\|(x_{kl})\|_q = \|\sum b_{nkl} x_n\|_q \leq 1$. If we define $b_n \in \mathcal{B}(\ell^2)$ by $b_n = (b_{nkl}) \in \mathcal{B}(\mathbb{C}^q) \subset \mathcal{B}(\ell^2)$ for $1 \leq n \leq N$, this restriction amounts to $\|\sum_n b_n \otimes x_n\|_{\mathcal{B}(\ell^2) \otimes X} \leq 1$. Also the expression being maximised turns out to be

$$\left\|\sum_n b_n \otimes a_n\right\|_{\mathcal{B}(\ell^2 \otimes \mathbb{C}^p)}$$

In this way $\|\sum \xi_n \otimes a_n\|$ works out to be the same as the right hand side of the second formula in the proposition, except that we still have to observe that the restriction to $b_1, b_2, \ldots, b_N \in \mathcal{B}(\ell^2)$ which are actually finitely nonzero matrics (which therefore can be regarded as in $M_q(\mathbb{C})$ for some q) does not affect the supremum in that second formula.

We can also pass from the case $K = \mathbb{C}^p$ to the case of general K by the method explained above.

Completion of the proof of the theorem. Returning to the proof of the Theorem, let $\xi_n \in OH(\mathbb{N})^*$ denote the biorthogonal sequence to T_n, so that the canonical identification of $OH(\mathbb{N})$ with $\overline{OH(\mathbb{N})^*}$ is the linear operator J that has $J(T_n) = \bar{\xi}_n$ or $J(x) = \sum \xi_n(x)\bar{\xi}_n$.

Take a finite version $J_n(x) = \sum_1^n \xi_i(x)\bar{\xi}_i$ and consider the operator space $OH^* \subset \mathcal{B}(L)$ for some Hilbert space L. By Proposition 10, we have

$$\|J_n\|_{\text{cb}} = \left\| \sum_1^n \xi_i \otimes \bar{\xi}_i \right\|_{OH^* \otimes \mathcal{B}(\bar{L})}$$

From Lemma 8, we have

$$\left\| \sum_1^n \xi_i \otimes \bar{\xi}_i \right\|_{\mathcal{B}(L \otimes \bar{L})} = \left\| \sum_1^n \xi_i \otimes T_i \right\|_{\mathcal{B}(L \otimes \mathfrak{H})}$$

If we define a sort of truncated identity map $I_n : OH(\mathbb{N}) \to OH(\mathbb{N})$ by $I_n(x) = \sum_1^n \xi_i(x)T_i$, Proposition 10 tells us that

$$\|I_n\|_{\text{cb}} = \left\| \sum_1^n \xi_i \otimes T_i \right\|_{OH^* \otimes \mathcal{B}(\mathfrak{H})} = \left\| \sum_1^n \xi_i \otimes T_i \right\|_{\mathcal{B}(L \otimes \mathfrak{H})}$$

Hence

$$\|J_n\|_{\text{cb}} = \|I_n\|_{\text{cb}}$$

Now, from the second formula in Proposition 10 we can see that $\|I_{n+1}\|_{\text{cb}} \geq \|I_n\|_{\text{cb}}$ (or alternatively we can see from Corollary 7 together with the above expression for $\|J_n\|_{\text{cb}}$ that $\|J_n\|_{\text{cb}}$ is monotone increasing in n).

From the definition of the completely bounded norm we can see that $\|I_n\|_{\text{cb}} \leq \|Id_{OH}\|_{\text{cb}} = 1$. Again from the definition of the norm and the density of $\bigcup_n \text{span}\,\{T_1, T_2, \ldots, T_n\}$ in $OH(\mathbb{N})$, we can see that $\|J_n\|_{\text{cb}} \to \|J\|_{\text{cb}}$ as $n \to \infty$. Thus $\|J\|_{\text{cb}} \leq 1$. Hence $\|J\|_{\text{cb}} = 1$ as $\|J\| = 1$.

This completes the proof of the theorem.

Pisier remaks that we can realise the operator space $OH(\mathbb{N}) \subset \mathcal{B}(\mathfrak{H})$ inside some $\mathcal{B}(H)$ with H separable by finding a separable subspace $H \subset \mathfrak{H}$ so that, for $P : \mathfrak{H} \to \mathfrak{H}$ the orthogonal projection onto H, the compression $x \mapsto PxP$ is a complete isometry on $OH(\mathbb{N})$. To do this take a dense countable sequence in $OH(\mathbb{N})$ and in $M_n(OH(\mathbb{N}))$ for each n. Then each of these countably many operators will realise its norm on some countable subset of $\mathfrak{H}^n$. Take H to be the closed subspace spanned by the union of all these countable number of elements of $\mathfrak{H}$ plus their images under the appropriate operators.

Note the key role played in the proof by Lemma 8. Combining this with Corollary 7 we get another formulation of that lemma, which Pisier also uses.

Lemma 11 *Let H be a Hilbert space and $a_0, a_2, \dots, a_n \in \mathcal{B}(H)$. Then*

$$\left\|\sum_0^n a_i \otimes T_i\right\|_{\mathcal{B}(H\otimes\mathfrak{H})} = \sup\left\{\left\|\sum_0^n a_i \otimes \bar{c}_i\right\|_{\mathcal{B}(H\otimes K)} : c_i \in \mathcal{B}(K), \left\|\sum_0^n c_i \otimes T_i\right\|_{\mathcal{B}(K\otimes\mathfrak{H})} \leq 1\right\}$$

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