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Quantisation Revisited: A Mathematical and Computational Model

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A nascent theory of near division is presented, from which an efficient quantisation algorithm for rhythm intervals is derived. Based on a number theoretic analysis of the uniqueness and convergence of this first algorithm, a generalised algorithm is presented. An empirical study of the algorithm's performance reveals a readily computable criterion within which the perceived ratio may reliably be produced on real performance data. Distribution properties are shown to be reasonable for computation.

Keywords: Inter-onset interval; quantisation; ratio; algorithm; mathematical computational model; round rational approximation

MCS/CCS/AMS Classification/CR Category numbers: 00A65 97M80 14E08 97F40 97F80

1. Introduction

Consider a sequence of Inter-Onset Interval (IOI)s in milliseconds. For each pair of consecutive IOIs, it will be claimed that there is a limited number of plausible ratios that may reasonably be perceived, without considering any other information than the onset times. At the core of the paper is an algorithm to generate the set of all such possible interpretations for each pair of consecutive intervals. Calculation is based on a formulation of approximation that is amenable to the constraints of the model.

The quantisation work presented here is perhaps unorthodox in that it does not presume a tactus or metre at the outset. It is in fact developed for a system for metre inference and beat tracking wherein higher-level structure is primarily composed of lower-level structure that has already been inferred (attempting to avoid the circular dependency of some earlier musical theories). Certainly some feedback from higher levels to lower levels

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may be present, but the micro-temporal structure is in place (albeit provisionally) before constructing the macro-level structure comprising it.

The modelling requirements are threefold: a need for (1) a complete set of plausible ratios, (2) computational feasibility, and (3) psychological plausibility. The paper will focus mainly on the first two criteria. The second criterion partly depends on requirements downstream of the work presented here: this will be discussed later but due attention is given to computability of the algorithm proposed herein. Discussion of the relevant literature from psychology of perception and cognitive science is beyond the scope of this paper [1–7].

The main modelling assumptions are: (1) a reliable stream of IOIs is available, (2) only ratios of a certain form may be perceived.

After a brief recollection of related research, the concept of a *plausible ratio* is defined and discussed. Calculation of plausible ratios is based on a nascent theory of *near division*, which is then introduced. The next section derives an algorithm for producing plausible ratios under certain criteria. Its behaviour is then analysed and used to explain a more general version of the algorithm. The adequacy and typical performance of the general algorithm on performance data are then presented and discussed.

2. Related Work

Contrary to the approach of this paper of constructing higher-level structure from lower-level structure, many published approaches to quantisation try to coerce the IOIs so as to align with previously computed higher level structure such as metre, phrase grouping, or repeated patterns. Others treat entire sections all at once, and as such are not psychologically plausible as models [8]. Such approaches will not be considered further: a far-reaching survey article of general computational approaches to metrical analysis is presented in [9].

Although the presented quantisation algorithm assumes the availability of reliable note onset times, it is worth noting other beat tracking systems that do not. Sethares et al. [10], operating on an audio stream, first compute a number of reduced streams or *rhythm tracks*. Each rhythm track is a signal processing feature offering an interpretation of what may constitute a phenomenal accent. No attempt is made to infer metrical structure. They compare two different stochastic models operating on the rhythm tracks: Bayesian particle filter and gradient descent. Quite another approach, [11], uses adaptive oscillators or resonators, operating directly on an audio source. Both approaches are quite successful without reliable onsets or inducing metre.

Longuet-Higgins [5, 12], presented a quantiser that is primed with the mean interval (and optionally subdivisions) of the meter. It reconciles observed onsets with expected beats within a fixed tolerance. A mean of the observed onset and the expected beat point for the current metrical subdivision is used to update the forthcoming beat-point expectations. Duple and triple metrical subdivisions are both available as alternatives. Both quantisation and a degree of metre inference are performed in tandem. It does not claim to be a comprehensive model, although it performs remarkably well in as far as it goes.

The connectionist quantisation model of Desain and Honing [13] requires priming with the tactus and adjusts floating point IOI values so as to reduce discrepancy between close values. (A quantiser such as the one presented here would still be necessary to produce whole number ratios.) Desain [14] argued for the importance of what he termed the decomposability of holistic theories into constituent low-level components, such that the theory may be used to estimate individual intervals and be compared with performance data. He showed how their earlier work [13] may be so decomposed. By contrast, the approach of this paper is bottom-up, facilitating construction of higher-level structure from the lower-level constructions presented.

Fraise [4] may have been the first to point out the role of low primes in rhythmic ratios. This relative simplicity facilitates a richness of gestural expressive timing. Resolving data into low numbers occurs for other aspects of musical structure [5, 15].

In the realm of perception, the first call for formalisation of metrical cognition may have been Simon [16]. Fraise distinguished between perception ("immediate" phenomena) and estimation (involving memory) of time, and identified a preferred interval, ~600 ms, for estimation: shorter estimates tend to be overestimated and vice versa [6]. The "law of low numbers" appears elsewhere, e.g. [7].

We will be concerned with *round* ratios. As discussed by Hardy & Wright [17], a number is called *round* if it is "the product of a considerable number of comparatively small [prime] factors". By natural analogy, this extends to ratios as consisting of two round factors. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime decomposition of $n \in \mathbf{N}$. Define $\omega(n) = k$ and $\Omega(n) = \sum_{i=1}^{k} a_i$. It is proved in [17] how $\omega(n)$ and $\Omega(n)$ are both approximated by $\log\log n$. Thus round ratios are very sparse in $\mathbf{Q}$: the analogy with plausible ratios being sparse among possible, approximate ratios is proposed.

3. Plausible Ratios and Near Division

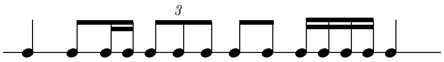
Table 1 presents sample performance data from the literature, e.g. [13]. IOIs are listed, along with their desired quantised values, $\langle q_i \rangle_i$, derived from the desired ratios, $\langle q_i/q_{i+1} \rangle$, which may be chosen from among the plausible ratios produced for each consecutive IOI pair. The criteria used to determine whether a given ratio is "plausible" or not are based on the devised criterion of *near-division*. The rationale is that the exact ratios of pairs of consecutive IOIs generally involve rather larger factors than the corresponding perceived ratios. The difference between the measured and perceived ratios may be accounted for by introducing some "slack", i.e. tolerating a certain amount of inaccuracy, in order to infer which ratio(s) could possibly have been intended. However, once any amount of slack is introduced into the concept of integer division, were this criterion to be used alone, then an unlimited number of close but different ratios could be claimed for any given pair of numbers. In keeping with the approach to modelling stated in § 1, this is restricted to favour the more round ratios. The most important constraint is Hyp. 3.1 below.

When considering rhythm of polyphonic music, the question arises as to whether to consider IOIs as they occur within a given voice, melodic line, or instrument, or whether to consider onsets grouped together across all such voices. The latter approach is taken and will be termed *pan voice*.

Hypothesis 3.1 *At least one of the factors of a perceived ratio between two pan-voice IOIs must be a power of two.*

This is a modelling hypotheses: the algorithms have been crafted to conform to it. While its validity has no bearing on the mathematical logic of the results that follow,

Table 1. Plausible ratios for consecutive IOI pairs



$\langle q_i/q_{i+1}\rangle_i$	2	2	1	3/4	1	1	2/3	1	2	1	1	1	1/4	
$\langle q_i\rangle_i$	12	6	3	3	4	4	4	6	6	3	3	3	3	12
IOIs	1177	592	288	337	436	337	387	600	634	296	280	296	346	1193
plausible ratios	4	4	2/3	2/3	2	4/5	2/3	4/3	4	4/3	4/3	2/3	2/7	
	4/3	4/3	4/5	4/5	4/3	3/4	4/5	4/5	4/3	4/5	4/5	4/5	4/13	
	9/4	9/4	3/4	3/4	5/4	1	4/7	5/4	9/4	5/4	5/4	3/4	4/15	
	7/4	7/4	1	1/2	3/2		3/4	3/4	2	3/4	3/4	1	1/4	
	2	2		1	1		1/2	1		1	1		1/2	

Chopin, Mazurka, Op. 63, No. 2

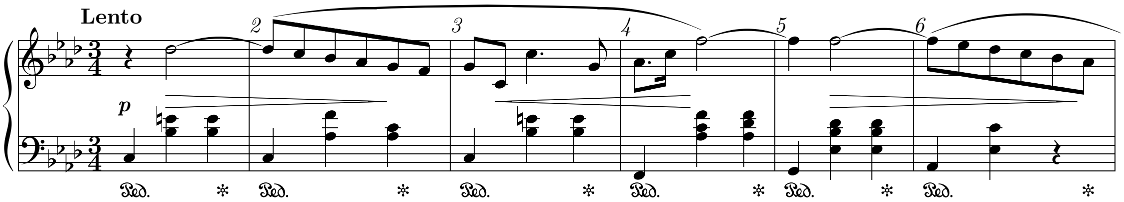


Figure 1. Counterexample for Hyp. 3.1 for intra-voice IOIs

practical application of the algorithms (which apply these results) will be compromised if it should turn out to be false. In practice this would mean that not all desired ratios would be produced. It is simply a matter of tentative observation that this hypothesis appears to hold without exception for the metrical intervals of tonal music considered. The most round ratios rejected by this assumption are 3:5 and 5:3.

Hypothesis 3.1 does not necessarily hold for IOIs within a given voice of polyphonic music: Fig. 1 gives a counterexample. The  and  of the right hand of bars 4–6 are in the ratio 6:5, yet each consecutive pair of pan-voice IOIs is in one of the simple ratios 1:1, 1:2, 2:1, 2:3, 3:1 or 1:4. An unaccompanied, monophonic melody bears the onus of outlining the metre and so, under this hypothesis, a ratio of 6:5 would rather be interpreted as, e.g. 1:1 with expressive timing or an accelerando.

Another constraint on the number of plausible ratios is given by limiting the range of the aforementioned power of 2. In general, as the sequence $1, 2, 4, 8, 16, \ldots$ increases, a ratio involving the latest member as a factor will tend to become less round if the opposite factor increases correspondingly in order to approximate the original ratio more accurately.

To define near division, let $a, b \in \mathbf{N}$, and recall that a divides b, written $a \mid b$, iff $\exists\, q \in \mathbf{N}$ such that $aq = b$. For $a \geq 1$, $b \in \mathbf{Z}$, recall also that $r = (b \bmod a)$ means that $r \in \mathbf{N}$, $0 \leq r < a$, and $\exists\, q \in \mathbf{Z}$ such that $b = aq + r$. The two concepts are related by the statement $a \mid b$ iff $b = (0 \bmod a)$. Exact integer division may be relaxed by considering "a divides b within a factor of Δ" as follows.

Definition 3.2 *Let $a, b \in \mathbf{N}$ and fix a constant $\Delta \in [0, \frac{1}{2}) \subset \mathbf{R}$. We write $a \wr_\Delta b$ iff $a \mid c$ for some c satisfying $|b - c| \leq \Delta a$.*

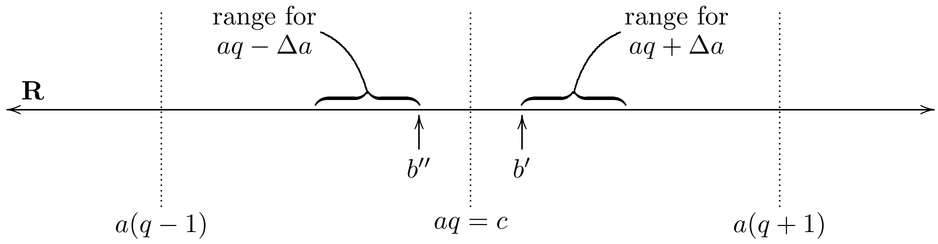


Figure 2. Sketch of cases in proof of Theorem 3.3

Definition 3.2 plainly formulates the desired intuitive concept, but it gives rise to the question of how to establish the existence or otherwise of such a "dividend" c given a, b. This is addressed by the following result.

THEOREM 3.3 *Let* $a, b \in \mathbf{N}$ *and* $\Delta \in [0, \frac{1}{2})$. *Then* $a \wr_\Delta b$ *iff either* $(b \bmod a) \leq \Delta a$ *or* $(b \bmod a) \geq (-\Delta a \bmod a)$.

Proof: To prove the first part ($\Rightarrow$), say that $aq = c$ and $|b - c| \leq \Delta a$. Then $-\Delta a \leq b - aq \leq \Delta a$, which gives

$$aq - \Delta a \leq b \leq aq + \Delta a. \tag{1}$$

Say we have $b \geq aq$, as depicted in Fig. 2 by $b = b'$. Then $aq \leq b \leq aq + \Delta a$ and, since $\Delta a < a$, we must have $(b \bmod a) \leq \Delta a$, by definition of mod. Otherwise it must be the case that $aq - \Delta a \leq b < aq$, depicted by $b = b''$ in Fig. 2, and since $\Delta a < a$ we have

$$a(q-1) < aq - \Delta a \leq b < aq \tag{2}$$

which proves the first part.

To prove the converse ($\Leftarrow$), say that $(b \bmod a) \leq \Delta a$. Then $\exists\, q, r \in \mathbf{N}$ such that $aq + r = b$, $0 \leq r < a$, and $r \leq \Delta a$. That is, $0 \leq b - aq = r \leq \Delta a$ and the required dividend c is given by aq.

Now say that $(b \bmod a) \geq (-\Delta a \bmod a)$. This means that $\exists\, q, r \in \mathbf{N}$ such that

$$aq + r = b, \quad 0 \leq r < a, \tag{3}$$

and

$$r \geq (-\Delta a \bmod a) = a - \Delta a \tag{4}$$

since $0 \leq \Delta a < a$. So we have

$$0 < a(q+1) - b = a - r \qquad \text{by (3),} \tag{5a}$$

$$\leq \Delta a \qquad \text{by (4),} \tag{5b}$$

and the required dividend c is given by $a(q+1)$. ■

Definition 3.2 and Theorem 3.3 require that $0 \leq \Delta < \frac{1}{2}$. In fact, the proof of Theorem 3.3 holds for $\Delta \geq \frac{1}{2}$, but then $a \wr_\Delta b\ \forall\, a, b \in \mathbf{N}$ and the concept degenerates as it stands. We will see that there may be sufficient metrical structure to allow occasional deviations of b exceeding $\pm a/2$, but that the integrity of the near division concept is nonetheless sufficient to be instrumental in the calculation for the case of arbitrary Δ.

4. Calculation of Plausible Ratios (Converging)

A simplified version of the algorithm for generating plausible ratios follows. It takes a pair of IOI values (such as those in Table 1) and returns a list of all the plausible ratios between them (listed below each pair of consecutive IOI values in Table 1). This is applied in turn to consecutive pairs of IOIs.

Near division has been formulated by Definition 3.2 in such a way as to allow one factor of a prospective ratio to be set while Theorem 3.3 tests all possible matches in the other factor within the corresponding range specified by Δ. This formulation facilitates direct calculation of round ratio approximations subject to Hyp. 3.1 and the parameters Δ, M. Only integer arithmetic is necessary and so the algorithm permits a very efficient implementation.

Algorithm 4.1 For $\Delta \in [0, \frac{1}{2})$, $M, n_1, n_2 \in \mathbf{N}$, $n_1, n_2 > 0$, $\texttt{PlausibleRatios1}_{\Delta,M}(n_1, n_2)$ returns a set of plausible ratios. An annotated listing is given in the supplementary electronic appendix A.1.

Algorithm 4.2 For $a, b, \Delta_a \in \mathbf{N}$, $a, b > 0$, $\texttt{NearDivides}(a, b, \Delta_a)$ returns c such that $a \,|\, c$ and $|b - c| \leq \Delta a$, or else null. A listing is given in the supplementary electronic appendix A.1.

Figure 3 plots a geometrical interpretation of Algorithm 4.1 for (3,7) with $\Delta = \frac{1}{3}$, $M = 2$. The green bands mark the acceptable ranges about the exact ratio as a small red circle. The values produced, prior to "cancelling down", are indicated by the red crosses. The three values returned, viz., $\{\frac{1}{2}, \frac{2}{5}, \frac{4}{9}\}$, have their slopes plotted to the points that produced them. Two superfluous points (i.e. $c \neq \emptyset$ and $Q_i = Q_{i-1}$ in Steps 5,6) have their slopes as dotted lines. At $2^0 \cdot 7$ on the abscissa, it can be seen that there was no multiple of 7 within 3 ± 2 along the ordinate. The decreasing angles subtended by the green bands in Fig. 3 suggest the following result.

Lemma 4.3 *Taking n_1, n_2, m as in Algorithm 4.1, the values produced by (Step 6 of) Algorithm 4.1$_\Delta(n_1, n_2)$ converge to n_1/n_2 with increasing m:*

$$\lim_{m\to\infty} \frac{2^m n_1}{c_m} = \frac{n_1}{n_2} = \lim_{m\to\infty} \frac{c'_m}{2^m n_1}$$

whenever c, c' are defined for m, where c is the value produced by Algorithm 4.2 during the first pass of the outer loop ($p = 1$) and c' that for the second pass ($p = -1$).

Proof: The proof is given in the supplementary electronic appendix A.3. ■

Corollary 4.4 *The rate of convergence of Algorithm 4.1$_\Delta(n_1, n_2)$ decreases with Δ.*

Proof: The proof is given in the supplementary electronic appendix A.3. ■

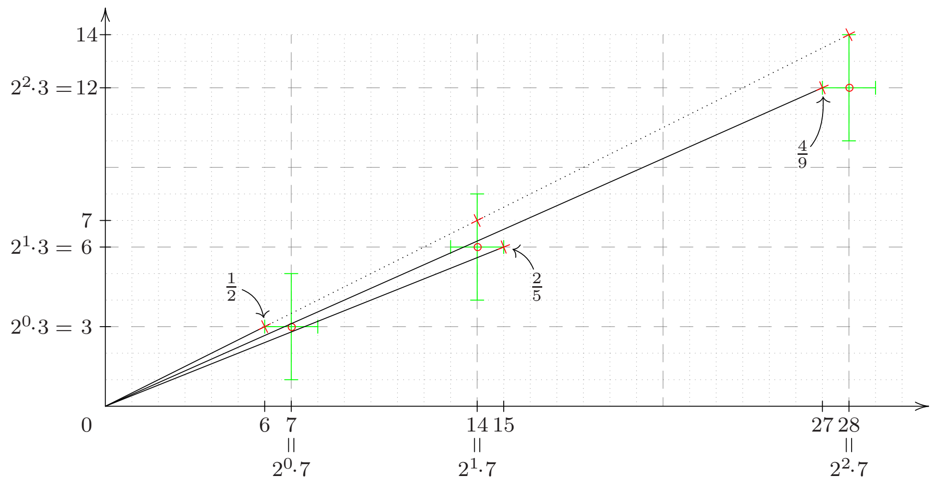


Figure 3. Geometrical interpretation of Algorithm $4.1_{\frac{1}{3}}(3, 7)$.

Note that Lemma 4.3 is contingent upon the existence of plausible ratios. In theory, this may or may not be the case. The case of extreme convergence is given by $n_1 \mid n_2$ (or $n_2 \mid n_1$) when $n_1 \wr_\Delta n_2$ (or $n_2 \wr_\Delta n_1$) for $M = \Delta = 0$ and convergence is immediate. The opposite extreme is given by the austere case of the following lemma.

Lemma 4.5 *Given primes $p_1, p_2 > 2$, $p_1 \neq p_2$ and $\Delta = 0$, then $\nexists M \in \mathbf{N}$ for which Algorithm $4.1_\Delta(p_1, p_2)$ returns a value.*

Proof: Assume, to the contrary and without loss of generality, that $\exists\, m \in \mathbf{N}$ such that $p_1 \mid 2^m p_2$. Then $\exists\, c_m \in \mathbf{N}$ such that $c_m p_1 = 2^m p_2$; that is,

$$c_m \frac{p_1}{p_2} = 2^m, \tag{6}$$

and c_m must be a multiple of p_2 in order that the left hand side be a natural number. Thus the scenario reduces to the statement that $p_1 \mid 2^m$ which cannot be the case by the Fundamental Theorem of Arithmetic. ■

Whenever a plausible ratio does exist, there is a sense in which it is unique. This will be of crucial importance in § 6.

Theorem 4.6 *Let $a, b \in \mathbf{N}$ and $\Delta \in [0, \frac{1}{2})$. There is at most one $c \in \mathbf{N}$ such that $a \mid c$ and $|b - c| \leq \Delta a$.*

Proof: If $a \mid c$ and $|b - c| \leq \Delta a$ then $\exists\, q \in \mathbf{N}$ such that $aq = c$. Suppose also that $\exists\, c' \in \mathbf{N}$, $c' \neq c$, $|b - c'| \leq \Delta a$ and $aq' = c'$ for some $q' \in \mathbf{N}$. If $a = 0$ or $\Delta = 0$ then $c = b = c'$.

Suppose then that $a, \Delta > 0$. Since $|b - c| < a/2 > |b - c'|$, it follows that $|c - c'| < a$ by the triangle inequality. That is, $|aq - aq'| < a$, whence $a|q - q'| < a$ and $|q - q'| < 1$. But then $q = q'$ since $q, q' \in \mathbf{N}$, giving $c = c'$. ■

5. Behaviour (Converging)

The number of plausible ratios generated is critical: the algorithm must be liberal enough to always include the preferred interpretation yet conservative enough to prevent computational explosion when the ratios are considered in combination. Consequently, the behaviour of Algorithm $4.1_{\Delta,M}$ will now be examined, first in terms of all possible ratios within a given range and then in terms of the specific values arising from the test corpus.

5.1. Test Corpus

The Kostka-Payne (KP) corpus [18] was used for empirical evaluation. It was constructed by Temperley to provide the field with a means to quantify performance of beat trackers. It contains both quantised and performance data for pieces spanning the "common practice era of tonal music", taken from [19].

The MIDI files of the performance corpus are parsed and assembled into an internal data structure. Each MIDI file is supplemented (by the author [20]) with a control file specifying: (1) filtering of extra-metrical notes and (2) clustering of near-simultaneous onsets into single nominal onsets.

Flagging of notes as extra-metrical is typically used for grace notes occurring out of any metrical context such as turns and trills. Although such grace notes are usually highly perceptually salient, they typically have no bearing on the induced metre and are best ignored as far as beat tracking is concerned. This facility is also useful for cleaning key bounces and occasional spurious events (those neither in the score nor near any beat). These events marked for cleaning are of near-zero MIDI velocity so that they cannot be heard and are considered noise in the data.

After the removal of extra-metrical events (if any), the list of onset clusters is processed. Events specified as belonging together in a single chord are allocated a single nominal onset point. Grace notes such as acciaccature and mordents which are distributed tightly around a discernible metrical beat division are typically marked as belonging to the nominal onset of that beat division.

The timing of grace notes is something of a subject unto itself [21], tending to prefer certain interval ranges and expressive timing deviation rather than metrical conformance. Most grace notes in the KP performed corpus are not removed but rather blurred together with their associated nominal onset in the prepared files used.

Even disregarding extra-metrical notes, the temporal spread of recorded onsets considered as constituting a single nominal onset (such as arpeggiation) ranges up to 514 ms while nominally-distinct IOIs range down to 48 ms. This overlap of an entire order of magnitude across the corpus precludes such heuristics as used by [22] (a 70 ms threshold) or [23] (pre-quantising all events to the nearest 35 ms pip) if reliable nominal onset identification is desired, as is the case for testing this model.

5.2. Distribution Profile

It has been shown that there exist extreme cases under which the set of all plausible ratios generated is either trivial or empty. In practice, the concern will be with more

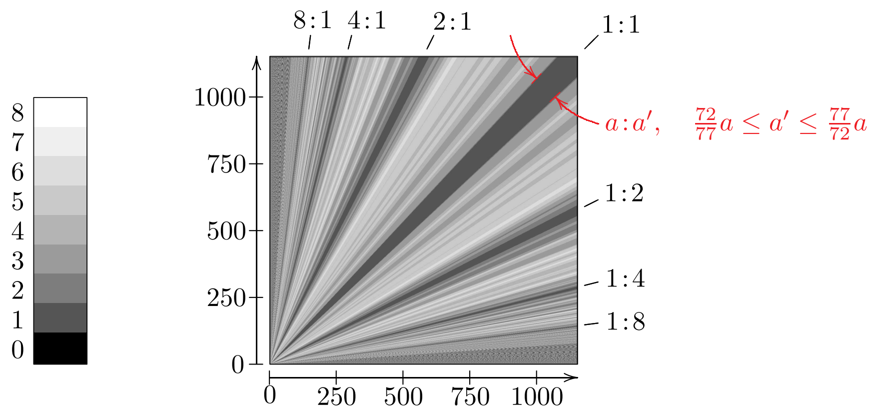


Figure 4. Each pixel in the graphic array represents the number of plausible ratios generated for its coordinates as a greyscale value from the key to the left. The array shown is for $M = 3$, $\Delta = \frac{4}{9}$, and the number of ratios range from 1–6.

reasonable cases that produce a variable number of ratios. Some preliminary statistics of the number of ratios generated are plotted in the supplementary electronic appendix B.1. As one might expect, the number of ratios generated increases monotonically both with increasing Δ and increasing M.

Figure 5 shows more specifically where the effect of the parameters Δ, M take place. Each panel represents the number of plausible ratios generated by Algorithm $6.1_{\Delta,M}$ on P_{1152} where $P_n = \{\langle i,j\rangle \in \mathbf{N}\times\mathbf{N} \mid 1 \le i,j \le n\}$. Values are plotted as a greyscale value according to the key given in Fig. 4. The constant values along radii from the origin account for the convergence with increasing n of the previous paragraph. Of particular note are the dark bands centred on $1\!:\!2^m$ for $m = -M,\dots,0,\dots,M$, and the variegated nature of the regions between them. It will be shown empirically in § 7 how these attributes are undesirable. For now, the reason for the troughs will be explained.

Let $a \in \mathbf{N}$ and $m \in \{0,\dots,M\}$. Clearly, $a \mid 2^m a$, so at least one ratio will always be generated for $\langle a, 2^m a\rangle \in P$. To see why not more than one ratio will be generated in this neighbourhood, consider $\langle a, 2^l a + h\rangle \in P$ for $l,h \in \mathbf{N}$, $0 \le l \le M$, $0 < h < a$ and let $k \in \{0,\dots,\lfloor\Delta a\rfloor\}$. Now $a \nmid 2^m(2^l a + h) + k$ if $2^m h + k < a$, which is certainly the case if $h < \lfloor(1-\Delta)a\rfloor/2^M$. The case for $2^l a - h - k$ is identical but for sines, giving a sector of approximately $|h| < 2^{-M}(1-\Delta)a$. For the second pass of the algorithm, when $p = -1$, consider $\langle 2^l a + h, a\rangle \in P$. Now $2^l a + h \mid 2^m a + k$ if $q(2^l a + h) = 2^m a + k$, i.e. if $(q-\Delta)h \approx \big(2^m + 2^l(\Delta - q)\big)a$, so that $h \approx \big[\big(2^m + 2^l(\Delta-q)\big)/\big(q-\Delta\big)\big]a$. The guaranteed ratio of 1:1 will be produced for $q = 2^{m-l}$, and it may be shown that the closest ratio thereto occurs for $\Delta = 4/9$, $M = 3$, $l = 0$ at $q = 9$, $m = 3$ with $h = -5a/77$. The first pass under the same conditions gives $|h| < 5a/72$. The two passes together thus yield a sector of $a{:}a'$ for $h_0 a \le a' \le h_0^{-1}a$, $h_0 = 72/77$ when $\Delta = 4/9$, $M = 3$, $l = 0$, as depicted in Fig. 4.

Every single preferred ratio is produced for the 3 465 pairs of consecutive pan-voice IOIs in the 41 KP quantised corpus excerpts. This supports Hyp. 3.1 as a sufficient condition. There is only a single instance of a ratio requiring $M = 3$. An analysis is given in the supplementary electronic appendix B.2.

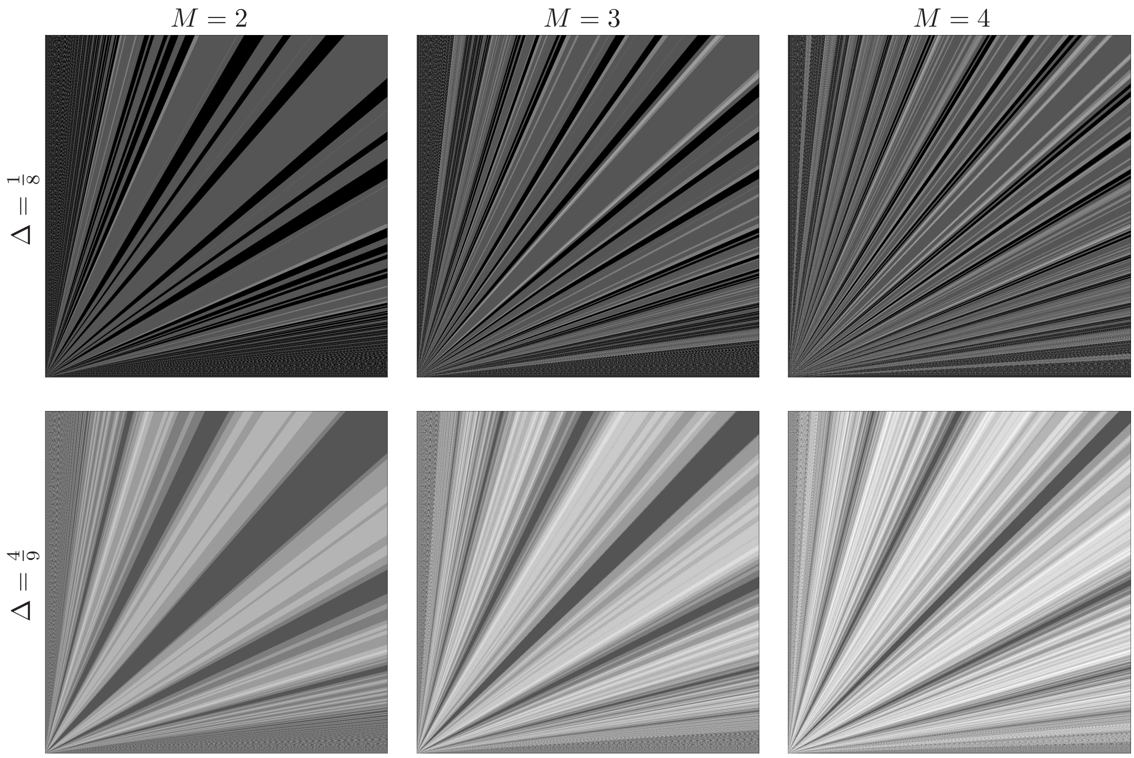


Figure 5. Distribution of number of plausible ratios for given Δ, M. Each panel is constructed according to the same greyscale key given in Fig. 4.

6. Calculation of Plausible Ratios (Diverging)

As mentioned at the end of § 3, it turns out that not all preferred, perceived ratios are produced by Algorithm 4.1$_\Delta$ with $0 \leq \Delta < \frac{1}{2}$ on performance data. (The preferred perceived ratios are indeed produced for a more metronomic rendition, but not in all instances of real performance data.) Algorithm 6.1 lists a version generalised for arbitrary $\Delta \geq 0$ and without the convergence explained in Lemma 4.3. Although this convergence may be seen as a limitation in terms of the spread of ratios generated, it may also be taken as a strength in terms of the uniqueness explained in Theorem 4.6. The uniqueness of each power-of-two factor (before cancellation) within its range is harnessed in Algorithm 6.1$_\Delta$. This range is extended at each power-of-two step so as the angle subtended by their ranges remains constant[1] at that of the case $m = 0$. The extension may be made in "window" steps of $2\lfloor \Delta a \rfloor + 1$ within these bounds, safe in the knowledge that at most one ratio may emerge each time. Thus, a complete set of ratios is produced in keeping with the modelling rationale of § 1. A geometrical interpretation is given in Fig. 6. By comparing with Fig. 3, it can be seen how the additional ratios of $\frac{1}{4}, \frac{3}{4}$ are generated.

ALGORITHM 6.1 For $\Delta \in \mathbf{R}^+$, $M, n_1, n_2 \in \mathbf{N}$, $n_1, n_2 > 0$, $\texttt{PlausibleRatios}_{\Delta,M}(n_1, n_2)$ returns a set of ratios. An annotated listing is given in the supplementary electronic appendix C.1.

[1]Semantically constant, that is. Technically, the un-rounded range θ is approached as $m \to \infty$, where $\theta = \tan^{-1}[a/(b - \Delta a)] - \tan^{-1}[a/(b + \Delta a)]$.

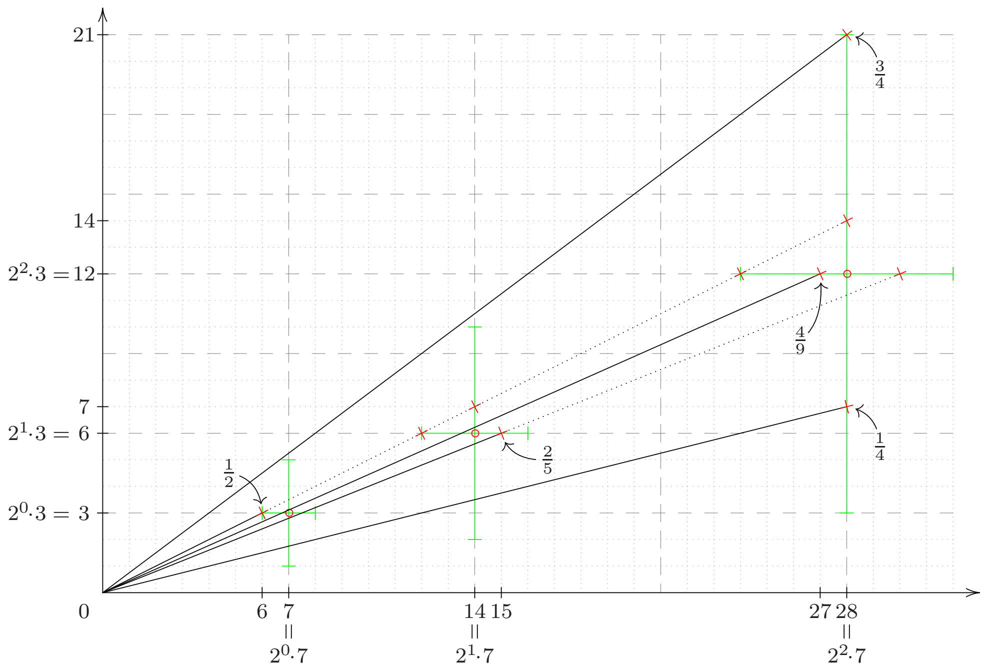


Figure 6. Geometrical interpretation of Algorithm 6.1$_{\frac{1}{3}}(3,7)$.

7. Behaviour (Diverging)

For a given ratio $r \in \mathbf{Q}^{\times}$, i.e. a positive rational p/q say, where $p, q \in \mathbf{Z}^{+}$, it will be convenient to define an idempotent unary operator on $\mathbf{Q}^{\times}$ by:

$$\|p/q\|_{\times} = \frac{\max\{p,q\}}{\min\{p,q\}}. \tag{7}$$

Thus $\|r\|_{\times} \geq 1$, and a rational less than unity maps to its reciprocal. This operator may be thought of as a measure of the disparity of the ratio's two factors: the closer the numerator and denominator are to each other in terms of their magnitude, the lower their disparity value. It could also be thought of as analogous to the absolute value operator —for the multiplicative group of $\mathbf{Q}$ instead of the additive group—hence the notation.

Notwithstanding Lemma 4.5, every single correct ratio is produced for the 1 554 pairs of consecutive pan-voice IOIs in the 15 KP performed corpus excerpts with a reasonable Δ. For each pair, Fig. 7 plots the barely sufficient Δ such that the pair (p, q) is produced, against its disparity $\|p/q\|_{\times}$. Almost all of the points plotted lie under a discernible curve approximated by ψ_0,

$$\psi_0(x) = b\log(x-a) + c \quad \text{for constants} \quad a = 0.85,\ b = 1/4,\ c = 1/2. \tag{8a}$$

As this curve is calculated as a function of the pre-quantised, performed IOIs, it gives a practical bound on the range of plausible ratios that need to be generated. The actual preferred values are not represented in Fig. 7, only how wide a range was necessary to generate them. The observation is that a suitable range may reliably and simply be

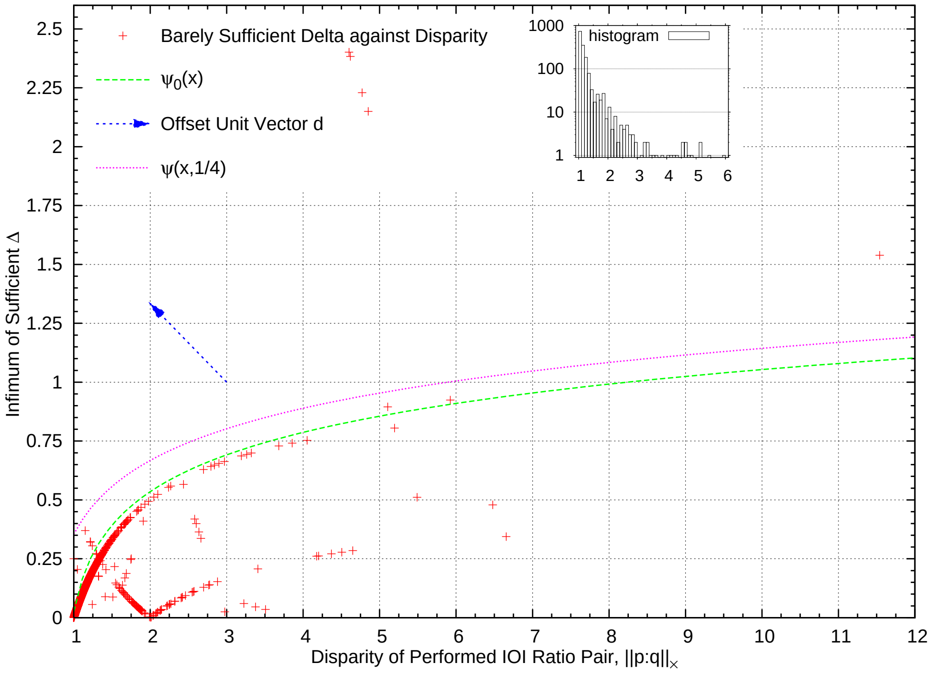


Figure 7. Distribution of barely sufficient Δ for pair r against pre-quantised ratio disparity $\|r\|_\times$.

calculated from the observed values so as to include the preferred value.

Note that this curve in Eqn (8a) and Fig. 7 exists independently of both the magnitude of the ratio factors ($p{:}q = np{:}nq$) and the sense of their disparity ($\|p{:}q\|_\times = \|q{:}p\|_\times$), and indeed any other parameters apparent at this stage. This means that, conveniently for computation, the curve holds irrespective of tempo (or of the sense of direction through time).

The data in Fig. 7 is dense for observed IOI pairs $p{:}q$ in the range $\|p/q\|_\times \leq 2$, with over 96% of the data there (1 498 points out of a population of 1 554). A histogram of the distribution with logarithmic ordinate is shown in an inset of Fig. 7. With increasing $\|p/q\|_\times$, the data rapidly becomes sparse and unreliable, but a pattern, an upper bound on the lower bounds for Δ, can nonetheless be traced. It is not possible to say to what extent this may be characteristic of the performer's style (all excerpts were played by the same musician) or of the genre, or to what extent it may be a more universal phenomenon. Intuitively though, it would seem surprising if uncertainty were not somehow correlated with disparity. It is useful to know that there exists this certain range, which may be calculated in advance, within which the perceived ratio is reasonably sure to lie, despite the fact that may be impossible as yet to know which plausible ratio it will be.

There are only five substantial outliers: they all come from the 7 : 1 : 7 : 1 : 7 : 1 : 10 sequence of Bars 5–6 of the Beethoven piano sonata, Opus 10, of Fig. 8. They could be accounted for by interpreting the demi-semi-quavers of Bars 5–6 instead as grace notes, or by considering a cumulative effect of uncertainty (not present in Eqn (8)) due to the succession of such wide disparity (cf. the preferred interval of Fraise in § 2), or yet as being compensated for by the particularly decisive and unambiguous metrical interpretation that this passage possesses.

Of the fourteen cases that require $\psi(x, l)$ with $l > 0$, three of them are hemiolas,

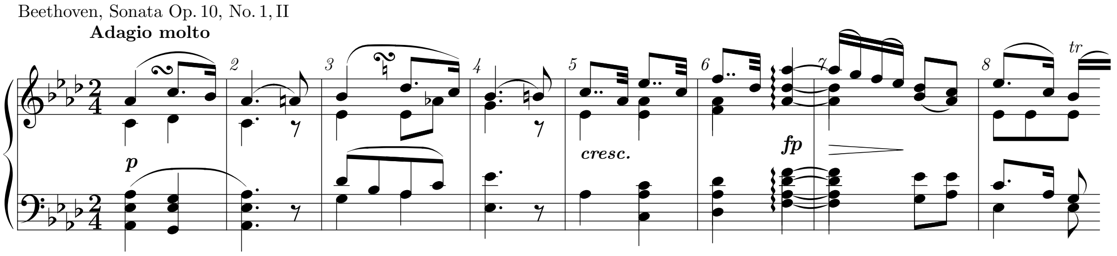


Figure 8. Slow tempo and straddling of metrical levels in consecutive IOIs require greater latitude.

although this number could be higher depending on interpretation as there were a few instances of hemiola with very short intervals that were marked as extra-metrical. This suggests that polyrhythms, unless executed very accurately, would be better interpreted as intra-voice intervals contrary to Hyp. 3.1 and Fig. 1. Otherwise there is a kind of temporal vernier effect under which small deviations from metronomic performance give rise to exaggeratedly deviant observed ratios. Polyrhythm is however beyond the scope of common practice tonal music under consideration. The remaining 11 of the 14 are due to rubato. Seven of these come from the Beethoven Sonata Op. 10 alone, discussed above. With $l = 0$, these fourteen will each be given a range of more metronomic but less metrical interpretations.

For practical application, *all* preferred ratios can be ensured by using a factor, l, to "translate the curve" by a variable amount in the direction of the (admittedly ad hoc) vector $\mathbf{d}$ illustrated in Fig. 7:

$$\psi(x, l) = \psi_0(x - l d_x) + l d_y, \quad \mathbf{d} = (d_x, d_y) = (-1, 1/3), \tag{8b}$$

i.e. putting $\Delta = \psi(\|p/q\|_\times, l)$ suffices in practice where l may be a specified rubato/ tolerance parameter. Of the consecutive interval pairs from the corpus, 99% have their preferred ratios produced with $l = 0$, finding ψ_0 inadequate in just 14 out of 1554 cases. Only five remain with $l = 0.18$.

Figures 4–5,B1 showed how the number of ratios generated increases with Δ and M; Fig. 9 shows graphically the number generated with $\Delta = \psi(\|p/q\|_\times, 0)$. By comparing with the local peaks and troughs of Fig. 5, one can see how a dynamic data-driven Δ yields a distribution that is both reasonably smooth and increasing with disparity as desired. By contrast, any locally fixed value of Δ for Algorithm 4.1_Δ would either fail to produce the perceived ratio for larger disparity or produce many spurious ratios for lower disparity.

8. Discussion and Conclusion

There is a certain kind of redundancy or "latency" in the algorithm's mechanism in that it sometimes tentatively generates a ratio that has already been a member of the output set. While a range of special cases such as the 1:2^n troughs could be catered for, it seems unlikely that the overhead of checking for their occurrence would be less than that of checking for membership of a rather small set. A full treatment would appear to require

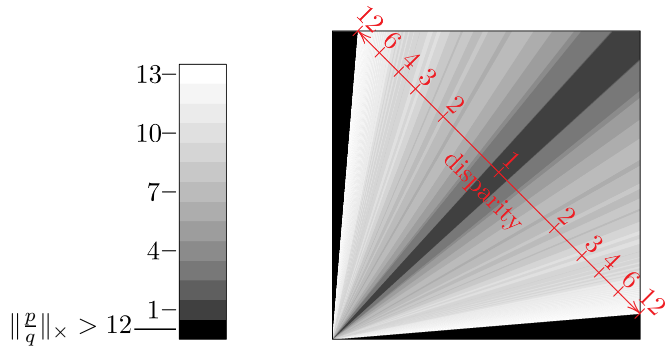


Figure 9. Distribution of number of plausible ratios over P_{1152} subject to $\|p/q\|_\times \leq 12$ for $\Delta = \psi(\|p/q\|_\times, 0)$, as in Eqn (8), and $M = 2$. Values range from 1–13 as per the key to the left.

a characterisation of the distribution of primes — a subject which remains the holy grail of number theory notwithstanding millennia of study [17].

Alternatively, all 2^m factor ratios could be generated and then filtered to within a specified tolerance. This would produce the same result but the presented algorithm is more efficient as it directly and efficiently generates only the required ratios.

Just under 2% of the 1 554 performance ratios strictly require $\Delta \geq 1/2$. Without the generalisation to $\Delta > 1/2$ of § 6–7, this would correspond to a beat tracker failing once every 50 onsets on average. For the more dense KP corpus excerpts, this corresponds to little more than two bars at a time, yielding unacceptably poor performance. The test corpus, including details of its preparation, along with implementations of the algorithms, is available from [20].

Algorithms for calculating round rational approximations exist in languages such as Haskell (`Ratio.approxRational`), Matlab/Octave (`rat`), and Mathematica (`Rationalize`). Each of them use continued fractions to produce a converging sequence of rationals, although there are implementation differences in how the continued fraction is expanded and how the converging sequence is derived from it. They each take a tolerance parameter, somewhat analogous to Δ_a of Algorithm 4.2. Haskell terminates at the *simplest* ratio; Octave terminates when the ratios have converged to within a supplied (or default) tolerance; Mathematica (closed source but documented) employs convergents but also tries to minimise the denominator.

Haskell's `approxRational` defines a partial order on ratios by $p{:}q$ is *simpler* than $p'{:}q'$ if $|p| \leq |p'| \wedge |q| \leq |q'|$. It is straightforward to show that, within a given interval, there exists a unique simplest rational; it is even easier to show that there exists a unique simplest ratio of the form $2^m{:}q$ or $p{:}2^m$. It is also possible to produce the output from Algorithm 6.1 from approxRational by judicious choice of interval for each ratio produced, and it is possible to substitute approxRational for Algorithm 4.2 in the code for Algorithm 6.1. The difficulty with this approach however lies in systematically constraining the factors of the ratios produced to the form 2^m while producing all such ratios within range. Algorithm 4.2 simultaneously harnesses Theorem 3.3 and Theorem 4.6 to cover an interval all at once with optimal efficiency, O(1). Conceptually, this is achieved by fixing one factor (thanks to Definition 3.2) while taking advantage of modular arithmetic with the other factor. The approach of continued fractions varies both factors at once. It should be possible to apply `approxRational` within a sufficiently small tolerance $(i \pm \frac{1}{2}, j \pm \frac{1}{2})$

about each coordinate (i, j) in range (the green band ranges in Fig. 6). When a ratio of the correct form is found, then Theorem 4.6 can be applied to skip some neighbouring coordinates. This involves testing most coordinates in range and invoking a recursive algorithm at each test — substantially less efficient than Algorithm 4.2. Possible computational alternatives to modify approxRational may include a recursively maintained version of the "counterbalance coefficient" mentioned in the notes to Algorithm 4.1, or a Prolog style backtracking algorithm with effective bailout heuristics for non-power-of-two factored ratios. Both alternatives seem unwieldy compared with Algorithm 4.2. The above remarks apply also to Octave and Mathematica.

Theorems for existence and uniqueness are offered for what has been termed near division. A few additional results characterise the behaviour of the algorithms presented. The theory would perhaps benefit from a completeness theorem. It may be noticed that the ratio 1:3 is not a member of the set produced by Algorithm $6.1_{\frac{1}{3}}(3, 7)$ as depicted in Fig. 6, despite being within range. This is because 3 does not divide any of the multiples of 7 employed (3 and 7 are coprime). In practice so far, this has not been problematic for real performance values with 3–4 significant figures and $M \geq 2$ as recommended. An appropriate completeness theorem could provide formal conditions under which all ratios of the stated form are guaranteed to be generated. Algorithm 6.1 could then be enhanced as needs be, e.g. by an additional step in the outermost loop for ratios of the required form. Similarly, if it should transpire that Hyp. 3.1 is too restrictive for a given application, then Algorithm 6.1 can easily be amended: Definition 3.2, Theorem 3.3 and Theorem 4.6 still apply.

A curious reader may wonder to what extent the bottom-up approach of this paper would have on the instability of a beat tracker employing it. It turns out that a meter detection layer and beat tracking layer driven by the plausible ratios of this paper performs well compared with other systems, being both stable and adaptable, while adhering to the modelling requirements of § 1. Demonstration of this is the subject of follow-up papers [24, 25]. It will be shown how it is possible to choose the perceived ratio, q_i/q_{i+1}, most of the time, so that only occasional periodic recourse to metre for coercion is necessary for a beat tracker. Such an approach has precedent in the generate-and-test paradigm of AI and pattern recognition [2, 3, 26–28]. It is a hope of this paper that such a "functional" approach, as with functional programming languages, will preclude hidden side effects and circular dependencies.

In summary, a formulation of near division is developed to facilitate direct calculation of round approximate ratios subject to constraints of a certain form. Employing this, an algorithm is devised to produce all plausible ratios within a given range. The behaviour of this algorithm is analysed: first in terms of its mathematical properties and then empirically to establish bounds on its behaviour. A generalised algorithm is then devised that harnesses the uniqueness property of the previous section to address the shortcomings identified in the analysis. This generalised algorithm is tested in turn on performance data, and criteria for more reasonable behaviour are identified.

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Appendix A. Calculation of Plausible Ratios (Converging, § 4)

A.1. Pseudocode with Annotation

Algorithm A.1 (aka Algorithm 4.1) For $\Delta \in [0, \frac{1}{2})$, $M, n_1, n_2 \in \mathbf{N}$, $n_1, n_2 > 0$, $\texttt{PlausibleRatios1}_{\Delta,M}(n_1, n_2)$ returns a set of plausible ratios.

0. Put $Q_0 = \{\}$, $\quad i = 0$,
1. For $\{a = n_1,\ b_1 = n_2,\ p = 1\}$ and $\{a = n_2,\ b_1 = n_1,\ p = -1\}$:
2.   Put $\Delta_a = \lfloor \Delta a \rfloor$,
3.   For $m = 0, 1, \ldots, M$:
4.   Put $b = 2^m b_1$, $\quad c = \texttt{NearDivides}(a, b, \Delta_a)$, $\quad$ increment i,
5.   If $c \neq \emptyset$ then:
6.   Put $Q_i = Q_{i-1} \cup \{\frac{n_1}{n_2}(\frac{b}{c})^p\}$.
7.   otherwise
8.   Put $Q_i = Q_{i-1}$.
9. Return Q_i.

Algorithm A.2 (aka 4.2) For $a, b, \Delta_a \in \mathbf{N}$, $a, b > 0$, $\texttt{NearDivides}(a, b, \Delta_a)$ returns c such that $a \,|\, c$ and $|b - c| \leq \Delta a$, or else null.

0. Put $r = (b \bmod a)$,
1. If $r \leq \Delta_a$ then:
2.   Return $b - r$.
3. else if $r \geq a - \Delta_a$ then:
4.   Return $a + b - r$.
5. Return $\emptyset$.

Notes on the steps of Algorithm A.1, $\texttt{PlausibleRatios1}$, follow:

(1) In keeping with Hyp 3.1, the first pass of the outer loop gathers all ratios within bounds of the form $n{:}2^m$ while the second pass adds those of the form $2^m{:}n$.
(2) The parameter Δ is set externally, $0 \leq \Delta < \frac{1}{2}$. There is nothing to be lost, and a slight gain in efficiency, by rounding Δa down at this stage.
(3) The parameter M determines the upper limit on the set of powers of two considered. It is also set externally.
(4) If $a \wr_\Delta b$ then c is set such that $a \,|\, c$ and $|b - c| \leq \Delta a$, as given explicitly in the proof of Theorem 3.3.
(6) On the first pass of Step 1,

$$\frac{n_1}{n_2}\left(\frac{b}{c}\right)^p = \frac{n_1 2^m n_2}{n_2 c} = \frac{2^m}{q_1} \quad \text{for} \quad q_1 = \frac{c}{n_1} \in \mathbf{N};$$

on the second pass,

$$\frac{n_1}{n_2}\left(\frac{b}{c}\right)^p = \frac{n_1 c}{n_2 2^m n_1} = \frac{q_2}{2^m} \quad \text{for} \quad q_2 = \frac{c}{n_2} \in \mathbf{N}.$$

Thus, a ratio with 2^m as one or other factor is formed each time. The ratio $(b/c)^p$ may be thought of as a kind of "counterbalance" coefficient, one which brings n_1/n_2 back to a round ratio that might have been intended.

A.2. Haskell Implementation

There follows an implementation of Algorithm A.1 (aka 4.1) and Algorithm A.2 (aka 4.2) in Haskell. See also § C.1 for the full algorithm and download details.

```
import Maybe
import Ratio
import List

type IOI = Int
type Delta = Rational
type PowerOfTwo = Int
type PlausibleRatio = Ratio IOI

data Pass = First | Second


nearDivides :: IOI -> IOI -> Int -> Maybe Int
nearDivides a b da = let r = b `mod` a in
                     if r <= da then Just (b - r)
                     else if r >= a - da then Just (a + b - r)
                          else Nothing


plausibleRatios1 :: PowerOfTwo -> Delta -> IOI -> IOI -> [PlausibleRatio]
plausibleRatios1 mM d n1 n2 = nub $ gather First n1 n2 ++ gather Second n2 n1
    where gather p a b1 = let da = floor $ d * fromIntegral a in
                          mapMaybe (gather_m p da a b1) [0..mM]
          gather_m p da a b1 m = let b = 2^m*b1
                                     n = nearDivides a b da
                                 in case n of
                                      Just c -> Just $ (n1%n2)*r
                                          where r = case p of
                                                      First -> b%c
                                                      Second -> c%b
                                      _ -> Nothing
```

A.3. Lemmas

Lemma A.3 *Taking n_1, n_2, m as in Algorithm A.1, the values produced by (Step 6 of) Algorithm A.1$_\Delta(n_1, n_2)$ converge to n_1/n_2 with increasing m:*

$$\lim_{m\to\infty} \frac{2^m n_1}{c_m} = \frac{n_1}{n_2} = \lim_{m\to\infty} \frac{c'_m}{2^m n_1}$$

whenever c, c' are defined for m, where c is the value produced by Algorithm A.2 during the first pass of the outer loop ($p = 1$) and c' that for the second pass ($p = -1$).

Proof: Assuming that c exists, we have

$$|2^m n_2 - c_m| \leq \Delta n_1. \tag{A1}$$

Multiplying by $n_1/(n_2 c_m)$ gives

$$\left|\frac{2^m n_1}{c_m} - \frac{n_1}{n_2}\right| \leq \frac{\Delta n_1^2}{n_2 c_m}. \tag{A2}$$

Given any $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$, put

$$m = \min\left\{ l \in \mathbf{N} \;\middle|\; l > \log_2\left[\frac{\Delta}{\varepsilon}\left(\frac{n_1}{n_2}\right)^2 + \Delta\frac{n_1}{n_2}\right] \text{ and } c_l \text{ exists}\right\}. \tag{A3}$$

Then

$$2^m > \frac{\Delta}{\varepsilon}\left(\frac{n_1}{n_2}\right)^2 + \Delta\frac{n_1}{n_2}$$

and

$$2^m n_2 - \Delta n_1 > \frac{\Delta n_1^2}{\varepsilon n_2}. \tag{A4}$$

It follows from (A1) that $2^m n_2 - \Delta n_1 \leq c_m$; therefore, by (A4),

$$\frac{\Delta n_1^2}{\varepsilon n_2} < c_m \qquad \text{and} \qquad \frac{\Delta n_1^2}{n_2 c_m} < \varepsilon. \tag{A5}$$

The result for the first pass now follows from (A2):

$$\lim_{m\to\infty} \frac{2^m n_1}{c_m} = \frac{n_1}{n_2}. \tag{A6}$$

Similarly for the second pass we have

$$|2^m n_1 - c'_m| \leq \Delta n_2; \tag{A7}$$

multiplying by $n_2/(n_1 c'_m)$ gives

$$\left|\frac{2^m n_2}{c'_m} - \frac{n_2}{n_1}\right| \leq \frac{\Delta n_2^2}{n_1 c'_m}. \tag{A8}$$

Given $\varepsilon > 0$, put

$$m = \min\left\{ l \in \mathbf{N} \;\middle|\; l > \log_2\left[\frac{\Delta}{\varepsilon}\left(\frac{n_2}{n_1}\right)^2 + \Delta\frac{n_2}{n_1}\right] \text{ and } c'_l \text{ exists}\right\} \tag{A9}$$

so that

$$2^m n_1 - \Delta n_2 > \frac{\Delta n_2^2}{\varepsilon n_1}. \tag{A10}$$

By (A7), $2^m n_1 - \Delta n_2 \leq c'_m$ and so

$$\frac{\Delta n_2^2}{\varepsilon n_1} < c'_m \qquad \text{and} \qquad \frac{\Delta n_2^2}{n_1 c'_m} < \varepsilon. \tag{A11}$$

By (A8),

$$\lim_{m\to\infty} \frac{2^m n_2}{c'_m} = \frac{n_2}{n_1}, \tag{A12}$$

and the result follows from a standard property of limits:

$$\lim_{m\to\infty} \frac{c'_m}{2^m n_2} = \frac{n_1}{n_2}. \tag{A13}$$

■

Corollary A.4 *The rate of convergence of Algorithm A.1*$_\Delta(n_1, n_2)$ *decreases with* Δ.

Proof: This follows from Eqn (A3) and Eqn (A9). ■

Appendix B. Behaviour (Converging, § 5)

B.1. Distribution

Figure B1 plots some statistics of the number of ratios generated by Algorithm A.1 (aka 4.1) under varying parameters. As one might expect, the number of ratios generated increases monotonically both with increasing Δ and increasing M. For rather low values of Δ, not every pair of numbers receives a ratio; however, all pairs of numbers up to at least 1 152 receive a non-empty set of plausible ratios for $M \geq 1$ and $\Delta \geq \frac{13}{64}$. For increasing values of M, certain pairs of numbers receive increasingly many ratios with increasingly fine distance between them.

The two plots to the left show the mean number of plausible ratios generated for $P_n = \{\langle i,j\rangle \in \mathbf{N}\times\mathbf{N} \mid 1 \leq i,j \leq n\}$ and $Q_n = \{\langle i,j\rangle \in \mathbf{N}\times\mathbf{N} \mid 1 \leq i \leq j \leq n \text{ and } \gcd(i,j) = 1\}$ for $n = 12$ and $n = 1152$ (where Q lies very close to P); the two plots to the right similarly show standard deviation. While the former set, P (plotted in red), calculates over all pairs of numbers in range, the latter set, Q (in green), calculates over all unique ratios $i:j$ for $i \leq j$ in range. The symmetry of the algorithm ensures that $i{:}j$ and $j{:}i$ produce isomorphic sets: each member is a reciprocal of a corresponding member of the other (dual) set. Hence the condition that $i \leq j$ for uniqueness in Q. This duality is of use in § 7. The lower plot shows how the mode and max behave.

As n increases, so does the mean while the variance decreases. In both cases, this change is slow and the rate of change decreases: they both appear to converge. For $n = 12$, the statistics are not substantially different; for large n, there is no significant difference. In

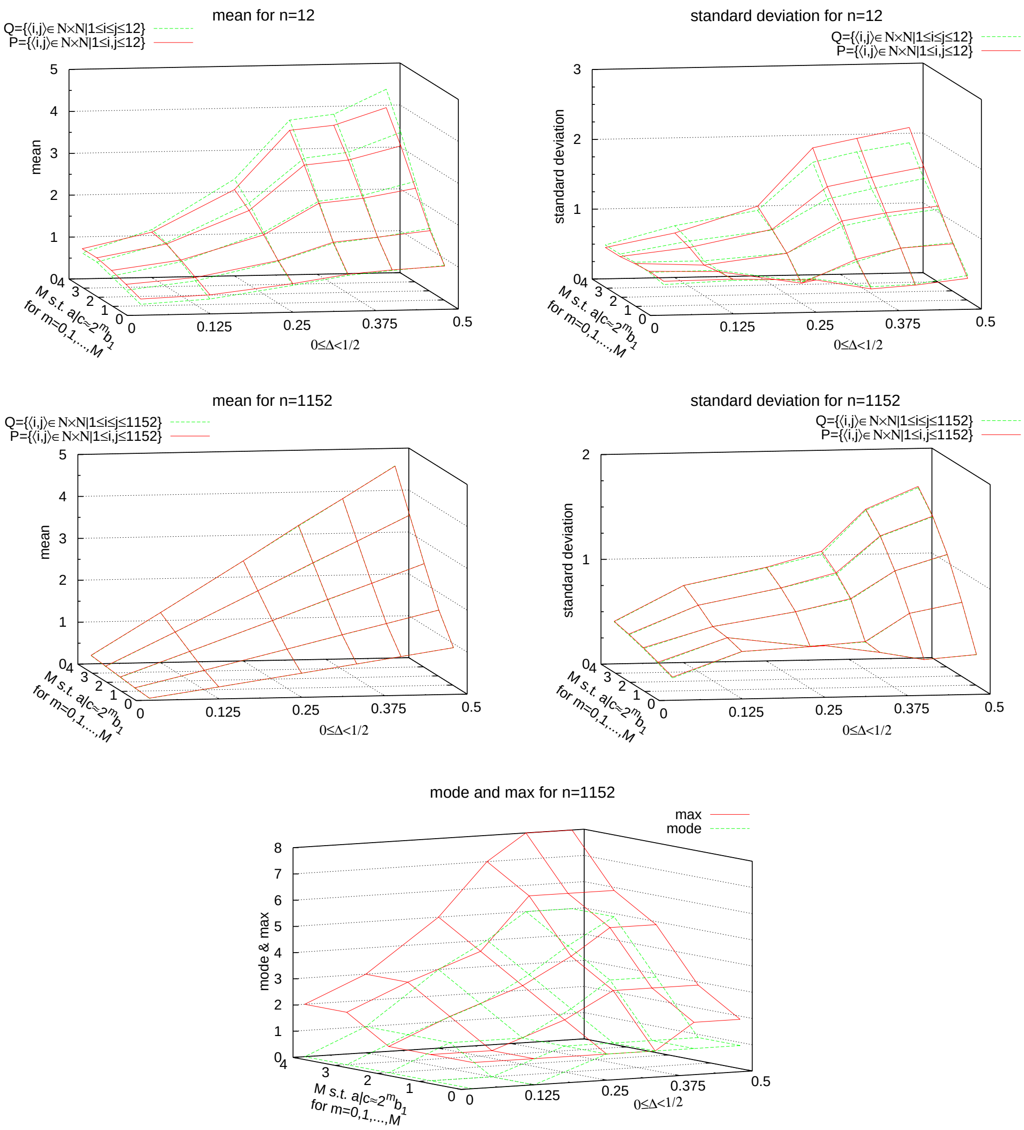

Figure B1. Distribution of mean number of plausible ratios from Algorithm A.1 for given Δ, M

keeping with Corollary A.4, Algorithm A.1's behaviour becomes increasingly stable with increasing n. In practice then, 3–4 significant figures or a temporal resolution of ~1 ms may be optimally adequate. There is only slight difference between the measurements for P and Q for $n = 1152$ and this difference vanishes as n increases. Indeed, it can be shown that $|Q_n|/|P_n| = \phi(n^2)/n^2 \sim 6/\pi^2$, where $\phi(n)$ is Euler's totient function.

B.2. Outlier Requiring $M = 3$

There is only a single instance of a ratio requiring $M = 3$. Although the interval subdivisions used are perfectly orthodox, what makes this instance unusual is the fact that

Figure B2. The only instance of a ratio requiring $M = 3$ in the corpus: that of 8:3 across bars 36–37, where $8 = 2^3$.

it proceeds immediately from an interval of two tactus (crochet) beats to an interval implicitly requiring a subdivision of the tactus level. Normally in common practice tonal music, when proceeding directly from interval level $2L$ to $L/2$ say, i.e. skipping level L, the $L/2$ interval level will be marked explicitly. In this instance, the $L/2$ (quaver) level is straddled. The strong parallelism in the first violin part, echoed in the second violin and viola, helps to carry an otherwise weaker sense of metre over this hurdle. The minim interval conspicuously skips a tactus beat (like what London [29] calls a "loud rest" except that it may be sustained). The effect is somewhat analogous to that of a fermata in giving a sense of the piece being momentarily suspended, but discretely rather than continuously and for tactus rather than for tempo, as the timing may be steady. All other preferred ratios in the corpus were produced with $M = 2$.

Appendix C. Calculation of Plausible Ratios (Diverging, § 6)

C.1. Pseudocode with Annotation

Algorithm C.1 (aka Algorithm 6.1) For $\Delta \in \mathbf{R}^+$, $M, n_1, n_2 \in \mathbf{N}$, $n_1, n_2 > 0$, $\texttt{PlausibleRatios}_{\Delta,M}(n_1, n_2)$ returns a set of ratios.

0\. Put $Q_0 = \{\}, \quad i = 0,$
1\. For $\{a = n_1,\, b = n_2,\, p = 1\}$ and $\{a = n_2,\, b = n_1,\, p = -1\}$:
2\. Put $h' = \lceil a/2 \rceil - 1,$
3\. For $m = 0, 1, \ldots, M$:
4\. Put $h = \lfloor 2^m \Delta a \rfloor, \quad \Delta_a = \min\{h, h'\},$
5\. If $h > h'$ then put $s = 2\Delta_a + 1$;
6\. otherwise put $s = \emptyset$.
7\. If $s \neq \emptyset$ then put $b' = 2^m b + h - \Delta_a, \quad b_s = 2^m b - h + \Delta_a$;
8\. otherwise put $b' = b_s = 2^m b$.
9\. Put $b'_s = b_s,$
10\. Until $b'_s = \emptyset$:
11\. Put $c = \texttt{NearDivides}(a, b'_s, \Delta_a), \quad$ increment i,
12\. If $c \neq \emptyset \wedge c \neq 0$ then put $Q_i = Q_{i-1} \cup \left\{ \left(\frac{2^m a}{c}\right)^p \right\}$;
13\. otherwise put $Q_i = Q_{i-1}$.
14\. If $s \neq \emptyset$ then put $b_s = b_s + s$;
15\. otherwise put $b_s = \emptyset$.
16\. If $b' = b'_s$ then put $b'_s = \emptyset$;
17\. else if $b_s > b'$ then put $b'_s = b'$;
18\. otherwise put $b'_s = b_s$.
19\. Return Q_i.

(2) If range extension is to take place at any subsequent step, then $\pm h'$ is the full extent within which the uniqueness of Theorem 4.6 applies.

(4) h is the full range of possibility for a "dividend" about $2^m b$; $\pm\Delta_a$ is the extent within which the existence of a dividend will be queried. Δ_a effects a crossover from h to h' as m (or Δ) increases.

(5–6) If range extension is going to take place, then s is the maximum amount by which the local window may step.

(7–8) b_s is set to the initial window centre; b' is the terminal window centre.

(9–11) b'_s is the local window centre, stepping as it may.

(12) This is a simplified but functionally equivalent version of Step 6 in Algorithm A.1, $\texttt{PlausibleRatios1}_\Delta$. It lacks the symmetry of the previous version and perhaps conceals the semantics of its derivation.

(14–18) b_s is the stepping window centre at full extent; b'_s is the stepping window centre not exceeding h in range. The local window, $b'_s \pm \Delta_a$, steps as necessary, its centre never exceeding b'.

C.2. Haskell Implementation

There follows an implementation of Algorithm C.1 (aka Algorithm 6.1) in Haskell. The code of § A.2 is assumed to exist in the same file. This code, along with implementations in Common Lisp and C, can be downloaded from `ftp://ftp.maths.tcd.ie/pub/dec/quantisation/src.tar.gz` [20].

```
mapGather :: (a -> Maybe b -> (a,Maybe b,Maybe c)) -> a -> Maybe b -> [c]
mapGather f a b = case f a b of
                    (_,Nothing,Nothing) -> []
                    (_,Nothing,Just c') -> [c']
                    (a',b'@(Just _),Nothing) -> mapGather f a' b'
                    (a',b'@(Just _),Just c') -> c' : mapGather f a' b'
```

```
plausibleRatios :: PowerOfTwo -> Delta -> IOI -> IOI -> [PlausibleRatio]
plausibleRatios mM d n1 n2 = nub $ gather First n1 n2 ++ gather Second n2 n1
    where gather p a b = let h' = ceiling (a%2) - 1 in
                         concatMap (gather_m p h' a b) [0..mM]
          gather_m p h' a b m =
              let h  = floor $ (2^m * numerator d * fromIntegral a) % denominator d
                  da = min h h'
                  js = if h > h' then Just (2*da+1) else Nothing
                  bm = 2^m*b
                  hd = h-da
                  b' = if isJust js then bm+hd else bm
                  bs = if isJust js then bm-hd else bm
                  gather_b's b_s jb'_s =
                      case jb'_s of
                        Just b'_s -> (b_si,b'_si,pr)
                          where pr = case nearDivides a b'_s da of
                                       Just c | c /= 0 -> Just $ case p of
                                                            First -> (2^m*a)%c
                                                            Second -> c%(2^m*a)
                                       _ -> Nothing
                                b_si = case js of
                                        Just s -> b_s+s
                                        _ -> undefined
                                b'_si = if b' == b'_s
                                        then Nothing
                                        else Just $ if b_si > b' then b' else b_si
                        _ -> (b_s,Nothing,Nothing)
              in mapGather (gather_b's) bs (Just bs)
```

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