A NORMAL FORM FOR A REAL 2-CODIMENSIONAL SUBMANIFOLD IN $\mathbb{C}^{N+1}$ NEAR A CR SINGULARITY

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Abstract. We construct a complete formal normal form for a real 2-codimensional submanifold $M \subset \mathbb{C}^{N+1}$ near a CR singularity approximating the sphere. This result gives a higher dimensional extension of Huang-Yin normal form in $\mathbb{C}^2$.

1. Introduction and the main result

The study of the real submanifolds in the complex space near an isolated complex tangent point goes back to Bishop (see [1]). A point $p \in M$ with the property that the map $M \ni q \mapsto \dim_{\mathbb{C}} T_q^c M$ defined near p is not continuous at p is called a CR singularity. Here $T_q^c M := T_q M \cap J\left(T_q M\right)$, where $J : \mathbb{C}^{N+1} \longrightarrow \mathbb{C}^{N+1}$ is the standard complex structure.

Bishop considered the case when there exist coordinates (z, w) in $\mathbb{C}^2$ such that near a CR singularity $p = 0$, a real 2-codimensional submanifold $M \subset \mathbb{C}^2$ is given locally by

$$w = z\overline{z} + \lambda\left(z^2 + \overline{z}^2\right) + \mathrm{O}(3), \quad \text{or} \quad w = z^2 + \overline{z}^2 + \mathrm{O}(3), \tag{1.1}$$

where and $\lambda \in [0, \infty]$ is a holomorphic invariant called the Bishop invariant. When $\lambda = \infty$, M is understood to be defined by the second equation from (1.1). If $\lambda \notin \left\{0, \frac{1}{2}, \infty\right\}$, Moser and Webster (see [15]) proved that there exists a formal transformation that sends M into the normal form

$$w = z\overline{z} + \left(\lambda + \epsilon u^q\right)\left(z^2 + \overline{z}^2\right), \quad \epsilon \in \{0, -1, +1\}, \quad q \in \mathbb{N}, \tag{1.2}$$

where $w = u + iv$. When $\lambda = 0$ Moser (see [14]) derived the following partial normal form (the Moser normal form):

$$w = z\overline{z} + 2\mathsf{Re}\left\{\sum_{j \geq s} a_j z^j\right\}. \tag{1.3}$$

Here $s := \{j \in \mathbb{N}^\star;\ a_j \neq 0\}$ is the simplest higher order invariant known as the Moser invariant. This partial normal form was completed by Huang and Yin (see [9]). They proved that (1.3) is either a quadric or it can be formally transformed into the following normal form

$$w = z\overline{z} + 2\mathsf{Re}\left\{\sum_{j \geq s} a_j z^j\right\}, \quad a_s = 1, \quad a_j = 0, \quad \text{if} \quad j = 0, 1 \bmod s, \quad j > s. \tag{1.4}$$

In this paper we construct a higher dimensional analogue of the Huang-Yin normal form. If $(z, w) = (z_1, \dots, z_N, w)$ are coordinates of $\mathbb{C}^{N+1}$ and $M \subset \mathbb{C}^{N+1}$ a real 2-codimensional submanifold, we consider the case when there exists a holomorphic change of coordinates (see [7] or [10]) such that near $p = 0$, M is given by

$$w = z_1\overline{z}_1 + \cdots + z_N\overline{z}_N + \sum_{m+n \geq 3} \varphi_{m,n}(z, \overline{z}), \tag{1.5}$$

where $\varphi_{m,n}(z, \overline{z})$ is a bihomogeneous polynomial of bidegree (m, n) in $(z, \overline{z})$.

Some of our methods extend those from [9]. First, we give a generalization of the Moser normal form (1.3) (see [14]), called here the Extended Moser Lemma (Theorem 2.2), which uses the trace operator (see e.g. [16], [17]):

$$\mathrm{tr} := \sum_{k=1}^{N} \frac{\partial^2}{\partial z_k \partial \overline{z}_k}. \tag{1.6}$$

In $\mathbb{C}^2$ the Moser normal form eliminates the terms in the local defining equation of M of positive degree in both z and $\overline{z}$. The higher dimensional case considered here brings new difficulties. In $\mathbb{C}^{N+1}$ the Extended Moser Lemma eliminates only iterated traces of the coresponding terms. However, these terms can still contribute to higher order terms in the construction of the normal form. Similar normal forms were constructed recently for Levi-nondegenerate hypersurfaces in $\mathbb{C}^{N+1}$ by Zaitsev (see [17]).

The Extended Moser Lemma allows us to find just a partial normal form. This partial normal form is not unique but is only determined up to an action of an infinite dimensional group $\mathsf{Aut}_0(M_\infty)$, the formal self-transformation group of the quadric model $M_\infty := \{w = z_1\overline{z}_1 + \cdots + z_N\overline{z}_N\}$. The condition that (1.3) contains nontrivial higher order terms has the following natural generalization to higher dimension:

$$\sum_{k\geq 3}\varphi_{k,0}(z) \not\equiv 0, \tag{1.7}$$

where here and throughout the paper we use the abbreviation

$$\varphi_{k,0}(z) := \varphi_{k,0}(z,\bar{z})$$

as the latter polynomials do not depend on $\bar{z}$. As a consequence we obtain that $s := \min\{k \in \mathbb{N}^\star;\ \varphi_{k,0}(z) \not\equiv 0\} < \infty$. Then s is a biholomorphic invariant and $\varphi_{s,0}(z)$ is invariant (as tensor). We call the integer $s \geq 3$ the generalized Moser invariant. In course of this paper we will use the following notations

$$\Delta(z) := \varphi_{s,0}(z), \quad \Delta_k(z) := \partial_{z_k}\varphi_{s,0}(z), \quad k = 1,\ldots,N. \tag{1.8}$$

Definition 1.1. For a given homogeneous polynomial $V(z) = \sum\limits_{|I|=k} b_I z^I$ we consider the associated Fisher differential operator

$$V^\star = \sum_{|I|=k}\bar{b}_I\frac{\partial^{|I|}}{\partial z^I}. \tag{1.9}$$

The polynomial $\Delta(z)$ will be assumed to satisfy the following non-degeneracy condition:

Definition 1.2. The polynomial $\Delta(z)$ is called nondegenerate if for any linear forms $\mathcal{L}_1(z),\ldots,\mathcal{L}_N(z)$, one has

$$\mathcal{L}_1(z)\Delta_1(z) + \cdots + \mathcal{L}_N(z)\Delta_N(z) \equiv 0 \quad \Longrightarrow \quad \mathcal{L}_1(z) \equiv \cdots \equiv \mathcal{L}_N(z) \equiv 0. \tag{1.10}$$

In this paper we prove the following result:

Theorem 1.3. *Let $M \subset \mathbb{C}^{N+1}$ be a 2-codimensional real (formal) submanifold given near the point $0 \in M$ by the formal power series equation*

$$w = z_1\overline{z}_1 + \cdots + z_N\overline{z}_N + \sum_{m+n\geq 3}\varphi_{m,n}(z,\overline{z}), \tag{1.11}$$

where $\varphi_{m,n}(z,\overline{z})$ is a bihomogeneous polynomial of bidegree (m,n) in $(z,\overline{z})$. Assume that $\Delta(z)$ is nondenerate. Then there exists a unique formal map

$$(z',w') = (F(z,w),G(z,w)) = (z,w) + O(2), \tag{1.12}$$

that transforms M into the following normal form:

$$w' = z_1'\overline{z'}_1 + \cdots + z_N'\overline{z'}_N + \sum_{\substack{m+n\geq 3\\ m,n\neq 0}}\varphi'_{m,n}\left(z',\overline{z'}\right) + 2\mathsf{Re}\left\{\sum_{k\geq s}\varphi'_{k,0}\left(z'\right)\right\}, \tag{1.13}$$

where $\varphi'_{m,n}\left(z',\overline{z'}\right)$ is a bihomogeneous polynomial of bidegree (m,n) in $\left(z',\overline{z'}\right)$ satisfying the following normalization conditions

$$\begin{cases} tr^{m-1}\varphi'_{m,n}\left(z',\overline{z'}\right) = 0, & m\leq n-1, \quad m,n\neq 0;\\ tr^{n}\varphi'_{m,n}\left(z',\overline{z'}\right) = 0, & m\geq n, \quad m,n\neq 0.\end{cases} \tag{1.14}$$

$$\begin{cases} \left(\Delta^t\right)^\star\varphi'_{T,0}(z) = 0, \quad \text{if } T = ts+1;\ t\geq 1,\\ \left(\Delta_k\Delta^t\right)^\star\left(\varphi'_{T,0}(z)\right) = 0,\ k=1,\ldots,N, \quad \text{if } T = ts;\ t\geq 2.\end{cases} \tag{1.15}$$

A few words about the construction of the normal form. We want to find a formal biholomorphic map sending M into a formal normal form. This leads us to study an infinite system of homogeneous equations by truncating the original equation. We follow Huang-Yin strategy defining the weight of z_k to be 1 and the weight of $\overline{z}_k$ to be $s-1$, for all $k=1\ldots,N$. Since $\mathsf{Aut}_0(M_\infty)$ is infinite-dimensional, it follows that the homogeneous linearized normalization equations (see sections 3 and 4) have nontrivial kernel spaces. Using the preceding system of weights and a similar induction argument as in [9], we are able to trace precisely how the lower order terms arise in non-linear fashion: The kernel space of degree $2t+1$ is restricted by imposing a normalization condition on $\varphi'_{ts+1,0}(z)$ and the kernel space of degree $2t+2$ by imposing a normalization conditions on $\varphi'_{ts,0}(z)$. The non-uniqueness part of the lower degree solutions are uniquely determined in the higher order equations.

Our normal form is a natural generalization of the Huang-Yin normal form. Our normalization conditions are invariant under the linear change of coordinates that preserves the model $w=z_1\overline{z}_1+\cdots+z_N\overline{z}_N$, namely the unitary change of coordinates. Also, the non-degeneracy condition on $\Delta(z)$ is invariant under any linear change of coordinates.

A few words about the paper organization: In course of section 2 we will give a generalization of the Moser normal form and make further preparations for our normal form construction. The normal form construction will be presented in course of sections 3 and 4. In section 5 we prove the uniqueness of the formal transformation map.

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2. Preliminaries, notations and the Extended Moser Lemma

Let $(z_1,\ldots,z_N,w)$ be the coordinates from $\mathbb{C}^{N+1}$. Assume that there exists a holomorphic change of coordinates such that near the point $p=0$ M is defined by

$$w=z_1\overline{z}_1+\cdots+z_N\overline{z}_N+\sum_{m+n\geq 3}\varphi_{m,n}(z,\overline{z}), \tag{2.1}$$

where $\varphi_{m,n}(z,\overline{z})$ is a bihomogeneous polynomial of bidegree (m,n) in $(z,\overline{z})$, for all $m,n\geq 0$.

Let M' be another submanifold defined by

$$w'=z'_1\overline{z'}_1+\cdots+z'_N\overline{z'}_N+\sum_{m+n\geq 3}\varphi'_{m,n}\left(z',\overline{z'}\right), \tag{2.2}$$

where $\varphi'_{m,n}\left(z',\overline{z'}\right)$ is a bihomogeneous polynomial of bidegree (m,n) in $\left(z',\overline{z'}\right)$, for all $m,n\geq 0$. We define the hermitian product

$$\langle z,t\rangle=z_1\overline{t}_1+\cdots+z_N\overline{t}_N,\quad z=(z_1,\ldots,z_N)\,,\ t=(t_1,\ldots,t_N)\in\mathbb{C}^N. \tag{2.3}$$

Let $(z',w')=(F(z,w),G(z,w))$ be a formal map which sends M to M' and fixes the point $0\in\mathbb{C}^{N+1}$. Substituting this map into (2.2), we obtain

$$G(z,w)=\langle F(z,w),F(z,w)\rangle+\sum_{m+n\geq 3}\varphi'_{m,n}\left(F(z,w),\overline{F(z,w)}\right). \tag{2.4}$$

In the course of this paper we use the following notations

$$\varphi_{\geq k}(z,\overline{z})=\sum_{m+n\geq k}\varphi_{m,n}(z,\overline{z}),\quad \varphi_k(z,\overline{z})=\sum_{m+n=k}\varphi_{m,n}(z,\overline{z}),\quad k\geq 3. \tag{2.5}$$

Substituting in (2.4) $F(z,w)=\sum_{m,n\geq 0}F_{m,n}(z)w^n$, $G(z,w)=\sum_{m,n\geq 0}G_{m,n}(z)w^n$, where $G_{m,n}(z)$, $F_{m,n}(z)$ are homogeneous polynomials of degree m in z, using w satisfying (2.1) and notations (2.4), it follows that

$$\begin{aligned}\sum_{m,n\geq 0}G_{m,n}(z)\left(\langle z,z\rangle+\varphi_{\geq 3}\right)^n&=\left\|\sum_{m_1,n_1\geq 0}F_{m_1,n_1}(z)\left(\langle z,z\rangle+\varphi_{\geq 3}\right)^{n_1}\right\|^2\\&+\varphi'_{\geq 3}\left(\sum_{m_2,n_2\geq 0}F_{m_2,n_2}(z)\left(\langle z,z\rangle+\varphi_{\geq 3}\right)^{n_2},\overline{\sum_{m_3,n_3\geq 0}F_{m_3,n_3}(z)\left(\langle z,z\rangle+\varphi_{\geq 3}\right)^{n_3}}\right).\end{aligned} \tag{2.6}$$

Since our map fixes the point $0 \in \mathbb{C}^{N+1}$, it follows that $G_{0,0}(z) = 0$, $F_{0,0}(z) = 0$. Collecting the terms of bidegree $(1,0)$ in $(z,\overline{z})$ from (2.6), we obtain $G_{1,0}(z) = 0$. Collecting the terms of bidegree $(1,1)$ in $(z,\overline{z})$ from (2.6), we obtain

$$G_{0,1}\langle z, z\rangle = \langle F_{1,0}(z), F_{1,0}(z)\rangle . \tag{2.7}$$

Then (2.7) describes all the possible values of $G_{0,1}(z), F_{1,0}(z)$. Therefore $\operatorname{Im} G_{0,1} = 0$. Composing with an linear automorphism of $\operatorname{Re} w = \langle z, z\rangle$, we can assume that $G_{0,1}(z) = 1$, $F_{1,0}(z) = z$.

Using the same approach as in [17] (this idea was suggested me by Dmitri Zaitsev), the ,,good" terms that can help us to find the formal change of coordinates under some normalization conditions are

$$\varphi_{m,n}(z,\overline{z}), \quad \varphi'_{m,n}(z,\overline{z}), \quad G_{m,n}(z)\langle z,z\rangle^n, \quad \langle F_{m,n}(z), z\rangle \langle z,z\rangle^n, \quad \langle z, F_{m,n}(z)\rangle \langle z,z\rangle^n. \tag{2.8}$$

We recall the trace decomposition (see e.g. [17] , [16]):

Lemma 2.1. *For every bihomogeneous polynomial $P(z,\overline{z})$ and $n \in \mathbb{N}$ there exist $Q(z,\overline{z})$ and $R(z,\overline{z})$ unique polynomials such that*

$$P(z,\overline{z}) = Q(z,\overline{z})\langle z,z\rangle^n + R(z,\overline{z}), \quad tr^n R = 0. \tag{2.9}$$

Using the Lemma 2.1 and the ,,good" terms defined previously (see (2.8)) we develop a partial normal form that generalize the Moser Lemma. Let $\partial_z := (\partial_{z_1}, \ldots, \partial_{z_N})$. We prove the following statement:

Theorem 2.2 (Extended Moser Lemma)**.** *Let $M \subset \mathbb{C}^{N+1}$ be a 2-codimensional real-formal submanifold. Suppose that $0 \in M$ is a CR singular point and the submanifold M is defined by*

$$w = \langle z,z\rangle + \sum_{m+n\geq 3} \varphi_{m,n}(z,\overline{z}), \tag{2.10}$$

where $\varphi_{m,n}(z,\overline{z})$ is bihomogeneous polynomial of bidegree (m,n) in $(z,\overline{z})$, for all $m,n \geq 0$. Then there exists a unique formal map

$$(z', w') = \left(z + \sum_{m+n\geq 2} F_{m,n}(z) w^n, w + \sum_{m+n\geq 2} G_{m,n}(z) w^n \right), \tag{2.11}$$

where $F_{m,n}(z)$, $G_{m,n}(z)$ are homogeneous polynomials in z of degree m with the following normalization conditions

$$F_{0,n+1}(z) = 0, \quad F_{1,n}(z) = 0, \quad \textit{for all } n \geq 1, \tag{2.12}$$

that transforms M to the following partial normal form:

$$w' = \langle z', z'\rangle + \sum_{\substack{m+n\geq 3\\ m,n\neq 0}} \varphi'_{m,n}\left(z', \overline{z'}\right) + 2\operatorname{Re}\left\{ \sum_{k\geq 3} \varphi'_{k,0}\left(z'\right) \right\}, \tag{2.13}$$

where $\varphi'_{m,n}(z,\overline{z})$ are bihomogeneous polynomials of bidegree (m,n) in $(z,\overline{z})$, for all $m,n \geq 0$, that satisfy the following trace normalization conditions (1.14).

Proof. We construct the polynomials $F_{m',n'}(z)$ with $m' + 2n' = T - 1$ and $G_{m',n'}(z)$ with $m' + 2n' = T$ by induction on $T = m' + 2n'$. We assume that we have constructed the polynomials $F_{k,l}(z)$ with $k + 2l < T - 1$, $G_{k,l}(z)$ with $k + 2l < T$.

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $T = m + n$ from (2.6), we obtain

$$\varphi'_{m,n}(z,\overline{z}) = G_{m-n,n}(z)\langle z,z\rangle^n - \langle F_{m-n+1,n-1}(z), z\rangle \langle z,z\rangle^{n-1} - \langle z, F_{n-m+1,m-1}(z)\rangle \langle z,z\rangle^{m-1} + \varphi_{m,n}(z,\overline{z}) + \ldots, \tag{2.14}$$

where ,,. . ." represents terms which depend on the polynomials $G_{k,l}(z)$ with $k + 2l < T$, $F_{k,l}(z)$ with $k + 2l < T - 1$ and on $\varphi_{k,l}(z,\overline{z})$, $\varphi'_{k,l}(z,\overline{z})$ with $k + l < T = m + n$.

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $k := m + n \geq 3$ from (2.14), we have to study the following cases:

(1)Case $\mathbf{m < n - 1}$, $\mathbf{m, n \geq 1}$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (2.14) with $m < n - 1$ and $m,n \geq 1$, we obtain

$$\varphi'_{m,n}(z,\overline{z}) = -\langle z, F_{n-m+1,m-1}(z)\rangle \langle z,z\rangle^{m-1} + \ldots \tag{2.15}$$

We want to use the normalization condition $\operatorname{tr}^{m-1}\varphi'_{m,n}(z,\overline{z}) = 0$. This allows us to find the polynomial $F_{n-m+1,m-1}(z)$. By applying Lemma 2.1 to the sum of terms which appear in ,,. . .", we obtain

$$\varphi'_{m,n}(z,\overline{z}) = \left(-\langle z, F_{n-m+1,m-1}(z)\rangle + D_{m,n}(z,\overline{z})\right)\langle z,z\rangle^{m-1} + P_1(z,\overline{z}), \tag{2.16}$$

where $D_{m,n}(z,\overline{z})$ is a polynomial of degree $n-m+1$ in $\overline{z}_1,\dots,\overline{z}_N$ and 1 in $z_1,\dots,z_N$ with determined coefficients from the induction hypothesis and $\mathrm{tr}^{m-1}\left(P_1(z,\overline{z})\right)=0$. Then, using the normalization condition $\mathrm{tr}^{m-1}\varphi'_{m,n}(z,\overline{z})=0$, by the uniqueness of trace decomposition we obtain that $\langle z, F_{n-m+1,m-1}(z)\rangle = D_{m,n}(z,\overline{z})$. It follows that

$$F_{k,l}(z)=\overline{\partial_z\left(D_{l+1,k+l}(z,\overline{z})\right)},\quad \text{for all } k>2, l\geq 0. \tag{2.17}$$

(**2**)**Case $\mathbf{m>n+1,\ m,n\geq 1}$**. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (2.14) with $m>n+1$ and $m,n\geq 1$, we obtain

$$\varphi'_{m,n}(z,\overline{z})=\left(G_{m-n,n}(z)\langle z,z\rangle-\langle F_{m-n+1,n-1}(z),z\rangle\right)\langle z,z\rangle^{n-1}+\dots \tag{2.18}$$

In order to find the polynomial $G_{m-n,n}(z)$ we want to use the normalization condition $\mathrm{tr}^n\varphi'_{m,n}(z,\overline{z})=0$. By applying Lemma 2.1 to the sum of terms which appear in „$\dots$" and to $\langle F_{m-n+1,n-1}(z),z\rangle$, we obtain

$$\varphi'_{m,n}(z,\overline{z})=\left(G_{m-n,n}(z)-E_{m,n}(z)\right)\langle z,z\rangle^n+P_2(z,\overline{z}), \tag{2.19}$$

where $E_{m,n}(z)$ is a holomorphic polynomial with determined coefficients by the induction hypothesis and $\mathrm{tr}^n\left(P_2(z,\overline{z})\right)=0$. Then, using the normalization condition $\mathrm{tr}^n\varphi'_{m,n}(z,\overline{z})=0$, by the uniqueness of trace decomposition we obtain that $G_{m-n,n}(z)=E_{m,n}(z)$. It follows that

$$G_{k,l}(z)=E_{k+l,l}(z),\quad \text{for all } k\geq 2,\ l\geq 0. \tag{2.20}$$

(**3**)**Case $\mathbf{m=n-1,\ m,n\geq 1}$**. Collecting the terms of bidegree $(n-1,n)$ in $(z,\overline{z})$ from (2.14) with $n\geq 2$, we obtain

$$\varphi'_{n-1,n}(z,\overline{z})=\varphi_{n-1,n}(z,\overline{z})-\langle F_{0,n-1}(z),z\rangle\langle z,z\rangle^{n-1}-\langle z,F_{2,n-2}(z)\rangle\langle z,z\rangle^{n-2}+\dots \tag{2.21}$$

In order to find $F_{2,n-2}(z)$ we want to use the normalization condition $\mathrm{tr}^{n-2}\varphi'_{n-1,n}(z,\overline{z})=0$. By applying the Lemma 2.1 to the sum of terms from „$\dots$", we obtain

$$\varphi'_{n-1,n}(z,\overline{z})=-\left(\langle F_{0,n-1}(z),z\rangle\langle z,z\rangle+\langle z,F_{2,n-2}(z)\rangle-C_{n-1,n}(z,\overline{z})\right)\langle z,z\rangle^{n-2}+P_3(z,\overline{z}), \tag{2.22}$$

where $\mathrm{tr}^{n-2}\left(P_3(z,\overline{z})\right)=0$ and $C_{n-1,n}(z,\overline{z})$ is a determined polynomial of degree 1 in $z_1,\dots,z_N$ and degree 2 in $\overline{z}_1,\dots,\overline{z}_N$. We take $F_{0,n-1}(z)=0$ (see (2.12)). Next, using the normalization condition $\mathrm{tr}^{n-2}\varphi'_{n-1,n}(z,\overline{z})=0$, by the uniqueness of trace decomposition we obtain that $\langle z,F_{2,n-2}(z)\rangle=C_{n-1,n}(z,\overline{z})$. It follows that

$$F_{2,n-2}(z)=\overline{\partial_z\left(C_{n-1,n}(z,\overline{z})\right)}. \tag{2.23}$$

(**4**)**Case $\mathbf{m=n+1,\ m,n\geq 1}$**. Collecting the terms of bidegree $(n,n-1)$ in $(z,\overline{z})$ from (2.14) with $n\geq 2$, we obtain

$$\varphi'_{n,n-1}(z,\overline{z})=\left(G_{1,n-1}(z)\langle z,z\rangle-\langle F_{2,n-2}(z),z\rangle-\langle z,F_{0,n-1}(z)\rangle\langle z,z\rangle\right)\langle z,z\rangle^{n-2}+\varphi_{n,n-1}(z,\overline{z})+\dots \tag{2.24}$$

In order to find $G_{1,n-1}(z)$ we want to use the normalization condition $\mathrm{tr}^{n-1}\varphi'_{n,n-1}(z,\overline{z})=0$. Using (2.12) and by applying Lemma 2.1 to $\langle F_{2,n-2}(z),z\rangle$ (see (2.23)) and to the sum of terms from „$\dots$", we obtain

$$\varphi'_{n,n-1}(z,\overline{z})=\left(G_{1,n-1}(z)-B_{n,n-1}(z)\right)\langle z,z\rangle^{n-1}+P_4(z,\overline{z}), \tag{2.25}$$

where $\mathrm{tr}^{n-1}\left(P_4(z,\overline{z})\right)=0$ and $B_{n,n-1}(z)$ is a determined holomorphic polynomial. By the uniqueness of trace decomposition we obtain that $G_{1,n-1}(z)=B_{n,n-1}(z)$, for all $n\geq 2$.

(**5**)**Case $\mathbf{m=n,\ m,n\geq 1,\ m+n\geq 3}$**. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (2.14) with $m=n\geq 1$ and $m+n\geq 3$, we obtain

$$\varphi'_{n,n}(z,\overline{z})=G_{0,n}(z)\langle z,z\rangle^n-\langle F_{1,n-1}(z),z\rangle\langle z,z\rangle^{n-1}-\langle z,F_{1,n-1}(z)\rangle\langle z,z\rangle^{n-1}+\varphi_{n,n}(z,\overline{z})+\dots \tag{2.26}$$

By taking $F_{1,n-1}(z)=0$ (see (2.12)), we obtain $\varphi'_{n,n}(z,\overline{z})=G_{0,n}(z)\langle z,z\rangle^n+\dots$. In order to find $G_{0,n}(z)$ we use the normalization condition $\mathrm{tr}^n\varphi'_{n,n}(z,\overline{z})=0$. By applying the Lemma 2.1 to the sum of terms from „$\dots$" we obtain that $\varphi'_{n,n}(z,\overline{z})=\left(G_{0,n}(z)-A_n\right)\langle z,z\rangle^n+P_5(z,\overline{z})$, where A_n is a determined constant and $\mathrm{tr}^n\left(P_5(z,\overline{z})\right)=0$. By the uniqueness of trace decomposition we obtain that $G_{0,n}=A_n$, for all $n\geq 3$.

(**6**)**Case $(\mathbf{T},\mathbf{0})$ and $(\mathbf{0},\mathbf{T})$**. Collecting the terms of bidegree $(T,0)$ and $(0,T)$ in $(z,\overline{z})$ from (2.14), we obtain

$$\begin{cases} G_{T,0}(z)+\varphi'_{T,0}(z)=\varphi_{T,0}(z)+a(z)\\ \varphi'_{0,T}(\bar{z})=\varphi_{0,T}(\bar{z})+b(\bar{z})\end{cases}, \tag{2.27}$$

where $a(z)$, $b(\bar{z})$ are the sums of terms that are determined by the induction hypothesis . Using the normalization condition $\varphi'_{0,T}(\bar{z})=\overline{\varphi'_{T,0}(z)}$ we obtain that $g_{T,0}(z)=\varphi_{T,0}(z)+a(z)-\overline{b(\bar{z})}-\overline{\varphi_{0,T}(\bar{z})}$. ☐

The Extended Moser Lemma leaves undetermined an infinite number of parameters (see (2.12)). They act on the higher order terms. In order to determine them and complete our partial normal form we will apply in the course the sections 3 and 4 the following two lemmas:

Lemma 2.3. *Let $P(z)$ be a homogeneous pure polynomial. For every $k\in\mathbb{N}^\star$, there exist $Q(z)$, $R(z)$ unique polynomials such that*

$$P(z)=Q(z)\Delta(z)^k+R(z),\quad \left(\Delta^k\right)^\star(R(z))=0. \tag{2.28}$$

Lemma 2.4. *For every homogeneous polynomial $P(z)$ of degree $(t+1)s$ there exist a unique decomposition*

$$P(z)=L(z)+C(z),\quad \left(\Delta_k\Delta^t\right)^\star(C(z))=0,\quad k=1,\dots,N. \tag{2.29}$$

such that $L(z)=\left(\Delta_1(z)A_1(z)+\cdots+\Delta_N(z)A_N(z)\right)\Delta(z)^t$, where $A_1(z),\dots,A_N(z)$ are linear forms.

The lemmas 2.3 and 2.4 are consequences of the Fisher decomposition (see [16]).

Remark 2.5. The Lemma 2.4 is a particular case of the generalized Fisher decomposition. The polynomial $L(z)$ is uniquely determined, but the linear forms $A_1(z),\dots,A_N(z)$ are not necessary uniquely determined. In order to make them uniquely determined we consider a nondegenerate polynomial $\Delta(z)$ (see (1.8) and Definition 1.2).

The following proposition shows us the nondegeneracy condition on $\Delta(z)$ is invariant under any linear change of coordinates:

Proposition 2.6. *If $\Delta(z)$ is nondegenerate and $z\longmapsto Az$ is a linear change of coordinates, then $\Delta(Az)$ is also nondegenerate.*

Proof. Let $\widetilde{\Delta}(z)=\Delta(Az)$, where $A=\{a_{jk}\}_{1\leq j,k\leq N}$. Therefore $\widetilde{\Delta}_j(z)=\displaystyle\sum_{k=1}^{N}\Delta_k(Az)\,a_{jk}$, for all $j=1,\dots,N$. We consider $\mathcal{L}_1(z),\dots,\mathcal{L}_N(z)$ linear forms such that $\mathcal{L}_1(z)\widetilde{\Delta}_1(z)+\cdots+\mathcal{L}_N(z)\widetilde{\Delta}_N(z)\equiv 0$, or equivalently $\displaystyle\sum_{j,k=1}^{N}\Delta_k(Az)\,\mathcal{L}_j(z)a_{jk}\equiv 0$. Since $\Delta(z)$ is nondegenerate and $\{a_{jk}\}_{1\leq j,k\leq N}$ is invertible it follows that $\mathcal{L}_1(z)\equiv\cdots\equiv\mathcal{L}_N(z)\equiv 0$. □

The system of weights : Following the line of [9], we define a system of weights for $z_1,\overline{z}_1,\dots,z_N,\overline{z}_N$ as follows. We define $\text{wt}\{z_k\}=1$ and $\text{wt}\{\overline{z}_k\}=s-1$, for all $k=1,\dots,N$. If $A(z,\overline{z})$ is a formal power series we write $\text{wt}\{A(z,\overline{z})\}=k$ if $A\left(tz,t^{s-1}\overline{z}\right)=\mathrm{O}\left(t^k\right)$. We also write $\text{Ord}\{A(z,\overline{z})\}\geq k$ if $A(tz,t\overline{z})=t^kA(z,\overline{z})$. We denote by $\Theta_m^n(z,\overline{z})$ a series in $(z,\overline{z})$ of weight at least m and order at least n. We define the set of the normal weights

$$\text{wt}_{nor}\{w\}=2,\quad \text{wt}_{nor}\{z_1\}=\cdots=\text{wt}_{nor}\{z_N\}=\text{wt}_{nor}\{\overline{z}_1\}=\cdots=\text{wt}_{nor}\{\overline{z}_N\}=1.$$

Notations : If $h(z,w)$ is a formal power series with no constant term we introduce the following notations

$$\begin{aligned} h(z,w)&=\sum_{l\geq 1}h_{nor}^{(l)}(z,w),\quad \text{where } h_{nor}^{(l)}\left(tz,t^2w\right)=t^lh_{nor}^{(l)}(z,w),\\ h_{\geq l}(z,w)&=\sum_{k\geq l}h_{nor}^{(k)}(z,w),\quad h_{l<}(z,w)=\sum_{k<l}h_{nor}^{(l)}(z,w).\end{aligned} \tag{2.30}$$

3. Proof of Theorem 1.3-Case $T+1=ts+1,\ t\geq 1$

By applying Extended Moser Lemma we can assume that M is given by the following equation

$$w=\langle z,z\rangle+\sum_{m+n\geq 3}^{T+1}\varphi_{m,n}(z,\overline{z})+\mathrm{O}(T+2), \tag{3.1}$$

where $\varphi_{m,n}(z,\overline{z})$ satisfies 1.14), for all $3\leq m+n\leq T$.

We perform induction on $T\geq 3$. Assume that (1.15) holds for $\varphi_{k,0}(z)$, for all $k=s+1,\dots,T$ with $k=0,1 \bmod (s)$. If $T+1\not\in\{ts;\ t\in\mathbb{N}^\star-\{1,2\}\}\cup\{ts+1;\ t\in\mathbb{N}^\star\}$ we apply Extended Moser Lemma. In the case when $T+1\in\{ts;\ t\in\mathbb{N}^\star-\{1\}\}\cup\{ts+1;\ t\in\mathbb{N}^\star\}$, we will look for a formal map which sends our submanifold M to a new submanifold M' given by

$$w'=\langle z',z'\rangle+\sum_{m+n\geq 3}^{T+1}\varphi'_{m,n}\left(z',\overline{z'}\right)+\mathrm{O}(T+2), \tag{3.2}$$

where $\varphi'_{m,n}\left(z',\overline{z'}\right)$ satisfies (1.14), for all $3 \leq m+n \leq T$ and $\varphi'_{k,0}\left(z'\right)$ satisfies (1.15), for all $k = s+1, \dots, T$ with $k = 0, 1 \bmod (s)$. We will obtain that $\varphi'_{k,0}(z) = \varphi_{k,0}(z)$ for all $k = 3, \dots, T$.

In the course of this section we consider the case when $T+1 = ts+1$. We are looking for a biholomorphic transformation of the following type

$$(z',w') = (z + F(z,w), w + G(z,w))$$

$$F(z,w) = \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,w), \quad G(z,w) = \sum_{\tau=0}^{T-2t} G_{nor}^{(2t+1+\tau)}(z,w), \tag{3.3}$$

that maps M into M' up to the order $T+1 = ts+1$. In order for the preceding mapping to be uniquely determined we assume that $F_{nor}^{(2t+l)}(z,w)$ is normalized as in Extended Moser Lemma, for all $l = 1, \dots T$. Substituting (3.3) into (3.2) we obtain

$$w + G(z,w) = \langle z + F(z,w), z + F(z,w)\rangle + \sum_{m+n\geq 3}^{T+1} \varphi'_{m,n}\left(z + F(z,w), \overline{z + F(z,w)}\right) + \mathrm{O}\left(T+2\right), \tag{3.4}$$

where w satisfies (3.1). By making some simplifications in (3.4) using (3.1), we obtain

$$\begin{aligned} \sum_{\tau=0}^{T-2t} G_{nor}^{(2t+1+\tau)}\left(z, \langle z,z\rangle + \varphi_{\geq 3}(z,\overline{z})\right) = 2\mathsf{Re}\left\langle z, \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}\left(z, \langle z,z\rangle + \varphi_{\geq 3}(z,\overline{z})\right)\right\rangle + \left\|\sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}\left(z, \langle z,z\rangle + \varphi_{\geq 3}(z,\overline{z})\right)\right\|^2 \\ + \varphi'_{\geq 3}\left(z + \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}\left(z, \langle z,z\rangle + \varphi_{\geq 3}(z,\overline{z})\right), \overline{z + \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}\left(z, \langle z,z\rangle + \varphi_{\geq 3}(z,\overline{z})\right)}\right) \\ - \varphi_{\geq 3}(z,\overline{z}). \end{aligned} \tag{3.5}$$

Collecting the terms with the same bidegree from (3.5), we find $F(z,w)$ and $G(z,w)$ by applying Extended Moser Lemma. Since we don't have components of $F(z,w)$ of normal weight less than $2t$ and $G(z,w)$ with normal weight less than $2t+1$, collecting in (3.5) the terms with the same bidegree (m,n) in $(z,\overline{z})$ with $m+n < 2t+1$, we obtain that $\varphi'_{m,n}(z,\overline{z}) = \varphi_{m,n}(z,\overline{z})$.

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n = 2t+1$ (like in the Extended Moser Lemma proof) we find $G_{nor}^{(2t+1)}(z,w)$ and $F_{nor}^{(2t)}(z,w)$ as follows. We make the following claim:

Lemma 3.1. *$G_{nor}^{(2t+1)}(z,w) = 0$, $F_{nor}^{(2t)}(z,w) = aw^t - z\langle z,a\rangle w^{t-1}$, where $a = (a_1, \dots, a_N) \in \mathbb{C}^N$.*

Proof. Collecting the pure terms of degree $2t+1$ from (3.5), we obtain that $\varphi_{2t+1,0}(z) = \varphi'_{2t+1,0}(z)$. Collecting the terms of bidegree (m,n) with $m+n = 2t+1$ in $(z,\overline{z})$ and $0 < m < n-1$ (3.5), we obtain

$$\varphi'_{m,n}(z,\overline{z}) = -\left\langle z, F_{n-m+1,m-1}(z)\right\rangle \langle z,z\rangle^{m-1} + \varphi_{m,n}(z,\overline{z}). \tag{3.6}$$

Since $\varphi_{m,n}(z,\overline{z})$, $\varphi'_{m,n}(z,\overline{z})$ satisfy (1.14), by the uniqueness of the trace decomposition, we obtain $F_{n-m+1,m-1}(z) = 0$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n = 2t+1$ and $m > n+1$ from (3.5), we obtain

$$\varphi'_{m,n}(z,\overline{z}) = G_{m-n,n}(z)\langle z,z\rangle^n - \left\langle F_{m-n+1,n-1}(z), z\right\rangle \langle z,z\rangle^{n-1} + \varphi_{m,n}(z,\overline{z}). \tag{3.7}$$

Since $F_{m-n+1,n-1}(z) = 0$ it follows that $G_{m-n,n}(z) = 0$. Collecting the terms of bidegree $(t-1,t)$ and $(t,t-1)$ in $(z,\overline{z})$ from (3.5), we obtain the following two equations

$$\begin{aligned} \varphi'_{t-1,t}(z,\overline{z}) &= -\left(\left\langle F_{0,t-1}(z), z\right\rangle \langle z,z\rangle + \left\langle z, F_{2,t-2}(z)\right\rangle\right)\langle z,z\rangle^{t-2} + \varphi_{t-1,t}(z,\overline{z}), \\ \varphi'_{t,t-1}(z,\overline{z}) &= G_{1,t-1}(z)\langle z,z\rangle^{t-1} - \left(\left\langle F_{2,t-2}(z), z\right\rangle + \left\langle z, F_{0,t-1}(z)\right\rangle \langle z,z\rangle\right)\langle z,z\rangle^{t-2} + \varphi_{t,t-1}(z,\overline{z}). \end{aligned} \tag{3.8}$$

Using (3.8) it follows that $G_{1,t-1}(z) = 0$. We set $F_{0,t-1}(z) = a =: (a_1, \dots, a_N)$ and we write $F_{2,t-2}(z) = \left(F^1_{2,t-2}(z), \dots, F^N_{2,t-2}(z)\right)$. Since $\varphi_{m,n}(z,\overline{z})$, $\varphi'_{m,n}(z,\overline{z})$ satisfy (1.14), by the uniqueness of the trace decomposition, from (3.8) we obtain the equation $\langle z,a\rangle\langle z,z\rangle + \langle F_{2,t-2}(z), z\rangle = 0$, that can be solved as

$$F^k_{2,t-2}(z) = -\frac{\partial}{\partial \overline{z}_k}\left(\langle z,a\rangle\langle z,z\rangle\right) = -z_k\langle z,a\rangle, \quad k = 1, \dots, N. \tag{3.9}$$

Therefore $F_{nor}^{(2t)}(z,w) = aw^t - z\langle z,a\rangle w^{t-1}$, where $a = (a_1, \dots, a_N) \in \mathbb{C}^N$. ☐

By Lemma 3.1 we conclude that $F(z,w)=F^{(2t)}_{nor}(z,w)+F_{\geq 2t+1}(z,w)$ and $G(z,w)=G_{\geq 2t+2}(z,w)$ (see (2.30)). We also have $F_{\geq 2t+1}(z,w)=\sum\limits_{k+2l\geq 2t+1}F_{k,l}(z)w^{l}$, where $F_{k,l}(z)$ is a homogeneous polynomial of degree k. It follows that

$$\text{wt}\left\{F_{\geq 2t+1}(z,w)\right\}\geq \min_{k+2l\geq 2t+1}\{k+ls\}\geq \min_{k+2l\geq 2t+1}\{k+2l\}\geq 2t+1. \tag{3.10}$$

Next, we prove that $\text{wt}\left\{\overline{F_{\geq 2t+1}(z,w)}\right\}\geq ts+s-1$. Since $\text{wt}\left\{\overline{F_{\geq 2t+1}(z,w)}\right\}\geq \min\limits_{k+2l\geq 2t+1}\{k(s-1)+ls\}$, it is enough to prove that $k(s-1)+ls\geq ts+s-1$ for $k+2l\geq 2t+1$. Since we can write the latter inequality as $(k-1)(s-1)+ls\geq ts$, for $(k-1)+2l\geq 2t$, it is enough to prove that $k(s-1)+ls\geq ts$, for $k+2l\geq 2t$. Since $s\geq 3$ it follows that $ks-2k\geq 0$. Hence $2k(s-1)+2ls\geq ks+2ls$. It follows that $k(s-1)+ls\geq \frac{s}{2}(k+2l)\geq\frac{2ts}{2}=ts$.

Lemma 3.2. *Using the previous calculations, we give the following immediate estimates*

$$\begin{aligned}
&wt\{F_{\geq 2t+1}(z,w)\}\geq 2t+1,\quad wt\left\{\overline{F_{\geq 2t+1}(z,w)}\right\}\geq ts+s-1,\quad wt\left\{\|F_{\geq 2t+1}(z,w)\|^{2}\right\}\geq ts+2,\\
&wt\left\{F^{(2t)}_{nor}(z,w)\right\}\geq ts+2-s,\quad wt\left\{\overline{F^{(2t)}_{nor}(z,w)}\right\}\geq ts,\quad wt\left\{\left\|F^{(2t)}_{nor}(z,w)\right\|^{2}\right\}\geq ts+2,\\
&wt\left\{\left\langle F^{(2t)}_{nor}(z,w),F_{\geq 2t+1}(z,w)\right\rangle\right\},\quad wt\left\{\left\langle F_{\geq 2t+1}(z,w),F^{(2t)}_{nor}(z,w)\right\rangle\right\}\geq ts+2,
\end{aligned} \tag{3.11}$$

where w satisfies (3.1).

As a consequence of the preceding estimates, we obtain

$$\|F(z,w)\|^{2}=\left\|F^{(2t)}_{nor}(z,w)\right\|^{2}+2\text{Re}\left\langle F^{(2t)}_{nor}(z,w),F_{\geq 2t+1}(z,w)\right\rangle+\|F_{\geq 2t+1}(z,w)\|^{2}=\Theta^{2t+2}_{ts+2}(z,\overline{z}), \tag{3.12}$$

where w satisfies (3.1). We observe that the preceding power series $\Theta^{2t+2}_{ts+2}(z,\overline{z})$ has the property $\text{wt}\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\overline{z})}\right\}\geq ts+2$.

In order to apply Extended Moser Lemma in (3.5) we have to evaluate the weight and the order of the terms which appear and are not ,,good". Beside the previous weight estimates (see (3.11) and (3.12)) we also need to prove the following lemmas:

Lemma 3.3. *For all $m,n\geq 1$ and w satisfying (3.1), we have the following estimate*

$$\varphi'_{m,n}\left(z+F(z,w),\overline{z+F(z,w)}\right)=\varphi'_{m,n}(z,\overline{z})+2\text{Re}\left\langle\Theta^{2}_{s}(z,\overline{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle+\Theta^{2t+2}_{ts+2}(z,\overline{z}), \tag{3.13}$$

where $wt\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\overline{z})}\right\}\geq ts+2$.

Proof. We make the expansion $\varphi'_{m,n}\left(z+F(z,w),\overline{z+F(z,w)}\right)=\varphi'_{m,n}(z,\overline{z})+\dots$, where in ,,...” we have different types of terms involving $F_{k',l'}(z)$ with $k'+2l'<m+n$ and normalized terms $\varphi_{k,l}(z,\overline{z})$, $\varphi'_{k,l}(z,\overline{z})$ with $k+l<m+n$. In order to study the weight and the order of terms which can appear in ,,...” it is enough to study the weight and the order of the following particular terms

$$A_1(z,w)=F_1(z,w)z^{I}\overline{z}^{J},\quad A_2(z,w)=z^{I_1}\overline{z}^{J_1}\overline{F_1(z,w)},\quad B_1(z,w)=F_2(z,w)z^{I}\overline{z}^{J},\quad B_2(z,w)=\overline{F_2(z,w)}z^{I_1}\overline{z}^{J_1},$$

where $F_1(z,w)$ is the first component of $F^{(2t)}_{nor}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+1}(z,w)$. Here we assume that $|I|=m-1$, $|I_1|=m$, $|J_1|=n-1$, $|J|=n$.

Using (3.11) we obtain $\text{wt}\{A_1(z,w)\}\geq m-1+ts+2-s+n(s-1)\geq ts+2$. It is equivalent to prove that $m-1+s(n-1)-n\geq 0$. This is true because $m-1+s(n-1)-n\geq m-1+3(n-1)-n\geq m+3n-4-n\geq 3+n-4\geq 0$. On the other hand, we have $\text{Ord}\{A_1(z,w)\}\geq m-1+2t+n\geq 2t+2$.

Using (3.11) we obtain $\text{wt}\{A_2(z,w)\}\geq m+ts+(n-1)(s-1)\geq ts+2\iff m+(s-1)(n-1)\geq 2$. We have $m+(n-1)(s-1)\geq m+2(n-1)\geq m+2n-4\geq 0$, and this is true because $m+n\geq 3$ and $m,n\geq 1$. On the other hand we have $\text{Ord}\{A_2(z,w)\}\geq m+2t+n-1\geq 2t+2$.

In the same way we obtain that $\text{Ord}\{B_1(z,w)\}$, $\text{Ord}\{B_2(z,w)\}\geq 2t+1$. Using (3.11), every term from ,,...” that depends on $F_2(z,w)$ can be written as $\Theta^{2}_{s}(z,\overline{z})F_2(z,w)$. From here we obtain our claim. □

Lemma 3.4. *For w satisfying 3.1) and for all $k>s$, we have the following estimation*

$$\varphi'_{k}\left(z+F(z,w)\right)=\varphi'_{k}(z)+2\text{Re}\left\langle\Theta^{2}_{s}(z,\overline{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle+\Theta^{2t+2}_{ts+2}(z,\overline{z}), \tag{3.14}$$

where $wt\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\overline{z})}\right\}\geq ts+2$.

Proof. We make the expansion $\varphi_k'\left(z+F(z,w)=\varphi_k'(z)+\dots\right.$. To study the weight and the order of terms which can appear in „. . . ” it is enough to study the weight and the order of the following terms

$$A(z,w)=F_1(z,w)z^I,\quad B(z,w)=F_2(z,w)z^I,$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+1}(z,w)$. Here we assume that $|I|=m-1\geq s$. Then, by (3.11), we obtain that $\text{wt}\left\{A(z,w)\right\}\geq s+ts+2-s\geq ts+2$. On the other hand, we have $\text{Ord}\left\{A(z,w)\right\}\geq s+2t\geq 2t+2$. Using (3.11), every term from „. . . ” that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\overline{z})F_2(z,w)$. From here we obtain our claim. □

We want to evaluate the weight and the order of the other terms of (3.5). By Lemma 4.3 and Lemma 4.4, it remains to evaluate the order and the weight of the terms of the following expression

$$\begin{aligned}S(z,\overline{z})&=2\text{Re}\left\langle F(z,w),z\right\rangle+2\text{Re}\left\{\varphi_s'\left(z+F(z,w)\right)\right\},\\&=2\text{Re}\left\langle F_{nor}^{(2t)}(z,w)+F_{\geq 2t+1}(z,w),z\right\rangle+2\text{Re}\left\{\Delta\left(z+F_{nor}^{(2t)}(z,w)+F_{\geq 2t+1}(z,w)\right)\right\},\end{aligned}\tag{3.15}$$

where w satisfies (3.1).

Lemma 3.5. *For $F_{nor}^{(2t)}(z,w)$ given by Lemma 4.1 and w satisfying (3.1) we have*

$$2\text{Re}\left\langle F_{nor}^{(2t)}\left(z,w\right),z\right\rangle=2\text{Re}\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\}+\Theta_{ts+2}^{2t+2}(z,\overline{z}),\tag{3.16}$$

where $wt\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\}\geq ts+2.$

Proof. We compute

$$\begin{aligned}2\text{Re}\left\langle F_{nor}^{(2t)}\left(z,w\right),z\right\rangle&=2\text{Re}\left\{w^t\left\langle a,z\right\rangle\right\}-2\text{Re}\left\{\langle z,a\rangle\langle z,z\rangle w^{t-1}\right\},\\&=2\text{Re}\left\{\langle z,a\rangle\,w^t-\langle z,a\rangle\langle z,z\rangle w^{t-1}\right\}+\langle a,z\rangle\left(w^t-\overline{w}^t\right)+\langle z,a\rangle\left(\overline{w}^t-w^t\right),\\&=2\text{Re}\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\}+\Theta_{ts+2}^{2t+2}(z,\overline{z}),\end{aligned}\tag{3.17}$$

where $\text{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\}\geq ts+2.$ □

In course of our proof we will use the notation $\Delta'(z)=(\Delta_1(z),\dots,\Delta_N(z))$. It remains to prove the lemma

Lemma 3.6. *For w satisfying (3.1) we have the following estimate*

$$\begin{aligned}2\text{Re}\left\{\Delta\left(z+F(z,w)\right)\right\}=&2\text{Re}\left\{\Delta(z)-s\langle z,a\rangle\Delta(z)w^{t-1}\right\}\\&+2\text{Re}\left\langle\Delta'(z)+\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle+\Theta_{ts+2}^{2t+2}(z,\overline{z}),\end{aligned}\tag{3.18}$$

where $wt\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\}\geq ts+2.$

Proof. Using the Taylor expansion it follows that

$$2\text{Re}\left\{\Delta\left(z+F(z,w)\right)\right\}=2\text{Re}\left\{\Delta(z)+\sum_{k=1}^{N}\Delta_k(z)F_{\geq 2t}^k(z,w)+L(z,\overline{z})\right\},\tag{3.19}$$

where $F_{\geq 2t}^k(z,w)=\left(F_{\geq 2t}^1(z,w),\dots,F_{\geq 2t}^N(z,w)\right)$ and $L(z,\overline{z})=\left\langle\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle$. We compute

$$\begin{aligned}\sum_{k=1}^{N}2\text{Re}\left\{\Delta_k(z)F_{\geq 2t}^k(z,w)\right\}&=\sum_{k=1}^{N}2\text{Re}\left\{\Delta_k(z)\left(a_kw^t-z_k\langle z,a\rangle w^{t-1}+F_{\geq 2t+1}^k(z,w)\right)\right\},\\&=\Theta_{ts+2}^{2t+2}(z,\overline{z})-2s\text{Re}\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\}+2\text{Re}\left\langle\Delta'(z),\overline{F_{\geq 2t+1}(z,w)}\right\rangle,\end{aligned}\tag{3.20}$$

where $\text{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\}\geq ts+2.$ □

For w satisfying (3.1), by Lemma 3.5 and Lemma 3.6, we can rewrite (3.15) as follows

$$S(z,\overline{z}) = 2(1-s)\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\} + 2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta_s^2(z,\overline{z}), \overline{F_{\geq 2t+1}(z,w)}\right\rangle + \Theta_{ts+2}^{2t+2}(z,\overline{z}), \tag{3.21}$$

where wt $\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\} \geq ts+2$. By Lemmas 3.1 − 3.6 we obtain

$$\begin{aligned} G_{\geq 2t+2}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right) =& 2(1-s)\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)\left(\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right)^{t-1}\right\} \\ &+ 2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta_s^2(z,\overline{z}), \overline{F_{\geq 2t+1}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right)}\right\rangle \\ &+ \varphi_{\geq 2t+2}(z,\overline{z}) - \varphi'_{\geq 2t+2}(z,\overline{z}) + \Theta_{ts+2}^{2t+2}(z,\overline{z}), \end{aligned} \tag{3.22}$$

where wt $\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\overline{z})}\right\} \geq ts+2$.

Assume that $t=1$. Collecting the terms of total degree $k<s+1$ in $(z,\overline{z})$ from (3.22) we find the polynomials $\left(G_{nor}^{(k+1)}(z,w), F_{nor}^{(k)}(z,w)\right)$ for all $k<s$. Collecting the terms of total degree $m+n=s+1$ in $(z,\overline{z})$ from (3.22), we obtain

$$G_{nor}^{(s+1)}\left(z,\langle z,z\rangle\right) = 2(1-s)\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)\right\} + 2\mathsf{Re}\left\langle z, F_{nor}^{(s)}\left(z,\langle z,z\rangle\right)\right\rangle + \varphi'_{s+1}(z,\overline{z}) - \varphi_{s+1}(z,\overline{z}) + (\Theta_1)_{s+2}^{s+1}(z,\overline{z}). \tag{3.23}$$

By applying Extended Moser Lemma we find a solution $\left(G_{nor}^{(s+1)}(z,w), F_{nor}^{(s)}(z,w)\right)$ for the latter equation. We consider the following Fisher decompositions

$$\varphi_{s+1,0}(z) = Q(z)\Delta(z)+R(z), \quad \varphi'_{s+1,0}(z) = Q'(z)\Delta(z)+R'(z), \tag{3.24}$$

where $\Delta^\star\left(R(z)\right) = \Delta^\star\left(R'(z)\right) = 0$. We want to put the normalization condition $\Delta^\star\left(\varphi'_{s+1,0}(z)\right) = 0$. Collecting the pure terms of degree $s+1$ in (3.23), by (3.24) we obtain

$$\varphi'_{s+1,0}(z) = \varphi_{s+1,0}(z) - (1-s)\langle z,a\rangle\Delta(z) = \left(Q(z)-(1-s)\langle z,a\rangle\right)\Delta(z)+R(z), \tag{3.25}$$

where $Q(z)$ is a determined polynomial of degree 1 in $z_1,\ldots,z_N$. It follows that $Q'(z) = Q(z)-(1-s)\langle z,a\rangle$ and $R'(z) = R(z)$. Then the normalization condition $\Delta^\star\left(\varphi'_{s+1,0}(z)\right) = 0$ is equivalent to find a such that $Q'(z) = Q(z) - (1-s)\langle z,a\rangle = 0$. The last equation provides us the free parameter a.

Assuming that $t\geq 2$, we prove the following lemma (this is the analogue of the Lemma 3.3 from [9]):

Lemma 3.7. *Let $N_s := ts+2$. For all $0\leq j\leq t-1$ and $p\in[2t+j(s-2)+2, 2t+(j+1)(s-2)+1]$, we make the following estimate*

$$\begin{aligned} G_{\geq p}(z,w) =& 2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + 2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta_s^2(z,\overline{z}), \overline{F_{\geq p-1}(z,w)}\right\rangle \\ &+ \varphi'_{\geq p}(z,\overline{z}) - \varphi_{\geq p}(z,\overline{z}) + \Theta_{N_s}^{p}(z,\overline{z}), \end{aligned} \tag{3.26}$$

where wt $\left\{\overline{\Theta_{N_s}^{2t+2}(z,\overline{z})}\right\} \geq N_s$ *and w satisfies (3.1).*

Proof.

Step 1. When $s=3$ this step is obvious. Assume that $s>3$. Let $p_0 = 2t+j(s-2)+2$, where $j\in[0,t-1]$. We make induction on $p\in[2t+j(s-2)+2, 2t+(j+1)(s-2)+1]$. For $j=0$ (therefore $p=2t+2$) the lemma is satisfied (see equation (3.22)). Let $p\geq p_0$ such that $p+1\leq 2t+(j+1)(s-2)+1$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (3.26) with $m+n=p$, we obtain

$$G_{nor}^{(p)}\left(z,\langle z,z\rangle\right) = 2\mathsf{Re}\left\langle z, F_{nor}^{(p-1)}\left(z,\langle z,z\rangle\right)\right\rangle + \varphi'_p(z,\overline{z}) - \varphi_p(z,\overline{z}) + (\Theta_1)_{N_s}^{p}(z,\overline{z}). \tag{3.27}$$

By applying Extended Moser Lemma we find a solution $\left(F_{nor}^{(p-1)}(z,w), G_{nor}^{(p)}(z,w)\right)$ for (3.27). Assume that p is even. In this case we find $F_{nor}^{(p-1)}(z,w)$ recalling the cases 1 and 3 of the Extended Moser Lemma proof. Using the cases 2 and 4 of the Extended Moser Lemma proof we find $G_{nor}^{(p)}(z,w)$. Since wt $\left\{\overline{(\Theta_1)_{N_s}^{2t+2}(z,\overline{z})}\right\} \geq N_s$ we obtain wt $\left\{\left\langle F_{nor}^{(p-1)}\left(z,\langle z,z\rangle\right), z\right\rangle\right\}$, wt $\left\{\overline{\left\langle F_{nor}^{(p-1)}\left(z,\langle z,z\rangle\right), z\right\rangle}\right\} \geq N_s$. Also wt $\left\{G_{nor}^{(p)}\left(z,\langle z,z\rangle\right)\right\}$, wt $\left\{\overline{G_{nor}^{(p)}\left(z,\langle z,z\rangle\right)}\right\} \geq N_s$.

We can bring similarly arguments when p is even. We obtain the following estimates

$$\begin{gathered}\mathrm{wt}\left\{F^{(p-1)}_{nor}(z,w)\right\}\geq N_s-s+1,\ \mathrm{wt}\left\{\overline{F^{(p-1)}_{nor}(z,w)}\right\}\geq N_s-1,\\ \mathrm{wt}\left\{\overline{F^{(p-1)}_{nor}(z,w)-F^{(p-1)}_{nor}(z,\langle z,z\rangle)}\right\}\geq N_s-1,\quad \mathrm{wt}\left\{F^{(p-1)}_{nor}(z,w)-F^{(p-1)}_{nor}(z,\langle z,z\rangle)\right\}\geq N_s-s+1,\\ \mathrm{wt}\left\{G^{(p)}_{nor}(z,w)\right\}\geq N_s,\ \mathrm{wt}\left\{G^{(p)}_{nor}(z,w)-G^{(p)}_{nor}(z,\langle z,z\rangle)\right\}\geq N_s,\end{gathered}\tag{3.28}$$

where w satisfies (3.1). As a consequence of (3.27) we obtain

$$\begin{gathered}G^{(p)}_{nor}(z,w)-G^{(p)}_{nor}(z,\langle z,z\rangle)=\Theta^{p+1}_{N_s}(z,\overline{z})',\quad 2\mathsf{Re}\left\langle z,F^{(p-1)}_{nor}(z,w)-F^{(p-1)}_{nor}(z,\langle z,z\rangle)\right\rangle=\Theta^{p+1}_{N_s}(z,\overline{z})',\\ \left\langle\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(p-1)}_{nor}(z,w)}\right\rangle+\left\langle\overline{F^{(p-1)}_{nor}(z,w)},\Delta'(z)+\Theta^2_s(z,\overline{z})\right\rangle=\Theta^{p+1}_{N_s}(z,\overline{z})',\end{gathered}\tag{3.29}$$

and each of the preceding formal power series $\Theta^{p+1}_{N_s}(z,\overline{z})'$ has the property $\mathrm{wt}\left\{\overline{\Theta^{p+1}_{N_s}(z,\overline{z})'}\right\}\geq N_s$. Substituting $F_{\geq p-1}(z,w)=F^{(p-1)}_{nor}(z,w)+F_{\geq p}(z,w)$ and $G_{\geq p}(z,w)=G^{(p)}_{nor}(z,w)+G_{\geq p+1}(z,w)$ into (3.26), we obtain

$$\begin{aligned}G^{(p)}_{nor}(z,w)+G_{\geq p+1}(z,w)=&2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}\\ &+2\mathsf{Re}\left\langle\overline{z}+\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(p-1)}_{nor}(z,w)}+\overline{F_{\geq p}(z,w)}\right\rangle+\varphi'_p(z,\overline{z})-\varphi_p(z,\overline{z})\\ &+(\Theta_1)^p_{N_s}(z,\overline{z})+\varphi'_{\geq p+1}(z,\overline{z})-\varphi_{\geq p+1}(z,\overline{z})+\Theta^{p+1}_{N_s}(z,\overline{z}).\end{aligned}\tag{3.30}$$

Collecting the pure terms of degree p from (3.27), it follows that $\varphi_{p,0}(z)=\varphi'_{p,0}(z)+\dots$, where in „…" we have determined terms with the weight less than $p<N_s:=ts+2$. Therefore $\varphi_{p,0}(z)=\varphi'_{p,0}(z)$. We will obtain that $\varphi_{k,0}(z)=\varphi'_{k,0}(z)$, for all $k=3,\dots,T$. By making a simplification in (3.30) using (3.27), it follows that

$$\begin{aligned}G_{\geq p+1}(z,w)=&2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}+2\mathsf{Re}\left\langle\overline{z}+\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F_{\geq p}(z,w)}\right\rangle\\ &+\varphi'_{\geq p+1}(z,\overline{z})-\varphi_{\geq p+1}(z,\overline{z})+J(z,\overline{z})+\Theta^{p+1}_{N_s}(z,\overline{z}),\end{aligned}\tag{3.31}$$

where $\mathrm{wt}\left\{\overline{\Theta^{p+1}_{N_s}(z,\overline{z})}\right\}\geq N_s$ and

$$\begin{aligned}J(z,\overline{z})=&2\mathsf{Re}\left\langle z,F^{(p-1)}_{nor}(z,w)-F^{(p-1)}_{nor}(z,\langle z,z\rangle)\right\rangle+2\mathsf{Re}\left\langle\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(p-1)}_{nor}(z,w)}\right\rangle\\ &+G^{(p)}_{nor}(z,\langle z,z\rangle)-G^{(p)}_{nor}(z,w).\end{aligned}\tag{3.32}$$

Using (3.28) and (3.29) it follows that $J(z,\overline{z})=\Theta^{p+1}_{N_s}(z,\overline{z})$, where $\mathrm{wt}\left\{\overline{\Theta^{p+1}_{N_s}(z,\overline{z})}\right\}\geq N_s$.

Step 2. Assume that we have proved the Lemma 3.7 for $p\in[2t+j(s-2)+2,2t+(j+1)(s-2)+1]$ for $j\in[0,t-1]$. We prove the Lemma 3.7 for $p\in[2t+(j+1)(s-2)+2,2t+(j+2)(s-2)+1]$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (3.26) with $m+n=\Lambda+1:=2t+(j+1)(s-2)+1$, we obtain

$$\begin{aligned}G^{(\Lambda+1)}_{nor}(z,\langle z,z\rangle)=&2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\right\}+2\mathsf{Re}\left\langle z,F^{(\Lambda)}_{nor}(z,\langle z,z\rangle)\right\rangle\\ &+\varphi'_{\Lambda+1}(z,\overline{z})-\varphi_{\Lambda+1}(z,\overline{z})+(\Theta_1)^{\Lambda+1}_{N_s}(z,\overline{z}).\end{aligned}\tag{3.33}$$

Here $\mathrm{wt}\left\{\overline{(\Theta_1)^{\Lambda+1}_{N_s}(z,\overline{z})}\right\}\geq N_s$. We define the map

$$F^{(\Lambda)}_{nor}(z,w)=F^{(\Lambda)}_1(z,w)+F^{(\Lambda)}_2(z,w),\quad F^{(\Lambda)}_1(z,w)=-(1-s)^{j+1}\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}\left(z_1,\dots,z_N\right).\tag{3.34}$$

Substituting (3.34) into (3.33), we obtain

$$G^{\Lambda+1}_{nor}(z,\langle z,z\rangle)=2\mathsf{Re}\left\langle z,F^{(\Lambda)}_2(z,\langle z,z\rangle)\right\rangle+\varphi_{\Lambda+1}(z,\overline{z})-\varphi_{\Lambda+1}(z,\overline{z})+(\Theta_1)^{\Lambda+1}_{N_s}(z,\overline{z}).\tag{3.35}$$

By applying Extended Moser Lemma we find a solution $\left(G^{(\Lambda+1)}_{nor}(z,w), F^{(\Lambda)}_{2}(z,w)\right)$ for (3.35). Using the same arguments as in the Step 1 we obtain the following estimates

$$\begin{aligned} &\text{wt}\left\{G^{(\Lambda+1)}_{nor}(z,w)-G^{(\Lambda+1)}_{nor}\left(z,\langle z,z\rangle\right)\right\}, \text{wt}\left\{G^{(\Lambda+1)}_{nor}(z,w)\right\},\ \text{wt}\left\{G^{(\Lambda+1)}_{nor}\left(z,\langle z,z\rangle\right)\right\}\geq N_s,\\ &\text{wt}\left\{F^{(\Lambda)}_{2}(z,w)-F^{(\Lambda)}_{2}\left(z,\langle z,z\rangle\right)\right\},\ \ \text{wt}\left\{F^{(\Lambda)}_{2}(z,w)\right\}, \text{wt}\left\{F^{(\Lambda)}_{2,k}\left(z,\langle z,z\rangle\right)\right\}\geq N_s-s+1,\\ &\text{wt}\left\{\overline{F^{(\Lambda)}_{2}(z,w)-F^{(\Lambda)}_{2}\left(z,\langle z,z\rangle\right)}\right\}, \text{wt}\left\{\overline{F^{(\Lambda)}_{2}(z,w)}\right\},\ \text{wt}\left\{\overline{F^{(\Lambda)}_{2}\left(z,\langle z,z\rangle\right)}\right\}\geq N_s-1, \end{aligned} \tag{3.36}$$

where w satisfies (3.1). As a consequence of (3.36) we obtain

$$\begin{aligned} &\left\langle \Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{2}(z,w)}\right\rangle+\left\langle \overline{F^{(\Lambda)}_{2}(z,w)},\Delta'(z)+\Theta^2_s(z,\overline{z})\right\rangle=\Theta^{\Lambda+2}_{N_s}(z,\overline{z})',\\ &G^{(\Lambda+1)}_{nor}(z,w)-G^{(\Lambda+1)}_{nor}\left(z,\langle z,z\rangle\right)=\Theta^{\Lambda+2}_{N_s}(z,\overline{z})',\quad 2\mathsf{Re}\left\langle F^{(\Lambda)}_{2}(z,w)-F^{(\Lambda)}_{2}\left(z,\langle z,z\rangle\right),z\right\rangle=\Theta^{\Lambda+2}_{N_s}(z,\overline{z})', \end{aligned} \tag{3.37}$$

where w satisfies (3.1) and each of the preceding formal power series has the property wt $\left\{\overline{\Theta^{\Lambda+2}_{N_s}(z,\overline{z})}\right\}\geq N_s$. Substituting $F_{\geq\Lambda}(z,w)=F^{(\Lambda)}_{nor}(z,w)+F_{\geq\Lambda+1}(z,w)$ and $G_{\geq\Lambda+1}(z,w)=G^{(\Lambda+1)}_{nor}(z,w)+G_{\geq\Lambda+2}(z,w)$ in (3.26), we obtain

$$\begin{aligned} G^{(\Lambda+1)}_{nor}(z,w)+G_{\geq\Lambda+2}(z,w)=&\,2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}\\ &+2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{nor}(z,w)}+\overline{F_{\geq\Lambda+1}(z,w)}\right\rangle+\varphi_{\Lambda+1}(z,\overline{z})-\varphi'_{\Lambda+1}(z,\overline{z})\\ &+\varphi'_{\geq\Lambda+2}(z,\overline{z})-\varphi_{\geq\Lambda+2}(z,\overline{z})+\left(\Theta_1\right)^{\Lambda+1}_{N_s}(z,\overline{z})+\Theta^{\Lambda+2}_{N_s}(z,\overline{z}). \end{aligned} \tag{3.38}$$

By making a simplification in (3.38) with (3.33), and then using (3.34), we obtain

$$G_{\geq\Lambda+2}(z,w)=2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F_{\geq\Lambda+1}(z,w)}\right\rangle+\varphi'_{\geq\Lambda+2}(z,\overline{z})-\varphi_{\geq\Lambda+2}(z,\overline{z})+\Theta^{\Lambda+2}_{N_s}(z,\overline{z})+J(z,\overline{z}), \tag{3.39}$$

where

$$\begin{aligned} J(z,\overline{z})=&\,2\mathsf{Re}\left\langle z,F^{(\Lambda)}_{nor}(z,w)-F^{(\Lambda)}_{nor}\left(z,\langle z,z\rangle\right)\right\rangle+2\mathsf{Re}\left\langle \Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{nor}(z,w)}\right\rangle\\ &+2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}-\langle z,a\rangle\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\right\}\\ &+G^{(\Lambda)}_{nor}\left(z,\langle z,z\rangle\right)-G^{(\Lambda)}_{nor}(z,w),\\ =&\,2\mathsf{Re}\left\langle z,F^{(\Lambda)}_{1}(z,w)-F^{(\Lambda)}_{1}\left(z,\langle z,z\rangle\right)+F^{(\Lambda)}_{2}(z,w)-F^{(\Lambda)}_{2}\left(z,\langle z,z\rangle\right)\right\rangle\\ &+2\mathsf{Re}\left\langle \Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{1}(z,w)}+\overline{F^{(\Lambda)}_{2}(z,w)}\right\rangle+G^{(\Lambda)}_{nor}\left(z,\langle z,z\rangle\right)-G^{(\Lambda)}_{nor}(z,w)\\ &+2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}\left(w^{t-j-1}-\langle z,z\rangle^{t-j-1}\right)\right\}. \end{aligned} \tag{3.40}$$

Using (3.36) and (3.37) it follows that

$$\begin{aligned} J(z,\overline{z})=&\,2\mathsf{Re}\left\langle z,F^{(\Lambda)}_{1}(z,w)-F^{(\Lambda)}_{1}\left(z,\langle z,z\rangle\right)\right\rangle+2\mathsf{Re}\left\langle \Delta'(z)+\Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{1}(z,w)}\right\rangle\\ &+2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}\left(w^{t-j-1}-\langle z,z\rangle^{t-j-1}\right)\right\}+\Theta^{\Lambda+2}_{N_s}(z,\overline{z}), \end{aligned} \tag{3.41}$$

where wt $\left\{\overline{\Theta^{\Lambda+2}_{N_s}(z,\overline{z})}\right\}\geq N_s$. We observe that

$$\mathsf{Re}\left\langle z,F^{(\Lambda)}_{1}\left(z,\langle z,z\rangle\right)\right\rangle=-(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\langle z,z\rangle^{t-j-1}\Delta(z)^{j+1}\right\}. \tag{3.42}$$

Since wt $\left\{F^{(\Lambda)}_{1}(z,w)\right\}\geq N_s-s$ and wt $\left\{\overline{F^{(\Lambda)}_{1}(z,w)}\right\}\geq N_s$, it follows that

$$\mathsf{Re}\left\langle \Theta^2_s(z,\overline{z}),\overline{F^{(\Lambda)}_{1}(z,w)}\right\rangle=\Theta^{\Lambda+2}_{N_s}(z,\overline{z}), \tag{3.43}$$

where wt $\left\{\overline{\Theta^{\Lambda+2}_{N_s}(z,\overline{z})}\right\}\geq N_s$. Using (3.42) and (3.43), we can rewrite (3.41) as follows

$$J(z,\overline{z})=2\mathsf{Re}\left\langle z,F^{(\Lambda)}_{1}(z,w)\right\rangle+2\mathsf{Re}\left\langle \Delta'(z),\overline{F^{(\Lambda)}_{1}(z,w)}\right\rangle+2(1-s)^{j+1}\mathsf{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}+\Theta^{\Lambda+2}_{N_s}(z,\overline{z}), \tag{3.44}$$

where wt $\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\overline{z})}\right\} \geq N_s$. Substituting the formula of $F_1^{(\Lambda)}(z,w)$ in (3.44), we obtain

$$\begin{aligned} J(z,\overline{z}) &= -2(1-s)^{j+1}\mathrm{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}\left(\langle z,z\rangle+\left\langle (z_1,\ldots,z_N),\left(\overline{\Delta_1(z)},\ldots,\overline{\Delta_N(z)}\right)\right\rangle\right)\right\} \\ &\quad + 2(1-s)^{j+1}\mathrm{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + \Theta_{N_s}^{\Lambda+2}(z,\overline{z}), \\ &= -2(1-s)^{j+1}\mathrm{Re}\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}\left(\langle z,z\rangle + s\Delta(z) - w\right)\right\} + \Theta_{N_s}^{\Lambda+2}(z,\overline{z}), \\ &= 2(1-s)^{j+2}\mathrm{Re}\left\{\langle z,a\rangle\Delta(z)^{j+2}w^{t-j-2}\right\} + \Theta_{N_s}^{\Lambda+2}(z,\overline{z}), \end{aligned} \tag{3.45}$$

where w satisfies (3.1) and wt $\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\overline{z})}\right\} \geq N_s$. □

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n=ts+1$ and $t=j-1$ from (3.26), we obtain

$$\begin{aligned} G_{nor}^{(ts+1)}\left(z,\langle z,z\rangle\right) =& 2(1-s)^t\mathrm{Re}\left\{\langle z,a\rangle\Delta(z)^t\right\} + 2\mathrm{Re}\left\langle z, F_{nor}^{(ts)}\left(z,\langle z,z\rangle\right)\right\rangle \\ &+ \varphi'_{ts+1,0}(z,\overline{z}) - \varphi_{ts+1,0}(z,\overline{z}) + \left(\Theta_1\right)_{N_s}^{ts+1}(z,\overline{z}). \end{aligned} \tag{3.46}$$

By applying Extended Moser Lemma we find a solution $\left(G_{nor}^{(ts+1)}(z,w), F_{nor}^{(ts)}(z,w)\right)$ for (3.46). Collecting the pure terms from (3.46) of degree $ts+1$, it follows that

$$\varphi'_{ts+1,0}(z) - \varphi_{ts+1,0}(z) = (1-s)^t\langle z,a\rangle\Delta(z)^t. \tag{3.47}$$

The parameter a will help us to put the desired normalization condition (see (1.15)). By applying Lemma 2.4 for $\varphi'_{ts+1,0}(z)$ and $\varphi'_{ts+1,0}(z)$, it follows that

$$\varphi_{ts+1,0}(z) = (1-s)^tQ(z)\Delta(z)^t + R(z), \quad \varphi'_{ts+1,0}(z) = Q'(z)\Delta(z)^t + R'(z), \tag{3.48}$$

where $\left(\Delta^t\right)^\star(R(z)) = \left(\Delta^t\right)^\star(R'(z)) = 0$. We impose the normalization condition $\left(\Delta^t\right)^\star\left(\varphi'_{ts+1,0}(z)\right) = 0$. This is equivalent finding a such that $Q'(z) = 0$. Here $Q(z)$ is a determined holomorphic polynomial. We find a by solving the equation $Q'(z) = (1-s)^t\langle z,a\rangle - Q(z) = 0$.

Composing the map that sends M into (3.1) with the map (3.3) we obtain our formal transformation that sends M into M' up to degree $ts+1$.

4. Proof of Theorem 1.3-Case $T+1=(t+1)s,\ t\geq 1$

In this case we are looking for a biholomorphic transformation of the following type

$$\begin{aligned} &(z',w') = (z+F(z,w), w+G(z,w)) \\ &F(z,w) = \sum_{l=0}^{T-2t-1} F_{nor}^{(2t+l+1)}(z,w),\ G(z,w) = \sum_{\tau=0}^{T-2t} G_{nor}^{(2t+2+\tau)}(z,w)' \end{aligned} \tag{4.1}$$

that maps M into M' up to the degree $T+1=(t+1)s$. In order to make the mapping (4.1) uniquely determined we assume that $F_{nor}^{(2t+l+1)}(z,w)$ is normalized as in Extended Moser Lemma, for all $l=1,\ldots T$. Replacing (4.1) in (3.2), and after a simplification with (3.1), we obtain

$$\begin{aligned} &\sum_{\tau=0}^{T-2t-1} G_{nor}^{(2t+2+\tau)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right) = 2\mathrm{Re}\left\langle \sum_{l=0}^{T-2t-1} F_{nor}^{(2t+l+1)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right), z\right\rangle \\ &+ \left\| \sum_{l=0}^{T-2t-1} F_{nor}^{(2t+l+1)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right)\right\|^2 \\ &+ \varphi'_{\geq 3}\left(z+\sum_{l=-1}^{T-2t} F_{nor}^{(2t+l+2)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right), \overline{z+\sum_{l=-1}^{T-2t} F_{nor}^{(2t+l+2)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\overline{z})\right)}\right) - \varphi_{\geq 3}(z,\overline{z}). \end{aligned} \tag{4.2}$$

Collecting the terms with the same bidegree in $(z,\overline{z})$ from (4.2) we will find $F(z,w)$ and $G(z,w)$ by applying Extended Moser Lemma. Since $F(z,w)$ and $G(z,w)$ don't have components of normal weight less than $2t+2$, collecting in (4.2) the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n<2t+2$, we obtain $\varphi'_{m,n}(z,\overline{z}) = \varphi_{m,n}(z,\overline{z})$.

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n=2t+2$ from (4.2), we prove the following lemma:

Lemma 4.1. *$G^{(2t+2)}_{nor}(z,w)=(a+\overline{a})\,w^{t+1},\ F^{(2t+1)}_{nor}(z,w)=w^t\begin{pmatrix} a_{11} & \dots & a_{1N}\\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN}\end{pmatrix}\begin{pmatrix} z_1\\ \vdots\\ z_N\end{pmatrix}$, where a is the trace of the matrix $(a_{ij})_{1\leq i,j\leq N}$.*

Proof. Collecting the pure terms of degree $2t+2$ from (4.2), we obtain that $\varphi_{2t+2}(z)=\varphi'_{2t+2}(z)$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n=2t+2$ and $0<m<n-1$ from (4.2), we obtain

$$\varphi'_{m,n}(z,\overline{z})=-\left\langle z,F_{n-m+1,m-1}(z)\right\rangle\left\langle z,z\right\rangle^{m-1}+\varphi_{m,n}(z,\overline{z}). \tag{4.3}$$

Since $\varphi_{m,n}(z,\overline{z})$, $\varphi'_{m,n}(z,\overline{z})$ satisfy (1.14), by the uniqueness of trace decomposition, we obtain $F_{n-m+1,m-1}(z)=0$. Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n=2t+2$ and $m>n+1$ from (4.2), we obtain

$$\varphi'_{m,n}(z,\overline{z})=G_{m-n}(z)\langle z,z\rangle^{n}-\left\langle F_{m-n+1,n-1}(z),z\right\rangle\left\langle z,z\right\rangle^{n-1}+\varphi_{m,n}(z,\overline{z}). \tag{4.4}$$

Since $F_{n-m+1,m-1}(z)=0$ it follows that $G_{m-n}(z)=0$.

Collecting the terms of bidegree $(t+1,t+1)$ in $(z,\overline{z})$ from (4.2), we obtain

$$\varphi'_{t+1,t+1}(z,\overline{z})=\left(G_{0,t+1}(z)\langle z,z\rangle-\langle F_{1,t}(z),z\rangle-\langle z,F_{1,t}(z)\rangle\right)\langle z,z\rangle^{t}+\varphi_{t+1,t+1}(z,\overline{z}). \tag{4.5}$$

Then (4.5) can not provide us $F_{1,t}(z)$. Therefore $F_{1,t}(z)$ is undetermined. We obtain

$$F^{(2t+1)}_{nor}(z,w)=w^t\begin{pmatrix} a_{11} & \dots & a_{1N}\\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN}\end{pmatrix}\begin{pmatrix} z_1\\ \vdots\\ z_N\end{pmatrix},\quad a_{ij}\in\mathbb{C},\ 1\leq i,j\leq N. \tag{4.6}$$

The trace of the matrix a is determined by $a_{11},\dots,a_{NN}$. We can write $a_{11}=a+b_{11},\dots,a_{NN}=a+b_{NN}$. We also use the notations $b_{k,j}=a_{k,j}$, for all $k\neq j$. The matrix $B=(b_{k,j})_{1\leq k,j\leq N}$ represents the traceless part of the matrix A. By applying Lemma 2.1 to the polynomial $\langle F_{1,t}(z),z\rangle$, we obtain $\langle F_{1,t}(z),z\rangle=a\langle z,z\rangle+P(z,\overline{z})$ with $\operatorname{tr}\left(P(z,\overline{z})\right)=0$, where $P(z,\overline{z})=\sum_{i,j=1}^{N}b_{i,j}z_i\overline{z}_j$. Using the preceding decomposition we obtain

$$\varphi'_{t+1,t+1}(z,\overline{z})=\left(G_{0,t+1}(z)-a-\overline{a}\right)\langle z,z\rangle^{t+1}+\varphi_{t+1,t+1}(z,\overline{z})-2\mathsf{Re}\,\left(P(z,\overline{z})\langle z,z\rangle^{t}\right). \tag{4.7}$$

Since $\operatorname{tr}\left(P(z,\overline{z})\right)=0$ it follows that $\operatorname{tr}^{t+1}\left(\mathsf{Re}\,\left(P(z,\overline{z})\langle z,z\rangle^{t}\right)\right)=0$ (see Lemma 6.6 from [17]). $\square$

We can write $F(z,w)=F^{(2t+2)}_{nor}(z,w)+F_{\geq 2t+3}(z,w)$ and $G(z,w)=G_{\geq 2t+2}(z,w)$ (see (2.30)). We have $F_{\geq 2t+2}(z,w)=\sum_{k+2l\geq 2t+2}F_{k,l}(z)w^l$, where $F_{k,l}(z)$ is a homogeneous polynomial of degree k. Therefore $\operatorname{wt}\left\{F_{\geq 2t+2}(z,w)\right\}\geq\min_{k+2l\geq 2t+2}\{k+ls\}\geq\min_{k+2l\geq 2t+2}\{k+2l\}\geq 2t+2$. Next, we show that $\operatorname{wt}\left\{\overline{F_{\geq 2t+2}(z,w)}\right\}\geq ts+s-1$. Since $\operatorname{wt}\left\{\overline{F_{\geq 2t+2}(z,w)}\right\}\geq\min_{k+2l\geq 2t+2}\{k(s-1)+ls\}$, it is enough to prove that $k(s-1)+ls\geq ts+s-1$ for $k+2l\geq 2t+2$. Since we can write the latter inequality as $(k-1)(s-1)+ls\geq ts$ for $(k-1)+2l\geq 2t+1$, it is enough to prove that $k(s-1)+ls\geq ts$ for $k+2l\geq 2t+1>2t$. Continuing the calculations like in the previous case we obtain the desired result.

Lemma 4.2. *For w satisfying (3.1), we make the following immediate estimates*

$$\begin{gathered} wt\left\{F^{(2t+1)}_{nor}(z,w)\right\}\geq ts+1,\quad wt\left\{\overline{F^{(2t+1)}_{nor}(z,w)}\right\}\geq ts+s-1,\quad wt\left\{\left\|F^{(2t+1)}_{nor}(z,w)\right\|^2\right\}\geq ts+s+1,\\ wt\left\{F_{\geq 2t+2}(z,w)\right\}\geq 2t+2,\quad \left\{\overline{F_{\geq 2t+2}(z,w)}\right\}\geq ts+s-1,\quad wt\left\{\left\|F_{\geq 2t+2}(z,w)\right\|^2\right\}\geq ts+s+1,\\ wt\left\{\left\langle F^{(2t+1)}_{nor}(z,w),F_{\geq 2t+2}(z,w)\right\rangle\right\},\ wt\left\{\left\langle F_{\geq 2t+2}(z,w),F^{(2t+1)}_{nor}(z,w)\right\rangle\right\}\geq ts+s+1.\end{gathered} \tag{4.8}$$

As a consequence of the estimates (4.8) we obtain

$$\left\|F(z,w)\right\|^2=\left\|F^{(2t+1)}_{nor}(z,w)\right\|^2+2\mathsf{Re}\,\left\langle F^{(2t+1)}_{nor}(z,w),F_{\geq 2t+2}(z,w)\right\rangle+\left\|F_{\geq 2t+2}(z,w)\right\|^2=\Theta^{2t+3}_{ts+s+1}(z,\overline{z}), \tag{4.9}$$

where $\operatorname{wt}\left\{\overline{\Theta^{2t+3}_{ts+s+1}(z,\overline{z})}\right\}\geq ts+s+1$.

In order to apply Extended Moser Lemma in (4.2) we have to identify and weight and order evaluate the terms which are not ,,good". We prove the following lemmas:

Lemma 4.3. *For all $m, n \geq 1$ and w satisfying (3.1), we make the following estimate*

$$\varphi'_{m,n}\left(z+F(z,w),\overline{z+F(z,w)}\right)=\varphi'_{m,n}(z,\overline{z})+2\mathsf{Re}\left\langle\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+2}(z,w)}\right\rangle+\Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{4.10}$$

where $wt\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\}\geq ts+s+1$.

Proof. We have the expansion $\varphi'_{m,n}\left(z+F(z,w),\overline{z+F(z,w)}\right)=\varphi'_{m,n}(z,\overline{z})+\dots$ (see the proof of the Lemma 3.3). In order to prove (4.10), it is enough to study the weight and the order of the following particular terms

$$A_1(z,w)=F_1(z,w)z^I\overline{z}^J,\quad A_2(z,w)=z^{I_1}\overline{z}^{J_1}\overline{F_1(z,w)},\quad B_1(z,w)=F_2(z,w)z^I\overline{z}^J,\quad B_2(z,w)=z^{I_1}\overline{z}^{J_1}\overline{F_2(z,w)},$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t+1)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+2}(z,w)$. Here we assume that $|I|=m-1$, $|J|=n$, $|I_1|=m$, $|J_1|=n-1$.

Using (4.8) we obtain $\mathrm{wt}\left\{A_1(z,w)\right\}\geq m-1+ts+1+n(s-1)\geq ts+s+1 \iff m+ns-n\geq s+1 \iff m+s(n-1)\geq n+1$ and the latter inequality is true since $m+s(n-1)\geq m+3(n-1)\geq n+1$. On the other hand $\mathrm{Ord}\left\{A_1(z,w)\right\}\geq m-1+2t+1+n\geq 2t+3$.

Using (4.8) we obtain $\mathrm{wt}\left\{A_2(z,w)\right\}\geq m+(n-1)(s-1)+ts+s-1\geq ts+s+1$ and the last inequality is equivalent with $m+(n-1)(s-1)\geq 2$. The latter inequality can be proved with the same calculations like in Lemma 3.3 proof. On the other hand, we observe that $\mathrm{Ord}\left\{A_1(z,w)\right\}\geq m+2t+1+n-1\geq 2t+3$.

In the same way we obtain $\mathrm{Ord}\left\{B_1(z,w)\right\}$, $\mathrm{Ord}\left\{B_2(z,w)\right\}\geq 2t+2$. Using (4.8), every term from „…" that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\overline{z})F_2(z,w)$. This proves our claim. □

Lemma 4.4. *For all $k>s$ and w satisfying (3.1), we make the following estimate*

$$\varphi'_{k,0}\left(z+F(z,w)\right)=\varphi'_{k,0}(z)+2\mathsf{Re}\left\langle\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+2}(z,w)}\right\rangle+\Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{4.11}$$

where $wt\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\}\geq ts+s+1$.

Proof. We make the expansion $\varphi'_{k,0}\left(z+F(z,w)\right)=\varphi'_{k,0}(z)+\dots$. To study the weight and the order of terms which can appear in „…" it is enough to study the weight and order of the following terms

$$A(z,w)=F_1(z,w)z^I,\quad B(z,w)=F_2z^I,$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t+1)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+3}(z,w)$. Here we assume that $|I|=m-1\geq s$. From (4.8) we obtain $\mathrm{wt}\left\{A(z,w)\right\}\geq s+ts+1=ts+s+1$. On the other hand, we have $\mathrm{Ord}\left\{A(z,w)\right\}\geq 2t+s+1\geq 2t+3$. Using (4.8) each term from „…" that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\overline{z})F_2(z,w)$. This proves our claim. □

Lemma 4.5. *For w satisfying (3.1) we have the following estimate*

$$\begin{aligned}2\mathsf{Re}\left\{\Delta\left(z+F(z,w)\right)\right\}=&2\mathsf{Re}\left\{\Delta(z)+\sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1+\dots+a_{kN}z_N\right)w^t\right\}\\&+2\mathsf{Re}\left\langle\Delta'(z)+\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+2}(z,w)}\right\rangle+\Theta_{ts+s+1}^{2t+3}(z,\overline{z}),\end{aligned} \tag{4.12}$$

where $wt\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\}\geq ts+s+1$.

Proof. For w satisfying (3.1), we have the expansion

$$2\mathsf{Re}\left\{\Delta\left(z+F(z,w)\right)\right\}=2\mathsf{Re}\left\{\Delta(z)+\sum_{k=1}^{N}\Delta_k(z)F_{\geq 2t+1}^k(z,w)+L(z,\overline{z})\right\}+\Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{4.13}$$

where $F_{\geq 2t+1}(z,w)=\left(F_{\geq 2t+1}^1(z,w),\dots,F_{\geq 2t+1}^N(z,w)\right)$ and $L(z,\overline{z})=\left\langle\Theta_s^2(z,\overline{z}),\overline{F_{\geq 2t+2}(z,w)}\right\rangle$. We compute

$$\begin{aligned}\sum_{k=1}^{N}2\mathsf{Re}\left\{\Delta_k(z)F_{\geq 2t+1}^k(z,w)\right\}&=\sum_{k=1}^{N}2\mathsf{Re}\left\{\Delta_k(z)\left(w^t\sum_{j=1}^{N}a_{kj}z_j+F_{\geq 2t+2}^k(z,w)\right)\right\}\\&=2\mathsf{Re}\left\{w^t\sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1+\dots+a_{kN}z_N\right)\right\}+2\mathsf{Re}\left\langle\Delta'(z),\overline{F_{\geq 2t+2}(z,w)}\right\rangle.\end{aligned} \tag{4.14}$$

□

Lemma 4.6. *For w satisfying (3.1), we have the following estimate*

$$G_{nor}^{(2t+2)}(z,w) - 2\mathsf{Re}\left\langle F_{nor}^{(2t+1)}(z,w), z\right\rangle = 2(a+\overline{a})\mathsf{Re}\left\{\Delta(z)w^t\right\} + 2\mathsf{Re}\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{4.15}$$

where $P(z,\overline{z}) = \displaystyle\sum_{k,j=1}^{N} b_{k,j} z_k \overline{z}_j$ *and* $wt\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$.

Proof. For w satisfying (3.1), by Lemma 4.1 it follows that

$$\begin{aligned}
G_{nor}^{(2t+2)}(z,w) - 2\mathsf{Re}\left\langle F_{nor}^{(2t+1)}(z,w), z\right\rangle &= (a+\overline{a})w^{t+1} - 2\mathsf{Re}\left\langle w^t \begin{pmatrix} b_{11}+a & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & b_{NN}+a \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, z\right\rangle, \\
&= 2\mathsf{Re}\left\{aw^{t+1}\right\} - 2\mathsf{Re}\left\{aw^t\langle z,z\rangle + P(z,\overline{z})w^t\right\} + \overline{a}\left(w^{t+1} - \overline{w}^{t+1}\right), \\
&= 2\mathsf{Re}\left\{aw^t\left(w - \langle z,z\rangle\right)\right\} - 2\mathsf{Re}\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2\mathsf{Re}\left\{aw^t\left(\Delta(z) + \overline{\Delta(z)}\right)\right\} - 2\mathsf{Re}\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2(a+\overline{a})\mathsf{Re}\left\{\Delta(z)w^t\right\} - 2\mathsf{Re}\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned} \tag{4.16}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$. □

Substituting $F(z,w) = F_{nor}^{(2t+1)}(z,w) + F_{\geq 2t+2}(z,w)$ and $G(z,w) = G_{nor}^{(2t+2)}(z,w) + G_{\geq 2t+3}(z,w)$ (see (2.30)) into (4.2) and by Lemmas 4.2 – 4.6, we obtain

$$\begin{aligned}
G_{\geq 2t+3}(z,w) = {} & 2\mathsf{Re}\left\{\left(\sum_{k=1}^{N} \Delta_k(z)\left(a_{k1}z_1 + \dots + a_{kN}z_N\right) - (a+\overline{a})\Delta(z)\right)w^t\right\} + 2\mathsf{Re}\left\{P(z,\overline{z})\left(w^t - \langle z,z\rangle^t\right)\right\} \\
& + 2\mathsf{Re}\left\langle \overline{z} + \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle + \varphi'_{\geq 2t+3}(z,\overline{z}) - \varphi_{\geq 2t+3}(z,\overline{z}) + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned} \tag{4.17}$$

where w satisfies (3.1) and $\mathrm{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$. It remains to study the expression

$$E(z,\overline{z}) = 2\mathsf{Re}\left\{P(z,\overline{z})\left(w^t - \langle z,z\rangle^t\right)\right\}. \tag{4.18}$$

Lemma 4.7. *For w satisfying (3.1) we make the following estimate*

$$E(z,\overline{z}) = 2\mathsf{Re}\left\{\left(P(z,\overline{z}) + \overline{P(z,\overline{z})}\right)\Delta(z)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{4.19}$$

where $P(z,\overline{z}) = \displaystyle\sum_{k,j=1}^{N} b_{k,j} z_k \overline{z}_j$ *and* $wt\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$.

Proof. We compute

$$\begin{aligned}
E(z,\overline{z}) &= 2\mathsf{Re}\left\{P(z,\overline{z})\left(\Delta(z) + \overline{\Delta(z)}\right)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2\mathsf{Re}\left\{\left(P(z,\overline{z}) + \overline{P(z,\overline{z})}\right)\Delta(z)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned} \tag{4.20}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$. □

We consider the following notations

$$\mathcal{L}(z,\overline{z}) = P(z,\overline{z}) + \overline{P(z,\overline{z})} = \sum_{k,j=1}^{N}\left(b_{k,j}+\overline{b}_{j,k}\right) z_k\overline{z}_j,$$

$$Q(z) = \sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1+\cdots+a_{kN}z_N\right) - (a+\overline{a})\,\Delta(z), \quad Q_1(z) = \sum_{k,j=1}^{N}\left(b_{k,j}+\overline{b}_{j,k}\right) z_k\Delta_k(z). \tag{4.21}$$

Then, for w satisfying (3.1), by Lemma 4.7 and the notations (4.21), we can rewrite (4.17) as follows

$$\begin{aligned} G_{\geq 2t+3}(z,w) = {} & 2\mathsf{Re}\left\{Q(z)w^t\right\} + 2Re\left\{\mathcal{L}(z,\overline{z})\Delta(z)E_{t-1}\left(w,\langle z,z\rangle\right)\right\} + 2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta_s^2(z,\overline{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle \\ & + \varphi'_{\geq 2t+3}(z,\overline{z}) - \varphi_{\geq 2t+3}(z,\overline{z}) + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \end{aligned} \tag{4.22}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\overline{z})}\right\} \geq ts+s+1$. Here $E_{t-1}\left(w,\langle z,z\rangle\right) = \sum_{k+l=t-1} w^k\langle z,z\rangle^l$. For $p\geq 2t+3$ we prove the following lemma (the analogue of the Lemma 3.4 from [9]):

Lemma 4.8. *We define $\epsilon=0$ if $p<2t+s$ and $\epsilon=1$ if $p\geq 2t+s$, $\gamma=1$ if $p<ts+2$ and $\gamma=0$ if $p=ts+2$. Let $N'_s := ts+s+1$. For all $0\leq j\leq t$ and $p\in[2t+j(s-2)+3, 2t+(j+1)(s-2)+2]$, we have the following estimate*

$$\begin{aligned} G_{\geq p}(z,w) = {} & 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j w^{t-j}\right\} + 2\gamma(-1)^j\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j}w^{l_1}\langle z,z\rangle^{l_2}\right\} \\ & + 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\} + 2\mathsf{Re}\left\langle \overline{z}+\Delta'(z)+\Theta_s^2(z,\overline{z}), \overline{F_{\geq p-1}(z,w)}\right\rangle \\ & + \varphi'_{\geq p}(z,\overline{z}) - \varphi_{\geq p}(z,\overline{z}) + \Theta_{N'_s}^{p}(z,\overline{z}), \end{aligned} \tag{4.23}$$

where $\mathrm{wt}\left\{\overline{\Theta_{N'_s}^{p}(z,\overline{z})}\right\} \geq N'_s$ and w satisfies (3.1). Here E_{l_1,l_2}^{t-j} with $l_1+l_2=t-j-1$ and F_p^{t-j} with $l=1,\ldots,j-1$ are natural numbers depending on some binomial coefficients. Also $\beta_l\in\mathbb{N}$, for all $l=1,\ldots,j-1$.

Proof. For $j=0$ and $k=0$ we obtain $p=2t+3$. Therefore (4.23) becomes (4.22).

Step 1. We make a similarly approach as in the Step 1 of the Lemma 3.7.

Step 2. Assume that we proved the Lemma 4.8 for $m\in[2t+j(s-2)+3, 2t+(j+1)(s-2)+2]$, for $j\in[0,t-1]$. We want to prove that (4.23) holds for $m\in[2t+(j+1)(s-2)+3, 2t+(j+2)(s-2)+2]$. Collecting from (4.23) the terms of bidegree (m,n) in $(z,\overline{z})$ with $m+n=\Lambda+1:=2t+(j+1)(s-2)+2$, we obtain

$$\begin{aligned} G_{nor}^{(\Lambda+1)}\left(z,\langle z,z\rangle\right) = {} & 2\mathsf{Re}\left\langle z, F_{nor}^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\rangle + 2\gamma(-1)^j\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j}\right\} \\ & + 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j\langle z,z\rangle^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_{t-j}^{t-j}\right\} + 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\} \\ & + \varphi'_{\Lambda+1}(z,\overline{z}) - \varphi_{\Lambda+1}(z,\overline{z}) + \left(\Theta_1\right)_{N'_s}^{\Lambda+1}(z,\overline{z}), \end{aligned} \tag{4.24}$$

where $\mathrm{wt}\left\{\overline{\left(\Theta_1\right)_{N'_s}^{\Lambda+1}(z,\overline{z})}\right\} \geq N'_s$. We define the following mappings

$$\begin{aligned} F_{nor}^{(\Lambda)}(z,w) &= F_1^{(\Lambda)}(z,w) + F_2^{(\Lambda)}(z,w) + F_3^{(\Lambda)}(z,w) + F_4^{(\Lambda)}(z,w), \\ F_1^{(\Lambda)}(z,w) &= -(1-s)^j Q(z)\Delta(z)^j w^{t-j-1}\left(z_1,\ldots,z_N\right), \\ F_2^{(\Lambda)}(z,w) &= -\epsilon Q_1(z)\Delta(z)^j w^{t-j-1}\left(\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right)\left(z_1,\ldots,z_N\right), \\ F_3^{(\Lambda)}(z,w) &= -\gamma(-1)^j\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j}w^{t-j-1}\left(\sum_{j=1}^{N}\left(b_{l,1}+\overline{b}_{1,l}\right)z_l,\ldots,\sum_{l=1}^{N}\left(b_{l,N}+\overline{b}_{N,l}\right)z_l\right). \end{aligned} \tag{4.25}$$

Substituting (4.25) into (4.24), by making some simplifications it follows that

$$G_{nor}^{\Lambda+1}(z,\langle z,z\rangle) = 2\mathsf{Re}\left\langle z, F_4^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\rangle + \varphi'_{\Lambda+1}(z,\overline{z}) - \varphi_{\Lambda+1}(z,\overline{z}) + \left(\Theta_1\right)_{N'_s}^{\Lambda+1}(z,\overline{z}). \tag{4.26}$$

By applying Extended Moser Lemma we find a solution $\left(G_{nor}^{(\Lambda+1)}(z,w), F_4^{(\Lambda)}(z,w)\right)$ for (4.26). By repeating the procedure from the first case of the normal form construction, we obtain the following estimates

$$\begin{aligned}
&\mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w) - G_{nor}^{(\Lambda+1)}\left(z,\langle z,z\rangle\right)\right\},\ \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w)\right\},\ \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}\left(z,\langle z,z\rangle\right)\right\} \geq N'_s,\\
&\mathrm{wt}\left\{F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\},\ \mathrm{wt}\left\{F_4^{(\Lambda)}(z,w)\right\},\ \mathrm{wt}\left\{F_4^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\} \geq N'_s - s + 1,\\
&\mathrm{wt}\left\{\overline{F_4^{(\Lambda)}(z,w)}\right\},\ \mathrm{wt}\left\{\overline{F_4^{(\Lambda)}\left(z,\langle z,z\rangle\right)}\right\},\ \mathrm{wt}\left\{\overline{F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}\left(z,\overline{z}\right)}\right\} \geq N'_s - 1,
\end{aligned} \tag{4.27}$$

where w satisfies (3.1). As a consequence of (4.27) we obtain

$$\begin{aligned}
&\left\langle \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_4^{(\Lambda)}(z,w)}\right\rangle + \left\langle \overline{F_4^{(\Lambda)}(z,w)}, \Delta'(z) + \Theta_s^2(z,\overline{z})\right\rangle = \Theta_{N'_s}^{\Lambda+2}(z,\overline{z})',\\
&\mathsf{Re}\left\langle F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}\left(z,\langle z,z\rangle\right), z\right\rangle = \Theta_{N'_s}^{\Lambda+2}(z,\overline{z})',
\end{aligned} \tag{4.28}$$

where w satisfies (3.1) and each of $\Theta_{N'_s}^{\Lambda+2}(z,\overline{z})'$ has the property $\mathrm{wt}\left\{\overline{\Theta_{N'_s}^{2t+3}(z,\overline{z})}\right\} \geq N'_s$. Substituting $F_{\geq\Lambda}(z,w) = F_{nor}^{(\Lambda)}(z,w) + F_{\geq\Lambda+1}(z,w)$ and $G_{\geq\Lambda+1}(z,w) = G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w)$ in (4.23), it follows that

$$\begin{aligned}
G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w) =& 2\mathsf{Re}\left\langle \overline{z} + \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_{nor}^{(\Lambda)}(z,w)} + \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi'_{\Lambda+1}(z,\overline{z}) - \varphi_{\Lambda+1}(z,\overline{z})\\
&+ \left(\Theta_1\right)_{N'_s}^{\Lambda+1}(z,\overline{z}) + \varphi'_{>\Lambda+1}(z,\overline{z}) - \varphi_{>\Lambda+1}(z,\overline{z}) + \Theta_{N'_s}^{\Lambda+2}(z,\overline{z})\\
&+ 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j w^{t-j}\right\}\\
&+ 2\gamma\mathsf{Re}\left\{(-1)^j\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2}\right\}\\
&+ 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\},
\end{aligned} \tag{4.29}$$

where w satisfies (3.1). Simplifying the preceding equation using (4.24), it follows that

$$G_{\geq\Lambda+2}(z,w) = 2\mathsf{Re}\left\langle \overline{z} + \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi_{\geq\Lambda+2}(z,\overline{z}) - \varphi'_{\geq\Lambda+2}(z,\overline{z}) + \Theta_{N'_s}^{\Lambda+2}(z,\overline{z}) + J(z,\overline{z}), \tag{4.30}$$

where we have used the following notation

$$\begin{aligned}
J(z,\overline{z}) =& 2\mathsf{Re}\left\langle z, F_{nor}^{(\Lambda)}(z,w) - F_{nor}^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\rangle + 2\mathsf{Re}\left\langle \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_{nor}^{(\Lambda)}(z,w)}\right\rangle\\
&+ 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j w^{t-j} - Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\} + G_{nor}^{(\Lambda+2)}\left(z,\langle z,z\rangle\right) - G_{nor}^{(\Lambda+2)}(z,w)\\
&+ 2\gamma(-1)^j\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\left(\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2} - \langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\right)\right\}\\
&+ 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l\left(F_l^{t-j}w^{t-j} - F_l^{t-j}\langle z,z\rangle^{t-j}\right)\right\},
\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
J(z,\overline{z}) = {}& 2\mathsf{Re}\left\langle z, \sum_{k=1}^{3}\left(F_k^{(\Lambda)}(z,w) - F_k^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right)\right\rangle + 2\mathsf{Re}\left\langle \Delta'(z)+\Theta_s^2(z,\overline{z}), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle \\
&+ 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j\left(w^{t-j}-\langle z,z\rangle^{t-j}\right)\right\} + G_{nor}^{(\Lambda+2)}\left(z,\langle z,z\rangle\right) - G_{nor}^{(\Lambda+2)}(z,w) \\
&+ 2\gamma(-1)^j\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\left(\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2} - \sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\langle z,z\rangle^{t-j-1}\right)\right\} \\
&+ 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\left(w^{t-j}-\langle z,z\rangle^{t-j}\right)\right\}.
\end{aligned}
\tag{4.32}
$$

We observe that

$$
\begin{aligned}
\mathsf{Re}\left\langle F_1^{(\Lambda)}\left(z,\langle z,z\rangle\right), z\right\rangle &= -(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\}, \\
\mathsf{Re}\left\langle F_2^{(\Lambda)}\left(z,\langle z,z\rangle\right), z\right\rangle &= -\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j\langle z,z\rangle^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\}, \\
\mathsf{Re}\left\langle F_3^{(\Lambda)}\left(z,\langle z,z\rangle\right), z\right\rangle &= -(-1)^j\gamma\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\right\}.
\end{aligned}
\tag{4.33}
$$

Since $\mathrm{wt}\left\{F_k^{(\Lambda)}(z,w)\right\} \geq ts+1$ and $\mathrm{wt}\left\{\overline{F_k^{(\Lambda)}(z,w)}\right\} \geq ts+s-1$ for all $k\in\{1,2,3\}$, it follows that

$$
2\mathsf{Re}\left\langle \Theta_s^2(z,\overline{z}), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle = \Theta_{N_s'}^{\Lambda+2}(z,\overline{z}),
\tag{4.34}
$$

where $\mathrm{wt}\left\{\overline{\Theta_{N_s'}^{\Lambda+2}(z,\overline{z})}\right\} \geq N_s'$. Using (4.27), (4.28), (4.33), (4.34) we can rewrite (4.32) as follows

$$
\begin{aligned}
J(z,\overline{z}) = {}& 2\mathsf{Re}\left\langle z, \sum_{k=1}^{3}F_k^{(\Lambda)}(z,w)\right\rangle + 2\mathsf{Re}\left\langle \Delta'(z), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle \\
&+ 2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j w^{t-j}\right\} + 2(-1)^j\gamma\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j}\right\} \\
&+ 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\}.
\end{aligned}
\tag{4.35}
$$

Substituting the formulas of $F_1^{(\Lambda)}(z,w)$, $F_2^{(\Lambda)}(z,w)$ and $F_3^{(\Lambda)}(z,w)$ in (4.35) and using w satisfying (3.1), we obtain

$$
\begin{aligned}
J(z,\overline{z}) = {}& -2(1-s)^j\mathsf{Re}\left\{Q(z)\Delta(z)^j w^{t-j-1}\left(\langle z,z\rangle + s\Delta(z)\right) - Q(z)\Delta(z)^j w^{t-j}\right\} \\
&- 2(-1)^j\gamma\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j} w^{l_2}\left(w^{l_1}-\langle z,z\rangle^{l_1}\right)\right\} \\
&- 2(-1)^j\gamma\mathsf{Re}\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1}E_{l_1,l_2}^{t-j} w^{t-j-1}\right\} \\
&- 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j} w^{t-j-1}\left(\langle z,z\rangle + s\Delta(z) - w\right)\right\},
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
J(z,\overline{z}) = {} & 2(1-s)^{j+1}\mathsf{Re}\left\{Q(z)\Delta(z)^{j+1}w^{t-j-1}\right\}\\
& + 2\gamma(-1)^{j+1}\mathsf{Re}\left\{\mathcal{L}(z,\overline{z})\Delta(z)^{j+2}\sum_{l_1+l_2=t-j-2}E^{t-j-1}_{l_1,l_2}w^{l_2}\langle z,z\rangle^{l_1}\right\}\\
& + 2(-1)^{j+1}\mathsf{Re}\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1}E^{t-j}_{l_1,l_2}w^{t-j-1}\right\}\\
& + 2\epsilon\mathsf{Re}\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l=0}^{j-1}(-1)^{\beta_l+1}(1-s)^{l+1}F^{t-j}_{l}w^{t-j-1}\right\} + \Theta^{\Lambda+2}_{N'_s}(z,\overline{z}),
\end{aligned}
\tag{4.37}
$$

where $\mathrm{wt}\left\{\overline{\Theta^{\Lambda+2}_{N'_s}(z,\overline{z})}\right\} \geq N'_s$. □

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ from (4.23) with $m+n=ts+s$ and $t=j$, we obtain

$$
\begin{aligned}
G^{(ts+s)}_{nor}(z,\langle z,z\rangle) = {} & 2(1-s)^{t}\mathsf{Re}\left\{Q(z)\Delta(z)^{t}\right\} + 2K\mathsf{Re}\left\{Q_1(z)\Delta(z)^{t}\right\} + 2\mathsf{Re}\left\langle z, F^{(ts+s-1)}_{nor}(z,w)\right\rangle\\
& + \varphi'_{ts+s,0}(z,\overline{z}) - \varphi_{ts+s,0}(z,\overline{z}) + \left(\Theta_1\right)^{ts+s}_{N'_s}(z,\overline{z}).
\end{aligned}
\tag{4.38}
$$

By applying Extended Moser Lemma we find a solution $\left(G^{(ts+s)}_{nor}(z,w), F^{(ts+s-1)}_{nor}(z,w)\right)$ for (4.38). Collecting the pure terms of degree $ts+s$ from (4.38), it follows that

$$
\varphi'_{ts+s,0}(z) - \varphi_{ts+s,0}(z) = (1-s)^{t}Q(z)\Delta(z)^{t} + KQ_1(z)\Delta(z)^{t},
\tag{4.39}
$$

where $K=(-1)^{\beta_1}k_1(1-s)^{t-1}+\dots+(-1)^{\beta_{t-1}}k_{t-1}(1-s)+(-1)^{\beta_t}k_t$, with $k_1,\dots,k_t\in\mathbb{N}$. By the proof of the Lemma 4.8 (see (4.36) and (4.37)) we observe that $\beta_1=-1,\dots,\beta_t=(-1)^t$. Next, by applying Lemma 2.4 to $\varphi_{ts+s,0}(z)$ and $\varphi'_{ts+s,0}(z)$, it follows that

$$
\begin{aligned}
\varphi_{ts+s,0}(z) &= \left(A_1(z)\Delta_1(z)+\dots+A_N(z)\Delta_N(z)\right)\Delta(z)^{t} + C(z),\\
\varphi'_{ts+s,0}(z) &= \left(A'_1(z)\Delta_1(z)+\dots+A'_N(z)\Delta_N(z)\right)\Delta(z)^{t} + C'(z),
\end{aligned}
\tag{4.40}
$$

where $\left(\Delta_k\Delta^t\right)^{\star}(C(z)) = \left(\Delta_k\Delta^t\right)^{\star}(C'(z)) = 0$, for all $k=1,\dots,N$. We have

$$
\begin{aligned}
Q(z) &= \sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1+\dots+\left(a_{kk}-\frac{a+\overline{a}}{s}\right)z_k+\dots+a_{kN}z_N\right),\\
Q_1(z) &= \sum_{k=1}^{N}\Delta_k(z)\left(\left(a_{k1}+\overline{a}_{1k}\right)z_1+\dots+\left(a_{kk}+\overline{a}_{kk}-(a+\overline{a})\right)z_k+\dots+\left(a_{kN}+\overline{a}_{Nk}\right)z_N\right).
\end{aligned}
\tag{4.41}
$$

We impose the normalization condition $\left(\Delta_k\Delta^t\right)^{\star}\left(\varphi'_{ts+s,0}(z)\right)=0$, for all $k=1,\dots,N$. By Lemma 2.4 this is equivalent to find $(a_{ij})_{1\leq i,j\leq N}$ such that $A'_1(z)=\dots=A'_N(z)=0$. It follows that

$$
\begin{aligned}
&(1-s)^{t}a_{kj} + K\left(a_{kj}+\overline{a}_{jk}\right) = c_{k,j}, \quad \text{for all } k,j=1,\dots,N,\ k\neq j,\\
&(1-s)^{t}a_{kk} - \frac{a+\overline{a}}{s} + K\left(a_{kk}+\overline{a}_{kk}-(a+\overline{a})\right) = c_{kk}, \quad \text{for all } k=1,\dots,N,
\end{aligned}
\tag{4.42}
$$

where $c_{k,j}$ is determined, for all $k,j=1,\dots,N$. Here $Na=\sum_{k=1}^{N}a_{kk}$. Using the second equation from (4.42) we find $\mathsf{Im}\,a_{kk}$, for all $k=1,\dots,N$. By taking the reality part in the second equation from (4.42), we obtain

$$
\left(Ns(1-s)^{t}+2NKs\right)\mathsf{Re}\,a_{kk} - \left(2(1-s)^{t}+2Ks\right)\sum_{l=1}^{N}\mathsf{Re}\,a_{ll} = \mathsf{Re}\,c_{k,k}, \quad k=1,\dots,N.
\tag{4.43}
$$

By summing all the identities from (4.43), it follows that $(1-s)^{t}N(s-2)\sum_{l=1}^{N}\mathsf{Re}\,a_{ll}=\sum_{k=1}^{N}\mathsf{Re}\,c_{k,k}$. Next, going back to (4.43) we find $\mathsf{Re}\,a_{ll}$, for all $l=1,\dots,N$. Now, let $k\neq j$ and $k,j\in\{1,\dots,N\}$. By taking the real and the imaginary part in first equation from (4.42), we obtain

$$\begin{aligned}&\left((1-s)^t+K\right)\mathsf{Re}\,a_{kj}+K\mathsf{Re}\,a_{jk}=\mathsf{Re}\,c_{k,j}, \quad K\mathsf{Re}\,a_{kj}+\left((1-s)^t+K\right)\mathsf{Re}\,a_{jk}=\mathsf{Re}\,c_{j,k},\\ &\left((1-s)^t+K\right)\mathsf{Im}\,a_{kj}-K\mathsf{Im}\,a_{jk}=\mathsf{Im}\,c_{k,j}, \quad -K\mathsf{Im}\,a_{kj}+\left((1-s)^t+K\right)\mathsf{Im}\,a_{jk}=\mathsf{Im}\,c_{j,k},\end{aligned} \tag{4.44}$$

where $c'_{k,j}$ is determined, for all $k,j=1,\dots,N$ and $k\neq j$. In order to solve the preceding system of equations it is enough to observe that $(1-s)^t\left((1-s)^t+2K\right)\neq 0$. It is equivalent to observe that

$$\begin{aligned}(1-s)^t+2\left((-1)k_1(1-s)^{t-1}+\cdots+(-1)^tk_t\right)\neq 0,\\ (-1)^t\left((s-1)^t+2\left(k_1(s-1)^{t-1}+k_2(s-1)^{t-2}+\cdots+k_t\right)\right)\neq 0.\end{aligned} \tag{4.45}$$

Composing the map that sends M into (3.1) with the map (4.1) we obtain our formal transformation that sends M into M' up to degree $ts+s+1$.

5. Proof of Theorem 1.5-Uniqueness of the formal transformation map

In order to prove the uniqueness of the map (1.12) it is enough to prove that the following map is the identity

$$M'\ni(z,w)\longrightarrow\left(z+\sum_{k\geq 2}F^{(k)}_{nor}(z,w),w+\sum_{k\geq 2}G^{(k+1)}_{nor}(z,w)\right)\in M'. \tag{5.1}$$

Here M' is a manifold defined by the normal form from the Theorem 1.5. We have used the notations (2.30). We perform induction on $k\geq 2$.

Definition 5.1. The undetermined homogeneous parts of the map (5.1) by applying Extended Moser Lemma are called the free parameters.

We prove that $F^{(2)}_{nor}(z,w)=0$. Here we recall the first case of the normal form construction. We assume that $t=1$. By repeating the normalization procedures from the first case of the normal form construction, we find that all of the homogeneous components of $F^{(2)}_{nor}(z,w)$ except the free parameter are 0 and that $G^{(3)}_{nor}(z,w)=0$. Using the same approach as in the first case of the normal form construction (see (3.25)), it follows that

$$\varphi_{s+1,0}(z)-\varphi_{s+1,0}(z)=(1-s)\langle z,a\rangle\Delta(z)=0. \tag{5.2}$$

Here a is the free parameter of $F^{(2)}_{nor}(z,w)$. It follows that $a=0$. Therefore $F^{(2)}_{nor}(z,w)=0$.

We assume that $F^{(2)}_{nor}(z,w)=\cdots=F^{(k-2)}_{nor}(z,w)=0$, $G^{(3)}_{nor}(z,w)=\cdots=G^{(k-1)}_{nor}(z,w)=0$. We want to prove that $F^{(k-1)}_{nor}(z,w)=0$, $G^{(k)}_{nor}(z,w)=0$. First, we consider the case when $k=2t$, with $t\geq 2$. Let $a\in\mathbb{C}^N$ be the free parameter of the polynomial $F^{(2t)}_{nor}(z,w)$. By repeating all the normalization procedures from the first case of the normal form construction it follows that all of the homogeneous components of $F^{(2t)}_{nor}(z,w)$ except the free parameters are 0 and that $G^{(2t+1)}_{nor}(z,w)=0$. We are interested in the image of the manifold M through the map (5.1) to M up to degree $ts+1$. We repeat the normalization procedure done during Lemma 3.7 proof. In that case we have considered a particular mapping (see (3.3)). Here we have a general polynomial map with other free parameters. They generates terms of weight at least $ts+2$ that do not change their weight under the conjugation:

$$\begin{aligned}&\mathrm{wt}\left\{\langle F_{1,m}(z)w^m,z\rangle\right\},\quad \mathrm{wt}\left\{\langle z,F_{1,m}(z)w^m\rangle\right\}\geq ts+2,\ \text{for all } m>t;\\ &\mathrm{wt}\left\{\langle F_{0,r}(z)w^r,z\rangle\right\},\quad \mathrm{wt}\left\{\langle z,F_{0,r}(z)w^r\rangle\right\}\geq ts+2,\ \text{for all } r\geq t+2.\end{aligned} \tag{5.3}$$

Here $F_{1,m}(z)w^m$, $F_{0,r}(z)w^r$ are the free parameters of $F^{(2m+1)}_{nor}(z,w)$ and $F^{(2r)}_{nor}(z,w)$, for all $m>t$ and $r\geq t+2$. Therefore they can not interact with the pure terms of degree $ts+1$ (because of the higher weight). All the Lemmas $3.1-3.6$ remain the same in this general case.

Using the same approach as in the first case of the normal form construction (see (3.47)), it follows that

$$\varphi_{ts+1,0}(z)-\varphi_{ts+1,0}(z)=(1-s)^t\langle z,a\rangle\Delta^t(z)=0. \tag{5.4}$$

It follows that $a=0$. Therefore $F^{(2t)}_{nor}(z,w)=0$.

We assume that $k=2t+1$, with $t\geq 2$. Let $(a_{i,j})_{1\leq i,j\leq N}$ be the free parameter of $F^{(2t+1)}_{nor}(z,w)$. By repeating all the normalization procedures from the first case of the normal form construction, it follows that all of the homogeneous components of $F^{(2t+1)}_{nor}(z,w)$ except the free parameters are 0 and that $G^{(2t+2)}_{nor}(z,w)=0$.

We are interested of the image of the manifold M' through the map (3.3) to M' up to degree $ts+s+1$. The other free parameters of the map (5.1) generates terms of weight at least $ts+s+1$ that do not change their weight under the conjugation:

$$\begin{aligned} &\mathrm{wt}\left\{\langle F_{1,m}(z)w^m, z\rangle\right\}, \quad \mathrm{wt}\left\{\langle z, F_{1,m}(z)w^m\rangle\right\} \geq ts+s+1, \text{ for all } m > t+1; \\ &\mathrm{wt}\left\{\langle F_{0,r}(z)w^r, z\rangle\right\}, \quad \mathrm{wt}\left\{\langle z, F_{0,r}(z)w^r\rangle\right\} \geq ts+s+1, \text{ for all } r \geq t+3. \end{aligned} \tag{5.5}$$

All the Lemmas 4.1 − 4.7 remain true in this general case.

Using the same approach as in the second case of the normal form construction (see (4.39)), it follows that

$$\varphi_{ts+s,0}(z) - \varphi_{ts+s,0}(z) = (1-s)^t Q(z)\Delta^t(z) = 0. \tag{5.6}$$

It follows that $(a_{i,j})_{1\leq i,j\leq N} = 0$. Therefore

$$F_{nor}^{(2t+1)}(z,w) = 0, \quad G_{nor}^{(2t+2)}(z,w) = 0. \tag{5.7}$$

This proves the uniqueness of the formal transformation (1.12).

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