A note on a family of analytic discs attached to a real submanifold $M \subset \mathbb{C}^{N+1}$ with a CR singularity

Valentin Burcea

Abstract. In this note we construct a family of analytic discs attached to a real submanifold $M \subset \mathbb{C}^{N+1}$ of codimension 2 with a CR singularity. These discs are mutually disjoint and form a smooth hypersurface $\widetilde{M}$ with boundary M in a neighborhood of the CR singularity. As an application we prove that if p is a flat-elliptic CR singularity and if M is nowhere minimal at its CR points and does not contain a complex manifold of dimension $(n-2)$ then $\widetilde{M}$ is a Levi-flat hypersurface real-analytic across the boundary.

1. Introduction and Main Result

Let $M \subset \mathbb{C}^{N+1}$ be a real submanifold. A point $p \in M$ with the property that the map $M \ni q \mapsto \dim_{\mathbb{C}} T_q^c M$ defined near p is not continuous at p is called a CR singularity. Here $T_q^c M := T_q M \cap J(T_q M)$ and $J : \mathbb{C}^{N+1} \longrightarrow \mathbb{C}^{N+1}$ is the canonical complex structure.

We assume that $\operatorname{codim}_{\mathbb{R}} M = 2$. Bishop studied this situation in $\mathbb{C}^2$. He considered the case when there exist the coordinates (z, w) such that near the CR singularity $p = 0$ the submanifold $M \subset \mathbb{C}^2$ is defined locally by

$$w = z\overline{z} + \lambda\left(z^2 + \overline{z}^2\right) + \mathrm{O}(3), \tag{1.1}$$

where $\lambda \in [0, \infty)$ is a holomorphic invariant called the Bishop invariant. When $\lambda \in \left[0, \frac{1}{2}\right)$ Kenig and Webster (see [**14**]) proved the existence of an unique family of 1-dimensional analytic disks shrinking to the CR singularity $p = 0$. The real-analytic case was studied by Huang and Krantz (see [**13**]).

Let $(z_1, \dots, z_N, w)$ be the coordinates from $\mathbb{C}^{N+1}$. We consider the higher dimensional case of (1.1) when M is defined near $p = 0$ by

$$w = z_1\overline{z}_1 + \lambda\left(z_1^2 + \overline{z}_1^2\right) + Q\left(z_1, \overline{z}_1, z_2, \overline{z}_2, \dots, z_N, \overline{z}_N\right) + \mathrm{O}(3), \tag{1.2}$$

where $Q\left(z_2, \overline{z}_2, \dots, z_N, \overline{z}_N\right)$ is a quadratic form depending on $z_2, \overline{z}_2, \dots, z_N, \overline{z}_N$ and combinations between $z_2, \overline{z}_2, \dots, z_N, \overline{z}_N$ and $z_1, \overline{z}_1$. We say that λ is elliptic if $\lambda \in \left[0, \frac{1}{2}\right)$.

In this paper we extend the work done by Kenig and Webster in $\mathbb{C}^2$. We prove the following result:

Theorem 1.1. *Let $M \subset \mathbb{C}^{N+1}$ be a smooth submanifold defined locally near $p = 0$ by (1.2) such that λ is elliptic. Then there exists a family of regularly embedded analytic discs with boundaries on M that are mutually disjoint and that forms a smooth hypersurface $\widetilde{M}$ with boundary M in a neighborhood of $p = 0$.*

The hypersurface $\widetilde{M}$ is not necessary Levi-flat as in the case of $\mathbb{C}^2$. The existence of a Levi-flat hypersurface with prescribed boundary S in $\mathbb{C}^{N+1}$, $N \geq 2$, requires certain compatibility conditions on S (see [**4**]):

(i) S is compact, connected and nowhere minimal at every CR point;
(ii) S does not contain a complex submanifold of dimension $(n-2)$;
(iii) S contains a finite number of flat elliptic CR singularities.

The CR singularity $p = 0$ is called elliptic if the quadratic part from (1.2) is positive definite. We say that $p = 0$ is a ,,flat" if the Definition 2.1 from [**4**] from is satisfied. Under the assumptions (i),(ii),(iii) Dolbeault, Tomassini and Zaitsev proved the existence of a Levi-flat which bounds S in the sense of currents (see Theorem 1.2, [**4**]).

The graph case was studied in [**5**]. Let $\mathbb{C}^{N+1} = \left(\mathbb{C}_z^N \times \mathbb{R}_u\right) \times \mathbb{R}_v$, where $w = u + iv$, and let Ω be a bounded strongly convex domain of $\mathbb{C}_z^N \times \mathbb{R}_u$ with smooth boundary $b\Omega$. Let $S \subset \mathbb{C}^{N+1}$, $n \geq 3$, be the graph of a function $g : b\Omega \longrightarrow \mathbb{R}_v$

Keywords: CR singularity, hull of holomorphy, analytic disc.
This project was supported by Science Foundation Ireland grant $10/RFP/MTH2878$.

such that S satisfies the conditions of [[**4**], Theorem 3.1]. Under these assumptions Dolbeault-Tomassini-Zaitsev proved that the following result

THEOREM 1.2. *Let $q_1, q_2 \in b\Omega$ be the projections of the complex points p_1, p_2 of S, respectively. Then, there exists a Lipschitz function $f : \overline{\Omega} \longrightarrow \mathbb{R}_v$ which is smooth on $\overline{\Omega} - \{q_1, q_2\}$ and such that $f|_{b\Omega} = g$ and $N = graph(f) - S$ is a Levi-flat hypersurface of $\mathbb{C}^{N+1}$. Moreover, each complex leaf of M_0 is the graph of a holomorphic function $\varphi : \Omega' \longrightarrow \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and φ is smooth on Ω'.*

As an application Theorem 1.1 and Dolbeault-Tomassini-Zaitsev theorem we obtain the following result that gives a higher dimensional generalization of Theorem 0.1 from [**13**]:

THEOREM 1.3. *Let $M \subset \mathbb{C}^{N+1}$ be a smooth submanifold defined locally near $p = 0$ by (1.2)satisfying the conditions of Theorem* 2.1. *Then $\widetilde{M}$ is a Levi-flat hypersurface real-analytic across the boundary manifold M.*

We prove our result by following the lines of [**12**], [**13**], [**14**], [**15**]. First, we make a perturbation along the CR singularity and then we find a holomorphic change of coordinates depending smoothly on a parameter. Then, we will adapt the methods used in $\mathbb{C}^2$ by Huang-Krantz, Kenig-Webster to our case. We mention that versions of our result were obtained in a higher codimensional case by Huang in [**12**], Kenig-Webster in [**15**].

Acknowledgements. I am grateful to my supervisor Dmitri Zaitsev for useful conversations and for bringing me this problem to my attention.

2. Preliminaries

2.1. A perturbation along the CR singularity. Let Δ be the unit open disc from $\mathbb{C}$ and let S^1 be its boundary. A map $f : \overline{\Delta} \longrightarrow \mathbb{C}$ is called an analytic disc if $f|_{\overline{\Delta}}$ is continuous and $f|_{\Delta}$ analytic. We say that f is an analytic disc attached to M if $f\left(\mathrm{S}^1\right) \subset M$. We construct analytic discs attached to M depending smoothly on

$$X = (z_2, \dots, z_N) = (x_2 + iy_2, \dots, x_N + iy_N) \approx 0 \in \mathbb{C}^{N-2}. \tag{2.1}$$

By using the notation $z = z_1$, our manifold M is defined near $p = 0$ by

$$w = z\overline{z} + \lambda\left(z^2 + \overline{z}^2\right) + Q\left(z_1, \overline{z}_1, z_2, \overline{z}_2, \dots, z_N, \overline{z}_N\right) + \mathrm{O}(3), \tag{2.2}$$

or equivalently by

$$\begin{aligned} w =& H_{0,0}(X) + \overline{z}H_{0,1}(X) + zH_{1,0}(X) + z\overline{z}\left(1 + H_{1,1}(X)\right) \\ &+ \left(\lambda + H_{2,0}(X)\right)z^2 + \left(\lambda + H_{0,2}(X)\right)\overline{z}^2 + \mathrm{O}\left(|z|^3\right), \end{aligned} \tag{2.3}$$

where $H_{0,0}(X)$, $H_{1,0}(X)$, $H_{0,1}(X)$, $H_{1,1}(X)$, $H_{2,0}(X)$, $H_{0,2}(X)$ are smooth functions vanishing at $X = 0$.

We make the following remark:

LEMMA 2.1. *Let $M \subset \mathbb{C}^2$ be a real smooth submanifold defined near $p = 0$ by $w = az + b\overline{z} + O\left(|z|^2\right)$. Then*

$$T_0^c M \neq \emptyset \Longleftrightarrow b = 0. \tag{2.4}$$

PROOF. We need to solve the equations $\partial f = \overline{\partial} f = 0$ at the point $z = w = 0$. We compute:

$$\partial f|_0 = \frac{\partial f}{\partial z}(0)dz + \frac{\partial f}{\partial w}(0)\,dw = -dw + adz, \quad \overline{\partial} f|_0 = \frac{\partial f}{\partial \overline{z}}(0)d\overline{z} + \frac{\partial f}{\partial \overline{w}}(0)d\overline{w} = bd\overline{z}. \tag{2.5}$$

We obtain $adz = dw$ and $bd\overline{z} = 0$. It follows that $p = 0$ is a CR singularity if and only if $b = 0$. □

We make a change of coordinates depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ preserving the CR singularity $p = 0$:

PROPOSITION 2.2. *There exists a biholomorphic change of coordinates in (z, w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ that sends (2.3) to a submanifold defined by*

$$w = z\overline{z} + \lambda(X)\left(z^2 + \overline{z}^2\right) + O\left(|z|^3\right), \tag{2.6}$$

preserving the CR singularity $p = 0$. Here $0 \leq \lambda(X) < \frac{1}{2}$ for $X \approx 0 \in \mathbb{C}^{N-2}$ and $\lambda(0) = \lambda$.

PROOF. We consider a local defining function for M near $p = 0$

$$\begin{aligned} f(z, X, w) =& -w + H_{0,0}(X) + \overline{z}H_{0,1}(X) + zH_{1,0}(X) + z\overline{z}\left(1 + H_{1,1}(X)\right) \\ &+ \left(\lambda + H_{2,0}(X)\right)z^2 + \left(\lambda + H_{0,2}(X)\right)\overline{z}^2 + \mathrm{O}\left(|z|^3\right). \end{aligned} \tag{2.7}$$

Each fixed $X \approx 0 \in \mathbb{C}^{N-2}$ defines us a real submanifold in $\mathbb{C}^2$ which may not have a CR singularity at the point $z = w = 0$ because $H_{0,1}(X)$ may be different than 0 (see Lemma 2.1). Therefore we need to make a change of coordinates in (z, w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ that perturbs the CR singularity $p = 0$. We consider the following equation

$$0 = \frac{\partial f}{\partial \overline{z}} = H_{0,1}(X) + (1 + H_{1,1}(X)) z + B(z, \overline{z}, X), \tag{2.8}$$

where $B(z, \overline{z}, X)$ is a smooth function. Since $H_{1,1}(0) = 0$, by applying the implicit function theorem we obtain a smooth solution $z_0 = z_0(X)$ for (2.8). By the translation $(w', z') = (w, z + z_0(X))$ the equation (2.3) becomes

$$w = zC_{1,0}(X) + z\overline{z}(1 + C_{1,1}(X)) + (\lambda + C_{2,0}(X)) z^2 + (\lambda + C_{0,2}(X))\overline{z}^2 + \mathrm{O}\left(|z|^3\right), \tag{2.9}$$

where $C_{1,0}(X)$, $C_{1,1}(X)$, $C_{2,0}(X)$, $C_{0,2}(X)$ are smooth functions vanishing at $X = 0$. Let $\gamma(X) = 1 + C_{1,1}(X)$, $\Lambda_1(X) = \lambda + C_{2,0}(X)$, $\Lambda_2(X) = \lambda + C_{0,2}(X)$. In the new coordinates $(w, z) := ((w - C_{1,0}(X) z)/\gamma(X), z)$ the equation (2.9) becomes

$$w = z\overline{z} + \Lambda_1(X) z^2 + \Lambda_2(X)\overline{z}^2 + \mathrm{O}\left(|z|^3\right). \tag{2.10}$$

Next, we consider a map $\Theta(X)$ such that $\Lambda_2(X) e^{-2i\Theta(X)} \geq 0$. Changing the coordinates $(w, z) := \left(w, ze^{i\Theta(X)}\right)$, we can assume $\Lambda_2(X) \geq 0$. Changing again the coordinates by $(w, z) := \left(w + (\Lambda_1(X) - \Lambda_2(X)) z^2, z\right)$ we obtain (2.6). □

We write

$$M : w = z\overline{z} + \lambda(X)\left(z^2 + \overline{z}^2\right) + P(z, X) + iK(z, X), \tag{2.11}$$

where $P(z, X)$ and $K(z, X)$ are real smooth functions. We prove an extension of Lemma 1.1 from [**14**]:

PROPOSITION 2.3. *There exists a holomorphic change of coordinates in (z, w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ in which K and its partial derivatives in z and $\overline{z}$ of order less or equal to l vanish at $z = 0$.*

PROOF. By making the substitution $(z'(X), w'(X)) = (z, w + B(z, X, w))$ and by (2.11) it follows that

$$M : w' = q(z, X) + P(z, X) + iK(z, X) + \mathsf{Re}\, B(z, X, w) + i\mathsf{Im}\, B(z, X, w), \tag{2.12}$$

where $q(z, X) = z\overline{z} + \lambda(X)\left(z^2 + \overline{z}^2\right)$. We want to make the derivatives in z of order less than l of $i(K(z, X) + \mathsf{Im}\, B(z, X, w))$ vanish at $z = 0$. Multiplying (2.12) by $i = \sqrt{-1}$, our problem is reduced to the following general equation

$$\mathsf{Re}\, B(z, X, q(z, X) + P(z, X) + iK(z, X)) = f(z, \overline{z}, X), \tag{2.13}$$

where $f(z, \overline{z}, X)$ is a real formal power series in $(z, \overline{z}, X)$ with cubic terms in z and $\overline{z}$ with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. We write

$$\begin{gathered} f(z, \overline{z}, X) = \sum_{m=3}^{l} f_m(z, \overline{z}, X), \ f_m(z, \overline{z}, X) = \sum_{j_1 + j_2 = m} c^m_{j_1, j_2}(X) z^{j_1}\overline{z}^{j_2}, \ c^m_{j_1, j_2}(X) = \overline{c^m_{j_2, j_1}(X)}, \\ B(z, X, w) = \sum_{m=3}^{l} B_m(z, X, w), \quad B_m(z, X, w) = \sum_{j_1 + 2j_2 = m} b^m_{j_1, j_2}(X) z^{j_1} w^{j_2}. \end{gathered} \tag{2.14}$$

We solve inductively (2.13) by using the following remark:

LEMMA 2.4. *The equation (2.13) has a unique solution with the normalization condition* $\mathsf{Im}\, B_m(0, X, u) = 0$.

PROOF. We define the weight of z to be 1 and the weight of w to be 2. We say that the polynomial $B_m(z, X, w)$ has weight m if $B_m\left(tz, X, t^2 w\right) = t^m B_m(z, X, w)$. Let $\mathbb{B}_m$ be the space of all such homogeneous holomorphic polynomials in (z, w) of weight m satisfying the normalization condition with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ and let $\mathbb{F}_m$ be the space of all homogeneous polynomials $f_m(z, \overline{z}, X)$ of bidegree (k, l) in $(z, \overline{z})$ with $k + 2l = m$ with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. We can rewrite (2.13) as follows

$$B_m(z, X, q(z, X) + P(z, X) + iK(z, X)) = B_m(z, X, q(z, X)) + \mathrm{O}\left(|z|^{m+1}\right). \tag{2.15}$$

In order to solve (2.15) it is enough to prove that we have a linear invertible transformation

$$\varphi(X) : \mathbb{B}_m \ni B_m(z, X, w) \mapsto \mathsf{Re}\, B_m(z, X, q(z, X)) \in \mathbb{F}_m, \tag{2.16}$$

depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. By Lemma 1.1 from [**14**] it follows that $\varphi(X)$ is invertible for $X = 0 \in \mathbb{C}^{N-2}$. By the continuity it follows that $\varphi(X)$ is invertible. If it is necessary we shrink the range of $X \approx 0 \in \mathbb{C}^{N-2}$. □

The proof is completed now by induction and by using Lemma 2.4. □

2.2. Some preparations. Let $w=u+iv$ and $I_\epsilon:=(-\epsilon,\epsilon)\subset\mathbb{R}$, for $0\leq\epsilon<<1$. We assume that M is defined by (2.11) and satisfies Proposition 2.3 properties.

In order to define a family of attached discs to the manifold M, we define the following domain

$$D_{X,r}=\left\{z\in\mathbb{C};\ v=0,\ q\left(z,X\right)+P\left(z,X\right)\leq u<\epsilon\right\}, \tag{2.17}$$

where $u=r^2$, which is a simply connected bounded set of $\mathbb{C}$ (see [**12**] for similar details). Therefore there exists a unique mapping $r\sigma_{X,r}:\Delta\to D_{X,r}$ such that $\sigma_{X,r}(0)=0$ and $\sigma'_{X,r}(0)>0$. Then, for $0<r<<1$ we can define the following family of curves depending smoothly on $X\approx 0\in\mathbb{C}^{N-2}$

$$\gamma_{X,r}=\left\{z\in\mathbb{C};\ q\left(z,X\right)+P\left(z,X\right)=r^2\right\}. \tag{2.18}$$

Next, we define the following family of analytic discs

$$\left\{\left(r\sigma_{X,r},X,r^2\right)\right\}_{X\approx 0\in\mathbb{C}^{N-2},\ 0<r<<1}. \tag{2.19}$$

The family of analytic discs shrinks to $\{0\}\times\mathcal{O}\times\{0\}$ as $r\mapsto 0$, where $0\in\mathcal{O}\subset\mathbb{C}^{N-2}$ and fills up the following domain

$$\widetilde{M_0}=\left\{(z,X,u)\in\mathbb{C}\times\mathbb{C}^{N-2}\times\mathbb{R};\ \|X\|<<1,\ q\left(z,X\right)+P\left(z,X\right)\leq u\right\}. \tag{2.20}$$

2.3. The Hilbert transform on a variable curve. Let $\gamma_{X,r}$ given by (2.18), where r is taken very small. For a function $\varphi_{X,r}(\theta)$ defined on $\gamma_{X,r}$ we define its Hilbert transform $\mathrm{H}_{X,r}\left[\varphi_{X,r}\right]$ to be the boundary value of a function holomorphic inside $\gamma_{X,r}$, with its imaginary part vanishing at the origin.

For $\alpha\in(0,1)$ we define the following Banach spaces:

$$\begin{aligned}\mathcal{C}^{\alpha}&:=\left\{u:\gamma_{X,r}\longrightarrow\mathbb{R};\ \|u\|_{\alpha}:=\sup_{x\in\gamma_{X,r}}+\sup_{x\neq y,\ x,y\in\gamma_{X,r}}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}<\infty\right\},\\ \mathcal{C}^{k,\alpha}&:=\left\{u:\gamma_{X,r}\longrightarrow\mathbb{R};\ \|u\|_{k,\alpha}:=\sum_{|\beta|\leq k}\left\|\mathrm{D}^{\beta}u\right\|_{\alpha}<\infty\right\}.\end{aligned} \tag{2.21}$$

We denote by $X:=\{x_2,y_2,\ldots,x_N,y_N\}$. The following result can be proved by using the same line as in [**14**] (see Theorem 2.5) or [**15**]:

PROPOSITION 2.5. *As $r\to 0$ and $X\approx 0\in\mathbb{C}^{N-2}$ we have*

$$\left\|\mathcal{H}_{X,r}\right\|_{j,\alpha}=O(r),\ \textit{for all } j\leq l-2;\quad \left\|\left(\partial_X^{|I|}\partial_r^{s}\right)\mathcal{H}_{X,r}\right\|_{j,\alpha}=O(1),\ \textit{for all } j+2s\leq l-4,\ I\in\mathbb{N}^{N-2}. \tag{2.22}$$

2.4. An implicit functional equation. During this section we work in the Holder space $\left(\mathcal{C}^{j,\alpha},\|\cdot\|_{j,\alpha}\right)$. We employ ideas from [**12**], [**13**], [**14**], [**15**] and we define the following auxiliary hypersurface

$$M_0=\left\{(z,X,u)\in\mathbb{C}\times\mathbb{C}^{N-2}\times\mathbb{R};\ \|X\|<<1,\ q\left(z,X\right)+P\left(z,X\right)=u<\epsilon\right\}, \tag{2.23}$$

where $\epsilon>0$ is small enough and $w=u+iv$. We consider the following normalized map

$$T=T\left[X\right]:=\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),\mathcal{B}\left(z,X,r\right)\right) \tag{2.24}$$

such that $T\left(M_0\right)\subseteq M$. Here $\mathcal{F}$, $\mathcal{B}$ are holomorphic functions in z and smooth in (X,r). It follows that

$$\mathcal{B}\left(z,X,r\right)|_{\gamma_{X,r}}=\left(q+P+iK\right)\left(z+z\mathcal{F}\left(z,X,r\right),X\right)|_{\gamma_{X,r}}, \tag{2.25}$$

where $\gamma_{X,r}$ is the curve defined by (2.18). By using the Hilbert transform on the curve $\gamma_{X,r}$ and by dividing by r^2 the equation (2.25), there exists a smooth function $V\left(X,r\right)$ such that

$$\begin{aligned}q\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)|_{\gamma_{X,r}}&=-P\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)|_{\gamma_{X,r}}\\&\quad-\mathcal{H}_{X,r}\left[K\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)\right]|_{\gamma_{X,r}}+V\left(X,r\right).\end{aligned} \tag{2.26}$$

We follow Huang-Krantz strategy from [**13**] and we define

$$\Omega\left(\mathcal{F}\left(z,X,r\right),X,r\right)=\frac{q\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)+P\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)}{r^2}|_{\gamma_{X,r}}. \tag{2.27}$$

By linearizing in $\mathcal{F}=0$ the functional defined in (2.27), the equation (2.26) becomes

$$1+\Omega'\left(\mathcal{F}\left(z,X,r\right),X,r\right)+\Omega_1\left(\mathcal{F}\left(z,X,r\right),X,r\right)+\frac{1}{r^2}\mathcal{H}_{X,r}\left[K\left(z\left(1+\mathcal{F}\left(z,X,r\right)\right),X\right)\right]|_{\gamma_{X,r}}-\frac{V\left(X,r\right)}{r^2}=0, \tag{2.28}$$

where $\Omega_1(\mathcal{F}(z,X,r),X,r)$, represents terms that are coming from the Taylor expansion of $P(z,X)$ and

$$\Omega'(\mathcal{F},X,r)=\frac{2}{r^2}\mathsf{Re}\,\{(q+P)_z(z,X)\,z\mathcal{F}\}\,|_{\gamma_{X,r}}. \tag{2.29}$$

We put the normalization condition $V(X,r)=r^2$. In order to find a solution $\mathcal{F}$ in the Holder space $\left(\mathcal{C}^{j,\alpha},\|\cdot\|_{j,\alpha}\right)$ for (2.28), we need to study the regularity properties of the functional Ω. We consider the following notation

$$\mathcal{C}_{X,r}(z)=\frac{2}{r^2}\mathsf{Re}\,\{(q+P)_z(z,X)\,z\}\,|_{\gamma_{X,r}}. \tag{2.30}$$

Since $\mathcal{C}_{X,r}(z)\neq 0$ for $|r|<<1$, $X\approx 0\in\mathbb{C}^{N-2}$, we can write $\mathcal{C}_{X,r}(z)=\mathcal{A}(z,X,r)\,\mathcal{B}(z,X,r)$ with

$$\mathcal{A}(z,X,r)=|\mathcal{C}_{X,r}(z)|,\quad \mathcal{B}(z,X,r)=\frac{\mathcal{C}_{X,r}(z)}{|\mathcal{C}_{X,r}(z)|}. \tag{2.31}$$

Then $\ln\mathcal{B}(z,X,r)$ is a well-defined smooth function in (z,X,r). Among the lines of [**13**] we define

$$\mathcal{C}^\star(z,X,r)=\frac{e^{i\mathcal{H}_{X,r}(\ln\mathcal{B}(z,X,r))}}{\mathcal{A}(z,X,r)}. \tag{2.32}$$

Then $\mathcal{C}^\star$ is a smooth positive function and $D(z,X,r):=\mathcal{C}^\star(z,X,r)\,\mathcal{C}(z,X,r)$ is holomorphic in z, smooth in (X,r). We write $D(z,X,r)\,\mathcal{F}(z,X,r)\equiv U(z,X,r)+\sqrt{-1}\mathcal{H}_{X,r}[U(z,X,r)]$. Since $D(z,X,r)\neq 0$ we can rewrite (2.28) as follows

$$\begin{aligned}U(z,X,r)=&-C^\star(z,X,r)\left(\Omega_1\left(\frac{U(z,X,r)+i\mathcal{H}_{X,r}[U(z,X,r)]}{D(z,X,r)},X,r\right)\right)\\&-C^\star(z,X,r)\,\frac{1}{r^2}\mathcal{H}_{X,r}\left[K\left(z\left(1+\frac{U(z,X,r)+i\mathcal{H}_{X,r}[U(z,X,r)]}{D(z,X,r)}\right),X\right)\right].\end{aligned} \tag{2.33}$$

We summarize all the precedent computations and we obtain the following regularity result

THEOREM 2.6. *The equation (2.33) has a unique solution in the Banach space* $\left(\mathcal{C}^{j,\alpha},\ \|\cdot\|_{j,\alpha}\right)$ *such that*

$$\|U\|_{j,\alpha}=O\left(r^{l-2}\right),\ \textit{for all } j\leq l-2;\qquad \left\|\left(\partial_X^{|I|}\partial_r^s\right)U\right\|_{j,\alpha}=O\left(r^{l-s-2}\right),\ \textit{for all } j+2s\leq l-4,\ I\in\mathbb{N}^{N-2}. \tag{2.34}$$

PROOF. The solution U and its uniqueness follows by applying the implicit function theorem. We denote by $\Lambda_1(U,X,r)$ and $\Lambda_3(U,X,r)$ the first and the last term and by $\Lambda_2(X,r)$ the term that does not depend on U from (2.33). Then $\|U\|_{j,\alpha}\leq\|\Lambda_1(U,X,r)\|_{j,\alpha}+\|\Lambda_2(X,r)+\Lambda_3(U,X,r)\|_{j,\alpha}\leq\|\Lambda_1(U,X,r)\|_{j,\alpha}+\mathrm{O}\left(r^{l-2}\right)\leq C\|U\|_{j,\alpha}^2+\mathrm{O}\left(r^{l-2}\right)$, for some $C>0$. It follows that $\|U\|_{j,\alpha}=\mathrm{O}\left(r^{l-2}\right)$.

The proof of the second regularity property goes after the previous line. By differentiating with r (2.33) we obtain that $\partial_r U=\partial_x(\Lambda_1(U,X,r))+\partial_U\Lambda_1(U,X,r)[\partial_x U]+\partial_r(\Lambda_2(X,r)+\Lambda_3(U,X,r))$. By Proposition 2.5 we obtain that $\|\partial_r U\|_{j,\alpha}=\mathrm{O}\left(r^{l-2-1}\right)$. We can take higher derivatives of r and since $P(z,X)=\mathrm{O}\left(z^3\right)$, $K(z,X)=\mathrm{O}\left(z^l\right)$, the differentiation of any order with x does not affect the estimates. Therefore the second estimates follow immediately. □

We write that

$$\mathcal{F}_{X,r}[\varphi_{X,r}]=\frac{U(z,X,r)+i\mathcal{H}_{X,r}[U(z,X,r)]}{D(z,X,r)}:=\varphi_{X,r}+i\mathcal{H}_{X,r}[\varphi_{X,r}], \tag{2.35}$$

where $\|\varphi_{X,r}\|_{j,\alpha}=\mathrm{O}\left(r^{l-2}\right)$, for all $j\leq l-2$ and $\left\|\left(\partial_X^{|I|}\partial_r^s\right)\varphi_{X,r}\right\|_{j,\alpha}=\mathrm{O}\left(r^{l-s-2}\right)$, for all $j+2s\leq l-4$, $I\in\mathbb{N}^{N-2}$.

3. The family of analytic discs and Proofs of Theorem 1.1 and Theorem 1.3

3.1. The family of analytic discs. We construct a continuous mapping T defined on $\widetilde{M_0}$ into $\mathbb{C}^2$ that is holomorphic in z for each fixed $u=r^2$ and that maps slice by slice the hypersurface M_0 into M. Let $\varphi_{X,r}$ be the function defined by (2.35). Then

$$\mathcal{F}_{X,r}[\varphi_{X,r}]=\varphi_{X,r}+i\mathcal{H}_{X,r}[\varphi_{X,r}],\quad \mathcal{B}_{X,r}[\varphi_{X,r}]=(q+P+iK)\left(z+z\mathcal{F}_{X,r}[\varphi_{X,r}],X\right). \tag{3.1}$$

We extend these functions to $\widetilde{M_0}$ by the Cauchy integral as follows

$$\begin{aligned}\mathcal{F}(\zeta,X,r)&=\mathcal{C}\left(\mathcal{F}_{X,r}[\varphi_{X,r}]\right)(\zeta)\equiv\frac{1}{2\pi i}\int_0^{2\pi}\frac{\mathcal{F}_{X,r}[\varphi_{X,r}](\theta)\,z_\theta(\theta,X,r)}{z(\theta,X,r)-\zeta}d\theta,\\ \mathcal{B}(\zeta,X,r)&=\mathcal{C}\left(\mathcal{B}_{X,r}[\varphi_{X,r}]\right)(\zeta)\equiv\frac{1}{2\pi i}\int_0^{2\pi}\frac{\mathcal{B}_{X,r}[\varphi_{X,r}](\theta)\,z_\theta(\theta,X,r)}{z(\theta,X,r)-\zeta}d\theta,\end{aligned} \tag{3.2}$$

where $z = z(\theta, X, r)$ is a parameterization of the curve $\gamma_{X,r}$ (see (2.18)).

We define T by (3.1). Then T is continuous by construction up to the boundary on each slice r=constant and X =constant. In order to obtain the regularity of T, we have to bound the derivatives in (z, X, u) of $\mathcal{F}$ and $\mathcal{B}$. We state the following lemma:

Lemma 3.1. *For all $j + 2s \leq l - 4$, $I \in \mathbb{N}^{N-2}$ as $r \mapsto 0$, we have*

$$\partial_\theta^j \partial_X^{|I|} \partial_r^s \mathcal{F}(z, X, r) = O\left(r^{l-s-2}\right), \quad \partial_z^j \partial_X^{|I|} \partial_r^s \mathcal{B}(z, X, r) = O\left(r^{l-s}\right). \tag{3.3}$$

The proof of the last lemma goes exactly after the line of Lemma 4.1 proof from [**14**].

Theorem 3.2. *Let M defined by (2.11) with $P(z, X) = O\left(z^3\right)$, $K(z, X) = O\left(z^l\right)$, $l \geq 7$, T extended by (3.2). Then $\widetilde{M} = T\left(\widetilde{M_0}\right)$ is a complex manifold-with-boundary regularly foliated by discs embedded of class $\mathcal{C}^{\frac{l-7}{3}}$.*

Proof. Since $\partial_u = \frac{1}{2r}\partial_r$, it follows that

$$\partial_z^j \partial_X^{|I|} \partial_u^s \mathcal{F}_{X,r}(z, X, r) = \mathrm{O}\left(r^{l-2s-j-2}\right), \quad \partial_z^j \partial_X^{|I|} \partial_u^s \mathcal{F}_{X,r}(z, X, r) = \mathrm{O}\left(r^{l-2s}\right), \tag{3.4}$$

and these derivatives remain bounded for all $j + 2s \leq l - 4$, $I \in \mathbb{N}^{N-2}$. It follows that the jacobian matrix DT of $T = T(X)$ is the identity matrix. ☐

3.2. Proof of Theorem 1.1. Let M, $\widetilde{M}$, T as in Theorem 3.2. By applying an extended reflection principle (see [**15**]) we construct smooth extension of T past every point of $M_0 - \{0\}$. By similar arguments as in [**14**] we obtain that $M \cup \widetilde{M}$ is a smooth manifold-with-boundary in a neighborhood of the point $p = 0$.

3.3. Proof of Theorem 1.3. Since the hypersurface given by Theorem 2.1 is Levi-flat it follows each of our analytic discs is a reparameterization of an analytic disc contained inside of it. Conversely: Let $a \in N$, where N is as in Theorem 1.2. Since N is a manifold with boundary M and since M is foliated by a reparameterization the family of curves given by (2.18), it follows that there exists an reparameterized analytic disc that contains inside of it a. It follows that the under the hypothesis of Theorem 1.2, the hypersurfaces given by Theorem 1.1 and Theorem 1.2 are the same.

We can study now the hull of M near $p = 0$ when M is assumed to real-analytic. The hypersurface M_0 defined by (2.23) is foliated by a the family of analytic discs defined by (2.19). Then $\widetilde{M}$ is foliated by the family of analytic discs defined by (3.2). Then among the lines of [**12**], [**13**] the following map

$$\Psi : \widetilde{M_0} - \{0\} \longrightarrow \bigcup_{0<r,\ \|X\|<<1} T[X]\left(r, \overline{\Delta}\right), \quad \Psi(z, u) = T[X](r, \xi(u, z)), \tag{3.5}$$

where $\xi(u, z)$ is determined by the equations $z = r\sigma_{X,r}(\xi)$ and $r = \sqrt{u}$, has an extension that is $\mathcal{C}^1$ across $0 \in \widetilde{M_0}$ and real-analytic over $\widetilde{M_0} - M_0$. The proof of Theorem 1.3 follows immediately by the same arguments as in [**12**], [**13**].

References

[1] **Bedford, E.; Gaveau, B.** — Envelopes of holomorphy of certain 2-spheres in $\mathbb{C}^2$. *Amer. J. Math.* **105** (1983), 975–1009.

[2] **Bedford, E.; Klingenberg, W.** — On the envelopes of holomorphy of a 2-sphere in $\mathbb{C}^2$. *J. Amer. Math. Soc.* **4** (1991), 623–646.

[3] **Bishop, E.** — Differentiable Manifolds In Complex Euclidian Space. *Duke Math. J.* **32** (1965), no. 1, 1–21.

[4] **Dolbeault, P.; Tomassini, G.; Zaitsev, D.** — On Levi-flat hypersurfaces with prescribed boundary. *Pure and Applied Mathematics Quarterly* **6** (2010), no 3, (*Special Issue: In honor of Joseph J. Kohn. Part 1*), 725–753.

[5] **Dolbeault, P.; Tomassini, G.; Zaitsev, D.** — Boundary Problem for Levi-Flat Graphs, http://de.arxiv.org/pdf/0912.1332

[6] **Helmes, L.** — *Potential Theory.* Spriger-Verlag London, (2009).

[7] **Hormander, L.** — *An introduction to Complex Analysis in Several Variables. Second Revisited edition. North-Holland Library, Vol. 7.* North-Holland Publishing Co., Amsterdam-London; American Elvesier Publishing Co., Inc., New York, (1973).

[8] **Huang, X.; Yin, W.** — A Bishop surface with a vanshing Bishop invariant. *Invent. Math.* **176** (2010), no 3, 461-520.

[9] **Huang, X.; Yin, W.** — A codimension two CR singular submanifold that is formally equivalent to a symetric quadric. *Int. Math. Res. Not. IMRN* (2009), no 15, 2789-2828.

[10] **Huang, X.** — *Local Equivalence Problems for Real Submanifolds in Complex Spaces.* Lecture Notes in Mathematics, Spriger-Verlag, pp. 109-161, Berlin-Heidelberg-New York, (2004).

[11] **Huang, X.; Yin, W.** — Equivalence problem for Bishop surfaces. *Sci. China Math.* **43** (2010), no 3, 687-700.

[12] **Huang, X.** — On a n-manifold in $\mathbb{C}^n$ near an elliptic complex tangent. *J. Amer. Math. Soc.* **11** (1998), no 3, 669-692.

[13] **Huang, X.; Krantz, S.** — On a problem of Moser. *Duke Math. J.* **78** (1995), no 1, 213-228.

[14] **Kenig, C; Webster, S.** — On the local hull of holomorphy of an n-manifold in $\mathbb{C}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **19** (1984), no. 2, 261-280.

[15] **Kenig, C; Webster, S.** —The hull of holomorphy of a surface in the space of two complex variables. *Inv. Math.* **19** (1982), no. 2, 261-280.

[16] **Mather, J.** — *On Nirenberg's proof of Malgrange's preparation theprem.* Proceedings of Liverpool Singularities Symposium 1, Lec. Notes in Math., Springer-Verlag **192**, (1971), 116-120.

[17] **Moser, J.** —Analytic Surfaces in $\mathbb{C}^2$ and Their Local Hull of Holomorphy. *Ann. Acad. Sci. Fenn. Ser. A Math.*, **10** (1985), 397-410.

[18] **Moser, J.; Webster, S.** — Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface tranformations. *Acta Math.* **150** (1983), 255–296.

[19] **Nirenberg, L.; Webster, S.; Yang, P.** — Local Boundary Regularity of Holomorphic Mappings. *Comm. Pure Appl. Math.* **33** (1980), 305-338.

V. Burcea: School of Mathematics, Trinity College Dublin, Dublin 2, Ireland
E-mail address: valentin@maths.tcd.ie