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Super Yang-Mills Theory with Impurity Walls and Instanton Moduli Spaces

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Abstract

We explore maximally supersymmetric Yang-Mills theory with walls of impurities respecting half of its supersymmetry. The walls carry fundamental or bifundamental matter multiplets. We employ three-dimensional $\mathcal{N} = 2$ superspace language to identify the Higgs branch of this theory. We find that the vacuum conditions determining the Higgs branch are exactly the bow equations determining Yang-Mills instantons on a multi Taub-NUT space.

Under the electric-magnetic duality, the fundamental- and bifundamental-carrying impurity walls are interchanged, while the super Yang-Mills theory describing the bulk is mapped to itself. We perform a one-loop computation on the Coulomb branch of the dual theory to find the asymptotic metric on the original Higgs branch.

Contents

1 Introduction

A string theory realization of a quantum gauge theory can be very useful in analyzing it [1, 2, 3, 4]. Such a realization can relate an intrinsically quantum problem of the gauge theory to a different amenable classical problem. For example, three-dimensional quantum gauge theories were related to the the dynamics of monopoles in [5, 6] and singular monopoles [7]. In this work, however, we employ such a string theory realization in a reverse fashion: We study a supersymmetric quantum gauge theory to make statements about the classical moduli space of Yang-Mills instantons on multi Taub-NUT space with k centers (TN_k).

The gauge theory we are interested in is four-dimensional and possesses[1] $\mathcal{N}_{4d} = 2$ supersymmetry. In the four-dimensional bulk, the theory is given by the maximally supersymmetric, i.e. $\mathcal{N}_{4d} = 4$, Yang-Mills theory. Half of the bulk supersymmetry is however broken by the presence of defects. The defect walls are all parallel, each carrying either a fundamental or a bifundamental $\mathcal{N}_{3d} = 4$ hypermultiplet. We choose the gauge group in the each space bounded by the bifundamental-carrying impurity walls to be unitary. The rank of the gauge group can a priori differ on the two sides of such a wall.

Without the bifundamental defects, this theory was studied in [8, 9]. As in [9], we split four-dimensional space into a three-dimensional space parallel to the defects and a one-dimensional space perpendicular to these. The latter will be parameterized by the coordinate $s = x^6$. To work in a superspace framework we embed $\mathcal{N}_{3d} = 2$ superspace in a compatible way into $\mathcal{N}_{4d} = 2$ superspace. Our gauge theory is therefore formulated in terms of $\mathcal{N}_{3d} = 2$ superfields on $\mathbb{R}^{1,2}$ that depend on a parameter s. Correspondingly, the vacuum D- and F-flatness conditions, which are usually algebraic, are now differential equations in this variable s. We find that on the Higgs branch of the gauge theory these equations are exactly the bow equations of [10] which describe $U(n)$ instantons on a multi-Taub-NUT space, TN_k. The number of the Taub-NUT centers k equals to the number of the bifundamental impurity walls of the gauge theory, while the rank n of the instanton gauge group equals to the number of the fundamental impurity walls. The instanton charges are determined by the ranks of the gauge group of the gauge theory. This identifies the Higgs branch of the theory with n fundamental and k bifundamental impurity walls with the moduli space of $U(n)$ instantons on TN_k. Obtaining these differential equations from the D- and F-flatness conditions and relating them to the instanton bow data is one of the results of this work. The equations we obtain, however, are slightly more general and suggest an interpretation of the instanton problem as a part of a larger framework provided by the quantum gauge theory.

Once we identify the Higgs branch of our impurity theory with the moduli space of instantons on TN_k, we acquire an entirely new way of computing the metric on the latter: We can apply electric-magnetic duality [11, 12] (which in three-dimensional language corresponds to gauge theory mirror symmetry [13]), so that instead of considering the Higgs branch of our original theory, we study the Coulomb branch of the dual theory. In particular, the asymptotic form of the moduli space metric is determined by perturbative corrections to the propagator of the gauge the-

[1] To avoid confusion, we give the space-time dimension in which we count real supersymmetry generators explicitly. For example, a four-dimension theory with $\mathcal{N}_{4d} = 2$ has the same amount of conserved real supercharges as a three-dimensional theory with $\mathcal{N}_{3d} = 4$.

ory. Moreover, on the Coulomb branch in extreme infrared the theory is effectively three-dimensional. We compute the one-loop correction in the corresponding three-dimensional theory, generalizing the results of [14, 15]. This leads exactly to the Lee-Weiberg-Yi-type [16] asymptotic metric, which was found to be the asymptotic metric of the instanton moduli space in [17].

The underlying relation of the moduli space of vacua of the impurity gauge theory to the moduli space of instantons is not coincidental: A string theory realization of our impurity gauge theory is given by a Chalmers-Hanany-Witten (CHW) brane configuration in type IIB string theory [18, 19]. Applying T-duality along the D3 relative transverse direction parameterized by s, one maps this configuration to a type IIA string theory configuration with n D6-branes wrapped on TN_k and some D2-branes transverse to TN_k. The latter has been argued [20, 21] to be effectively described by instantons.

Our paper is structured as follows: Having derived the matter content from the analysis of the CHW configuration in Section 2, we give the gauge theory Lagrangian in Section 3. The vacuum conditions are discussed in Section 4, where we also compare them to the instanton data. Section 5 contains the calculation of the one-loop quantum corrections to the metric on the moduli space, and we conclude in Section 6. Our superfield and gauge algebra conventions are summarized in the appendix.

2 Chalmers-Hanany-Witten brane configurations and instantons

In order to make various key ingredients in our discussion transparent, let us begin with the description of the Chalmers-Hanany-Witten D-brane configuration [6, 18]. This configuration is the type IIB string theory background summarized in Table 1, whose background geometry is ten-dimensional Minkowski space with one spatial dimension compactified on a circle: $\mathbb{R}^{1,2} \times \mathbb{R}^3_Z \times S^1 \times \mathbb{R}^3_Y$. As coordinates on the various components of this product space, we use $(x_0, x_1, x_2) \in \mathbb{R}^{1,2}$, $\vec{z} \in \mathbb{R}^3_Z$, $s \in S^1$ and $\vec{y} \in \mathbb{R}^3_Y$. The space contains n parallel D5-branes with world-volumes $\mathbb{R}^{1,2} \times \mathbb{R}^3_Y$ located at[2] $\lambda_j \in S^1$, $j = 1, \ldots, n$ and at the origin of $\mathbb{R}^3_Z$, i.e. at $\vec{z} = 0$. We also have k distinct parallel NS5-branes with world-volumes $\mathbb{R}^{1,2} \times \mathbb{R}^3_Z$ positioned at $p_\sigma \in S^1$ and $\vec{\nu}_\sigma \in \mathbb{R}^3_Y$, $\sigma = 1, \ldots, k$. The last important ingredient is a collection of D3-branes that are either suspended between D5-branes, having world-volumes $\mathbb{R}^{1,2} \times [\lambda_j, \lambda_{j+1}]$, or wrap the circle factor entirely and have the world-volume $\mathbb{R}^{1,2} \times S^1$.

[2]Our notation is chosen to match that of [17] later in the discussion.

Such configurations were thoroughly analyzed using their effective description in terms of three-dimensional gauge theories and mirror symmetry [13, 22]. They also proved to be very useful in the exploration of singular monopoles [7, 23]. Here we shall focus on the effective gauge theory in the *four-dimensional world-volume* of the D3-brane describing its - still four-dimensional - low energy dynamics.

Away from any five-branes, the effective infrared description of the CHW configuration is given by maximally supersymmetric Yang-Mills theory. The presence of the five-branes will manifest itself in the form of two types of defects in this theory. It is important to emphasize the different geometric nature of the two kinds of defect walls that we consider. A fundamental wall, i.e. a defect wall carrying a fundamental matter multiplet, is contained in the four-dimensional space. A gauge transformation in the bulk acts on the fundamental multiplet by its value at the wall. A bifundamental defect positioned at $s = p_\sigma$, on the other hand, separates the space time into a half-space-time $s \geq p_\sigma$ to its right and another half-space-time $s \leq p_\sigma$ to its left.

In order to make this transparent, we introduce two distinct points p^L_σ and $p^R_{\sigma-1}$ which are located at the boundary of the s-semi-axis in the space on the right and on the left of the wall, respectively. With these conventions any field ϕ continuous on one side of the wall satisfies[3] $\lim_{p\nearrow p_\sigma} \phi(p) = \phi(p^R_{\sigma-1})$ and $\lim_{p\searrow p_\sigma} \phi(p) = \phi(p^L_\sigma)$. As a result, k bifundamental walls separate the space-time into k slices so that the σ^{th} slice corresponds to the interval $[p^L_\sigma, p^R_\sigma]$. And the gauge group acts independently at $p^R_{\sigma-1}$ and at p^L_σ. Due to the presence of k walls with bifundamental multiplets the bulk gauge theory is defined on k independent slabs $\mathbb{R}^{1,2} \times [p^L_\sigma, p^R_\sigma]$. In order to deal with the boundary terms and integration by parts[4] for the fields in a given slab $\mathbb{R}^{1,2} \times [p^L_\sigma, p^R_\sigma]$, we understand them to be extended by zero on $\mathbb{R}^{1,2} \times [p^L_\sigma - \epsilon, p^R_\sigma + \epsilon]$.

Altogether, the gauge theory action $S = S_{\text{bulk}} + S_f + S_b$ and consists of

- the bulk contribution S_{bulk}, given by the action of the maximally supersymmetric Yang-Mills written in terms of $\mathcal{N}_{3d} = 2$ superfields,
- the fundamental multiplet contribution S_f which, since these multiplets are localized at $s = \lambda_j$, has three-dimensional Lagrangian density with the bulk fields values at that point, and
- the bifundamental contribution S_b with its σ^{th} term of its three-dimensional Lagrangian density containing bulk fields at $p^L_{\sigma-1}$ and at p^R_σ.

As a result, any variational equation obtained by varying a bulk field contains contributions from the j^{th} fundamental multiplets with a factor $\delta(s - \lambda_j)$. Additionally,

[3] where $\lim_{p\nearrow p_\sigma}$ and $\lim_{p\searrow p_\sigma}$ denote lower and upper limits, respectively

[4] See discussion on page 9.

contributions from the σ^{th} bifundamental multiplet will appear with a factor of either $\delta(s - p^L_{\sigma-1})$ or $\delta(s - p^R_\sigma)$. We will present the action in Section 3 and we will discuss the variational equations in detail in Section 4.

	0	1	2	3	4	5	6	7	8	9
Coordinates	x^0	x^1	x^2	$\vec{z}$			s	$\vec{y}$		
Symmetries	$SO(1,2)$			$SO(3)_Z$				$SO(3)_Y$		
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	p_σ	$\vec{\nu}_\sigma$		
D5	$\times$	$\times$	$\times$	$\vec{0}$			λ_j	$\times$	$\times$	$\times$
D3	$\times$	$\times$	$\times$	$\vec{0}$			$\times$	$\vec{y}^{D3}$		
$\mathcal{N}=1$ fields/$\tilde{\Psi}$	$\mathcal{V}$			$\mathcal{Z}$		$\mathcal{V}$				
$\tilde{\Psi}$ components	v_0	v_1	v_2	Z		Z_3				
$\mathcal{N}=1$ fields/$\tilde{\Upsilon}$							$\mathcal{X}$		$\mathcal{Y}$	
$\tilde{\Upsilon}$ components							v_6	Y_1	Y	

Table 1: The CHW brane configuration, its symmetries, and the components of the supermultiplets in the effective gauge theory description. Here, $Z = Z_1 + \mathrm{i}Z_2$ and $Y = Y_2 + \mathrm{i}Y_3$. The superfields $\tilde{\Psi}$ and $\tilde{\Upsilon}$ denote linear combinations of $\mathcal{N}_{4d} = 2$ vector- and hypermultiplets as explained in the appendix.

A CHW brane configuration preserves 8 of the 16 real supercharges. That is, we expect the gauge theory to exhibit $\mathcal{N}_{3d} = 4$ supersymmetry with R-symmetry algebra $so(4) \simeq su(2)_Z \times su(2)_Y \simeq so(3)_Z \times so(3)_Y$. The Higgs field triplets (Z_1, Z_2, Z_3) and (Y_1, Y_2, Y_3) form vector representations of the factors $so(3)_Z$ and $so(3)_Y$. These algebras correspond to rotations in the spaces $\mathbb{R}^3_Z$ and $\mathbb{R}^3_Y$ respectively.

Since the D3-branes can end on D5-branes if their positions in $\mathbb{R}^3_Z$ agree, we can have a branch of the space of vacua of the theory where all of the D3-branes are positioned at $\vec{z} = 0$ and any separation between the D3-branes is along the $\mathbb{R}^3_Y$ space factor. We call this the *Y-branch* of the theory. Potentially, if any of the NS5-brane positions $\vec{\nu}_\sigma$ coincide, there is another branch, the *Z-branch*, with D3-branes ending on NS5-branes positioned at the associated values of $\vec{y} = \vec{\nu}_\sigma$ and arbitrary values of $\vec{z}$. There is also a *mixed branch* corresponding to at least some of the D3-branes having world-volumes $\mathbb{R}^{1,2} \times S^1$ positioned at nonzero $\vec{z}$ and $\vec{y}$ or some D3-branes separated along the Z- while others are separated along the Y-directions. Here, we assume that all of the NS5-brane positions $\vec{\nu}_\sigma$ are distinct and therefore neither the Z-branch nor the mixed branch arises. An example brane configuration on the Y-branch is depicted in Figure 1.

We deliberately name these branches according to the directions in which the D3-branes are separated from each other and not according to the types of the gauge theory supermultiplets that parameterize them. This avoids the potential confusion that can arise once the gauge theory mirror symmetry interchanging the roles of the supermultiplets enters the discussion. Once we consider the gauge theory description of the low energy D3-brane dynamics in this CHW configuration, the Y-branch, which is the main object of our study, will correspond to the Higgs branch of the gauge theory. The Z-branch, if it existed, would correspond to the Coulomb branch of that gauge theory. In Section 5, however, where we perform perturbative computations at one loop, we will find it convenient to work in the mirror or S-dual picture. We still study the Y-branch, but once the mirror symmetry is applied, the Y-branch is identified with the Coulomb branch of the mirror theory. It is the one loop computation involving the vector multiplet that gives us the asymptotic metric on the Y-branch.

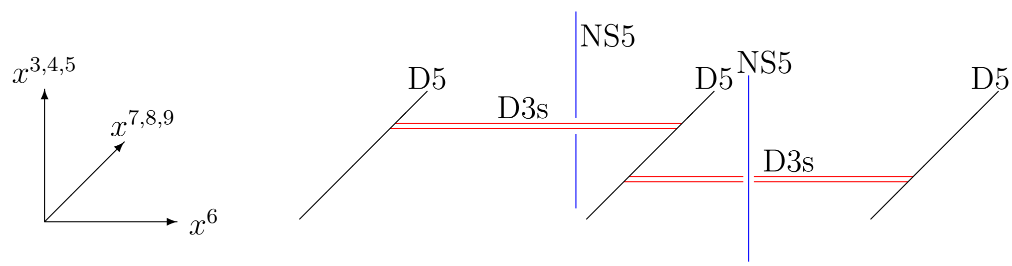

Figure 1: A picture of a Chalmers-Hanany-Witten configuration in the Y- or Higgs branch.

This brane configuration is directly related to Yang-Mills instantons as follows. Wherever the x^6 direction is compact, we can perform a T-duality along it. This yields a dual type IIA string theory brane configuration which contains the k-centered Taub-NUT space in the $6, 7, 8, 9$-directions, a number of D6-branes wrapping it with their world-volumes in the $0, 1, 2, 6, 7, 8, 9$ directions, and D2-branes (some of them fractional) with the world-volumes in the $0, 1, 2$ directions. Due to a slight modification of the argument of [20, 21], this configuration is described by instantons on the space TN_k. If the x^6 direction is noncompact or if some x^6 intervals have no D3-branes over them, the corresponding instanton on TN_k is independent of the isometric direction (i.e. the direction dual to x^6) and can be interpreted as a singular monopole. The relation between CHW configurations and instantons on a multi-Taub-NUT space was used in [10, 19] to describe such instanton configurations and their moduli spaces explicitly. Singular monopoles were constructed via this interpretation in [24, 25].

3 Gauge theory action

3.1 Field content

The bosonic degrees of freedom in the (x_0, x_1, x_2, x_6) bulk of the D3-brane are the gauge field (v_0, v_1, v_2, v_6), three hermitian Higgs fields Z_1, Z_2, Z_3 (corresponding to the relative-transverse directions along the NS5-branes) and three hermitian Higgs fields Y_1, Y_2, and Y_3 (corresponding to the relative-transverse directions along the D5-branes). They form the bosonic part of an $\mathcal{N}_{4d} = 4$ vector supermultiplet.

This $\mathcal{N}_{4d} = 4$ vector supermultiplet decomposes into an $\mathcal{N}_{4d} = 2$ vector multiplet and an $\mathcal{N}_{4d} = 2$ hypermultiplet. In the $\mathcal{N}_{4d} = 1$ language the $\mathcal{N}_{4d} = 4$ vector supermultiplet contains an $\mathcal{N}_{4d} = 1$ vector multiplet and three $\mathcal{N}_{4d} = 1$ chiral supermultiplets. For our discussion, it will be useful to switch to $\mathcal{N}_{3d} = 2$ superspace. We use the same embedding of $\mathcal{N}_{3d} = 2$ superspace into $\mathcal{N}_{4d} = 2$ superspace as in [9], providing the details in the appendix for completeness.

We arrange the bosonic fields listed above as follows into a vector superfield $\mathcal{V}$ (which will later give rise to the complex linear superfield Σ) and three chiral superfields $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$;

$$\begin{aligned} \mathcal{V} &: (v_0, v_1, v_2, Z_3, \lambda, D)\ , && \\ \mathcal{X} &: (X, \psi, G)\ , && \text{with } X = v_6 + \mathrm{i}Y_1\ , \\ \mathcal{U}^1 := \mathcal{Z} &: (Z, \chi^1, F^1)\ , && \text{with } Z = Z_1 + \mathrm{i}Z_2\ , \\ \mathcal{U}^2 := \mathcal{Y} &: (Y, \chi^2, F^2)\ , && \text{with } Y = Y_2 + \mathrm{i}Y_3\ . \end{aligned} \tag{1}$$

Explicitly, the superfield expansions read as

$$\begin{aligned} \mathcal{V} &= \mathrm{i}\theta_\alpha\bar{\theta}^\alpha Z_3 - \theta\sigma^\mu_{3\mathrm{d}}\bar{\theta}v_\mu + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D\ , \\ \mathcal{X} &= v_6(y) + \mathrm{i}Y_1(y) + \sqrt{2}\theta\psi(y) + \theta^2 G(y)\ , \\ \mathcal{U}^1 = \mathcal{Z} &= Z(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^1(y)\ , \\ \mathcal{U}^2 = \mathcal{Y} &= Y(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^2(y)\ . \end{aligned} \tag{2}$$

Here, $\sigma^\mu_{3\mathrm{d}}$ is the set of four-dimensional sigma matrices reduced to three dimensions. We use the convention $\sigma^\mu_{3\mathrm{d}} = (-\mathbb{I}, \sigma^1, \sigma^3)$. The complex scalar fields D, G, F^1 and F^2 are auxiliary fields and λ, ψ and χ are Weyl spinors in four dimensions. The four-dimensional gauge field (v_0, v_1, v_2, v_6) is spread over the two $\mathcal{N}_{3d} = 2$ superfields

$\mathcal{V}$ and $\mathcal{X}$. Correspondingly the real scalar field Z_3, which is expected to be part of the $\mathcal{N}_{3d} = 2$ chiral superfield $\mathcal{X}$, is part of the vector superfield. The reasons for this are given in the appendix.

Note that we will use the superfield conventions of [26]. In particular, we have

$$\theta\lambda := \theta^\alpha \lambda_\alpha \quad \text{and} \quad \bar{\theta}\bar{\lambda} := \bar{\theta}_\alpha \bar{\lambda}^\alpha \ . \tag{3}$$

For products of spinors of the form $\bar{\theta}\lambda$, we will always make the products explicit. A useful relation is

$$(\bar{\theta}_\alpha \theta^\alpha)^2 = \tfrac{1}{2}\theta^2\bar{\theta}^2 \ . \tag{4}$$

We will also denote the adjoint of a field λ by $\bar{\lambda}$ instead of $\lambda^\dagger$ to simplify notation.

Since v_6 is a gauge field along the direction $x^6 = s$, and each $\mathcal{N}_{3d} = 2$ superfield has to be understood as depending on the parameter s, gauge transformations will act on the superfields according to

$$\begin{aligned} \mathcal{U}^i &\to e^{-2\mathrm{i}\Lambda}\mathcal{U}^i e^{2\mathrm{i}\Lambda} \ , \\ \mathcal{X} &\to e^{-2\mathrm{i}\Lambda}\mathcal{X} e^{2\mathrm{i}\Lambda} + e^{-2\mathrm{i}\Lambda}\partial_s e^{2\mathrm{i}\Lambda} \ , \end{aligned} \tag{5}$$

where Λ is an $\mathcal{N}_{3d} = 2$ chiral superfield depending on the parameter s. In three dimensions, the vector superfield $\mathcal{V}$ gives rise to the complex linear superfield

$$\Sigma := \epsilon^{\alpha\beta}\bar{D}_\alpha(e^{2\mathrm{i}\mathcal{V}} D_\beta e^{-2\mathrm{i}\mathcal{V}}) \ . \tag{6}$$

Its field expansion reads as

$$\begin{aligned} \Sigma(x,\theta,\bar{\theta}) =& 4Z_3 + 4\theta_\alpha\bar{\lambda}^\alpha - 4\bar{\theta}_\alpha\lambda^\alpha - 4\mathrm{i}\bar{\theta}_\alpha\theta^\alpha D - 2\theta\sigma^{3\mathrm{d}}_\mu\bar{\theta}\epsilon^{\mu\nu\kappa}F_{\nu\kappa} \\ &- 2\mathrm{i}\bar{\theta}^2\theta\sigma^\mu_{3\mathrm{d}}\nabla_\mu\lambda + 2\mathrm{i}\theta^2\bar{\theta}\bar{\sigma}^\mu_{3\mathrm{d}}\nabla_\mu\bar{\lambda} - \theta^2\bar{\theta}^2\nabla_\mu\nabla^\mu Z_3 \ . \end{aligned} \tag{7}$$

Since gauge transformations act on the vector superfield as

$$e^{2\mathrm{i}\mathcal{V}} \to e^{-2\mathrm{i}\Lambda}e^{2\mathrm{i}\mathcal{V}}e^{2\mathrm{i}\bar{\Lambda}} \ , \tag{8}$$

the complex linear superfield transforms according to

$$\Sigma \to e^{-2\mathrm{i}\Lambda}\Sigma e^{2\mathrm{i}\Lambda} \ . \tag{9}$$

3.2 Action in superspace

The following bulk action is manifestly invariant under both $\mathcal{N}_{3d}=2$ supersymmetry and gauge symmetry:

$$S_{\text{bulk}} = \int \mathrm{d}s\, \mathrm{d}^3x\, \mathrm{tr}\Big[\int \mathrm{d}^4\theta \, \big(-\tfrac{1}{16}\Sigma^2 - \tfrac{1}{4}(e^{2\mathrm{i}\mathcal{V}}(\partial_s - \bar{\mathcal{X}})e^{-2\mathrm{i}\mathcal{V}} - \mathcal{X})^2 + \tfrac{1}{2}e^{2\mathrm{i}\mathcal{V}}\bar{\mathcal{U}}_i e^{-2\mathrm{i}\mathcal{V}}\mathcal{U}^i\big) + \tfrac{\mathrm{i}}{2}\int \mathrm{d}^2\theta \; \epsilon_{ij}\mathcal{U}^i[\partial_s + \mathcal{X}, \mathcal{U}^j] - \tfrac{\mathrm{i}}{2}\int \mathrm{d}^2\bar{\theta} \; \epsilon_{ij}\bar{\mathcal{U}}^i[\partial_s - \bar{\mathcal{X}}, \bar{\mathcal{U}}^j]\Big] \; . \tag{10}$$

Performing the Graßmann integrals, one obtains the component action of $\mathcal{N}_{4d}=4$ super Yang-Mills theory. The spinor fields appear in an unusual linear combination in the superfield expansions, cf. the appendix.

As we used four-dimensional superfields, there is only a $U(1)$ subgroup of the $SO(3)_Z \times SO(3)_Y$ R-symmetry group manifest. In component fields however, the full symmetry group is recovered after integrating out all auxiliary fields. There is an additional $SU(2)$ flavor symmetry acting on the doublet $(\mathcal{Z},\mathcal{Y})$. This symmetry partially mixes fields belonging to the $\mathcal{N}_{3d}=4$ vector and hypermultiplets and points towards the corresponding $SL(2,\mathbb{Z})$ symmetry.

Varying the action (10) with respect to the auxiliary fields yields

$$\frac{\delta S_{\text{bulk}}}{\delta D} = D - \partial_s Y_1 - [v_6, Y_1] - \tfrac{\mathrm{i}}{2}([Z,\bar{Z}] + [Y,\bar{Y}]) \; , \tag{11}$$

$$\frac{\delta S_{\text{bulk}}}{\delta G} = -\tfrac{1}{2}\bar{G} - \mathrm{i}[Y,Z] \; , \tag{12}$$

$$\frac{\delta S_{\text{bulk}}}{\delta F^1} = \tfrac{1}{2}\bar{F}_1 - \mathrm{i}\partial_s Y - \mathrm{i}[v_6 + \mathrm{i}Y_1, Y] \; , \tag{13}$$

$$\frac{\delta S_{\text{bulk}}}{\delta F^2} = \tfrac{1}{2}\bar{F}_2 + \mathrm{i}\partial_s Z + \mathrm{i}[v_6 + \mathrm{i}Y_1, Z] \; . \tag{14}$$

As we mentioned earlier, all the fields on a σ^{th} slab $\mathbb{R}^{1,2} \times [p^L_\sigma, p^R_\sigma]$ are extended by zero immediately outside the slab, thus the fields have discontinuities at the ends p^L_σ and p^R_σ and, potentially at any points λ_j within the interval. In the above equations, we understand the terms $\partial_s Y_1, \partial_s Y$, and $\partial_s Z$ as generalized derivatives. That is, at a point λ_j the derivatives $\partial_s Y_1$ and $\partial_s Y$ contain a discontinuity term

$$\delta(s-\lambda_j)\big(\lim_{r\searrow\lambda_j} Y_1(r) - \lim_{r\nearrow\lambda_j} Y_1(r)\big) \quad \text{and} \quad \delta(s-\lambda_j)\big(\lim_{r\searrow\lambda_j} Y(r) - \lim_{r\nearrow\lambda_j} Y(r)\big) \; , \tag{15}$$

respectively, while $\partial_s Z$ contains terms $\delta(s-p^L_\sigma)Z(p^L_\sigma) - \delta(s-p^R_\sigma)Z(p^R_\sigma)$. This convention automatically takes into account all boundary terms appearing from any integration by parts involved in obtaining Eqs. (11)-(14).

Only four of the eight supersymmetries of CHW brane configurations are manifest in our superspace formalism. It is therefore necessary to complement the equations $D = \bar{F}_i = \bar{G} = 0$ in such a way that they are invariant under the R-symmetry group, as we will discuss in Section 4.2.

By the classical argument of [27, 28], the (R-symmetry completed) D- and F-flatness conditions capture the full quantum corrected Higgs branch of the moduli space of vacua in the four-dimensional gauge theory.

3.3 Fayet-Iliopoulos terms

One should also allow for Fayet-Iliopoulos (FI) terms in the action. To preserve $SU(2)_Z$-invariance, we add both FI terms for the vector superfield $\mathcal{V}$ and the chiral superfield $\mathcal{Z}$:

$$S_{\mathrm{FI}} = \int \mathrm{d}s\, \mathrm{d}^3x\, \mathrm{tr} \left(\hat{r}_3(s) \int \mathrm{d}^4\theta\, 2\mathrm{i}\mathcal{V} - \hat{r}(s) \int \mathrm{d}^2\theta\, \mathcal{Z} - \bar{\hat{r}}(s) \int \mathrm{d}^2\bar{\theta}\, \bar{\mathcal{Z}} \right) \ . \tag{16}$$

These terms add the following contributions to the variations of the auxiliary fields:

$$\frac{\delta S_{\mathrm{FI}}}{\delta D} = \mathrm{i}\hat{r}_3(s)\mathbb{I}\ , \quad \frac{\delta S_{\mathrm{FI}}}{\delta F^1} = -\hat{r}(s)\mathbb{I}\ . \tag{17}$$

In the D- and F-flatness conditions (11) and (13), these contributions can be absorbed by the following shift of fields (cf. [29]):

$$Y_1 \to Y_1 + \mathrm{i}\mathbb{I} \int_{s_0}^{s} \mathrm{d}\tilde{s}\, \hat{r}_3(\tilde{s}) \quad \text{and} \quad Y \to Y + \mathrm{i}\mathbb{I} \int_{s_0}^{s} \mathrm{d}\tilde{s}\, \hat{r}(\tilde{s})\ . \tag{18}$$

The only effect of this shift is indeed the removal of the FI parameters in the bulk. It is only at boundaries, that they survive. Moreover, one can redefine Y_1 and Y by a constant shift that is different on different intervals. As in [17], using these two shifts one can reduce the above FI term to the form

$$\begin{aligned} S_{\mathrm{FI}} = \sum_{\sigma=1}^{k} \int \mathrm{d}^3x\, \mathrm{tr} \Bigg(& \nu_{3\sigma} \int \mathrm{d}^4\theta\, \mathrm{i} \left(\mathcal{V}(p^L_\sigma) - \mathcal{V}(p^R_{\sigma-1}) \right) \\ & - \nu_\sigma \frac{1}{2} \int \mathrm{d}^2\theta\, \left(\mathcal{Z}(p^L_\sigma) - \mathcal{Z}(p^R_{\sigma-1}) \right) - \bar{\nu}_\sigma \frac{1}{2} \int \mathrm{d}^2\bar{\theta}\, \left(\bar{\mathcal{Z}}_1(p^L_\sigma) - \bar{\mathcal{Z}}_1(p^R_{\sigma-1}) \right) \Bigg)\ . \end{aligned} \tag{19}$$

3.4 Defect walls with fundamental hypermultiplets

Each fundamental impurity wall carries an $SU(2)_Y$ doublet of complex scalars Q_i together with an $SU(2)_Z$ doublet of spinors ζ^i. These fields form a $\mathcal{N}_{3d} = 4$ hypermultiplet (cf. [18], [8]) and they are localized at the value $s = \lambda_j$, corresponding to the wall's position. We will combine them into $\mathcal{N}_{3d} = 2$ chiral superfields $\mathcal{Q}_{1j}$ and $\mathcal{Q}_{2j}$ with components $(Q_{1j}, \zeta_{1j}, J_{1j})$ and $(Q_{2j}, \zeta_{2j}, J_{2j})$ respectively. In doing this, we inevitably reduce the manifest R-symmetry group to the diagonal subgroup $SU(2)_D$ of the R-symmetry group $SU(2)_Z \times SU(2)_Y$. The full R-symmetry is manifest once the auxiliary fields are integrated out. The superfields $\mathcal{Q}_{1j}$ and $\mathcal{Q}_{2j}$ are coupled to the gauge potential on the D3-branes and transform in the fundamental and the antifundamental representation of $U(N)$, respectively:

$$g : \begin{pmatrix} Q_{1j} \\ Q_{2j} \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(\lambda_j) Q_{1j} \\ Q_{2j} g(\lambda_j) \end{pmatrix} \tag{20}$$

The kinetic term is given by

$$S_{f,1} = \tfrac{1}{2} \sum_{j=1}^{n} \int \mathrm{d}^3x \, \mathrm{d}^4\theta \, (\bar{\mathcal{Q}}_{1j} e^{-2\mathrm{i}\mathcal{V}(\lambda_j)} \mathcal{Q}_{1j} + \mathcal{Q}_{2j} e^{2\mathrm{i}\mathcal{V}(\lambda_j)} \bar{\mathcal{Q}}_{2j}) \ .$$

The full $SU(2)_D$-invariant Yukawa couplings in this action have to be of the form

$$\tfrac{1}{2} \epsilon_{\alpha\beta} \begin{pmatrix} \bar{\zeta}^{\alpha}_{1j} \\ \zeta^{\alpha}_{2j} \end{pmatrix}^T \begin{pmatrix} Z_3 & -\bar{Z} \\ Z & -Z_3 \end{pmatrix} \begin{pmatrix} \zeta^{\beta}_{1j} \\ \bar{\zeta}^{\beta}_{2j} \end{pmatrix} , \tag{21}$$

and to obtain this term, we have to add the following superpotential terms:

$$S_{f,2} = \tfrac{1}{2} \sum_{j=1}^{n} \int \mathrm{d}^3x \left(\int \mathrm{d}^2\theta \, \mathcal{Q}_{2j} \mathcal{Z}(\lambda_j) \mathcal{Q}_{1j} + \int \mathrm{d}^2\bar{\theta} \, \bar{\mathcal{Q}}_{1j} \bar{\mathcal{Z}}(\lambda_j) \bar{\mathcal{Q}}_{2j} \right) \ .$$

The contributions of the fundamental hypermultiplets to the action, $S_f = S_{f,1} + S_{f,2}$, yields the following contributions to the D- and F-flatness conditions:

$$\begin{aligned} \frac{\delta S_f}{\delta D} &= \tfrac{\mathrm{i}}{2} \sum_{j=1}^{n} (\bar{Q}_{2j} Q_{2j} - Q_{1j} \bar{Q}_{1j}) \delta(s - \lambda_j) \ , & \frac{\delta S_f}{\delta F_1} &= \tfrac{1}{2} \sum_{j=1}^{n} Q_{1j} Q_{2j} \delta(s - \lambda_j) \ , \\ \frac{\delta S_f}{\delta J_{1j}} &= \tfrac{1}{2} \left(\bar{J}_{1j} + Q_{2j} Z \right) \ , & \frac{\delta S_f}{\delta J_{2j}} &= \tfrac{1}{2} \left(\bar{J}_{2j} + Z Q_{1j} \right) \ . \end{aligned} \tag{22}$$

3.5 Defect walls with bifundamental hypermultiplets

As mentioned above, a bifundamental impurity wall located at p_σ cuts the space-time in two parts. This gives rise to two gauge group $U(N_{\sigma-1})$ and $U(N_\sigma)$, acting to the left and to the right of the impurity wall, respectively. There are two complex bifundamental scalars $B_{1\sigma}$ and $B_{2\sigma}$ confined to the world-volume of the wall: $B_{1\sigma}$ transforms in the $(N_{\sigma-1}, \bar{N}_\sigma)$, while $B_{2\sigma}$ transforms in the $(N_\sigma, \bar{N}_{\sigma-1})$ representation.

$$g : \begin{pmatrix} B_{1\sigma} \\ B_{2\sigma} \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(p^R_{\sigma-1}) B_{1\sigma} g(p^L_\sigma) \\ g^{-1}(p^L_\sigma) B_{2\sigma} g(p^R_{\sigma-1}) \end{pmatrix} \tag{23}$$

They are part of a bifundamental $\mathcal{N}_{3d} = 4$ hypermultiplet, which we decompose into two $\mathcal{N}_{3d} = 2$ chiral superfields $\mathcal{B}_{1\sigma}$ and $\mathcal{B}_{2\sigma}$ with components $(B_{1\sigma}, \xi_{1\sigma}, L_{1\sigma})$ and $(B_{2\sigma}, \xi_{2\sigma}, L_{2\sigma})$. The complex scalars $B_{1\sigma}$ and $\bar{B}_{2\sigma}$ in this hypermultiplet form again an $SU(2)_Y$ doublet, cf. [18].

Coupling the bifundamental superfields to the bulk gauge superfields yields the following terms:

$$S_{b,1} = \tfrac{1}{2} \sum_{\sigma=1}^{k} \int \mathrm{d}^3x \ \mathrm{tr}\Big(\int \mathrm{d}^4\theta \Big(e^{2\mathrm{i}\mathcal{V}(p^L_\sigma)} \bar{\mathcal{B}}_{1\sigma} e^{-2\mathrm{i}\mathcal{V}(p^R_{\sigma-1})} \mathcal{B}_{1\sigma} + e^{2\mathrm{i}\mathcal{V}(p^R_{\sigma-1})} \bar{\mathcal{B}}_{2\sigma} e^{-2\mathrm{i}\mathcal{V}(p^L_\sigma)} \mathcal{B}_{2\sigma} \Big)\Big) \ . \tag{24}$$

Again, the Yukawa couplings determine via $SU(2)_D$-invariance the superpotential couplings. We need to find the superfield expressions giving rise to the following terms:

$$\tfrac{1}{2}\epsilon_{\alpha\beta} \mathrm{tr}\left[\begin{pmatrix} \bar{\xi}^\alpha_{1\sigma} & \xi^\alpha_{2\sigma} \end{pmatrix} \begin{pmatrix} Z^R_3 & -\bar{Z}^R \\ Z^R & -Z^R_3 \end{pmatrix} \begin{pmatrix} \xi^\beta_{1\sigma} \\ \bar{\xi}^\beta_{2\sigma} \end{pmatrix} - \begin{pmatrix} \bar{\xi}^\alpha_{2\sigma} & \xi^\alpha_{1\sigma} \end{pmatrix} \begin{pmatrix} Z^L_3 & -\bar{Z}^L \\ Z^L & -Z^L_3 \end{pmatrix} \begin{pmatrix} \xi^\beta_{2\sigma} \\ \bar{\xi}^\beta_{1\sigma} \end{pmatrix} \right] \ . \tag{25}$$

This is done by adding the superpotential term

$$S_{b,2} = \tfrac{1}{2} \sum_{\sigma=1}^{k} \int \mathrm{d}^3x \ \mathrm{tr}\Big(\int \mathrm{d}^2\theta (\mathcal{B}_{2\sigma} \mathcal{U}^1(p^R_{\sigma-1}) \mathcal{B}_{1\sigma} - \mathcal{B}_{1\sigma} \mathcal{U}^1(p^L_\sigma) \mathcal{B}_{2\sigma}) + \int \mathrm{d}^2\bar{\theta} (\bar{\mathcal{B}}_{1\sigma} \bar{\mathcal{U}}^1(p^R_{\sigma-1}) \bar{\mathcal{B}}_{2\sigma} - \bar{\mathcal{B}}_{2\sigma} \bar{\mathcal{U}}^1(p^L_\sigma) \bar{\mathcal{B}}_{1\sigma}) \Big) \ . \tag{26}$$

Finally, the positions $\vec{\nu}_\sigma$ of the NS5-branes give rise to Fayet-Iliopoulos terms in the bulk theory, cf. [18]. As discussed above, FI-terms can be absorbed by a shift linear in s of the scalars in the $\mathcal{N}_{3d} = 4$ vector multiplet. However, on the boundaries, these terms survive in the boundary contributions of the shifted scalars. These contributions, in turn, correspond to the positions of the NS5-branes relative to the D3-branes. If the σ-th NS5-brane at $s = p_\sigma$ is positioned at $\vec{\nu}_\sigma \in \mathbb{R}^3_Y$, let $\nu = \nu_1 + \mathrm{i}\,\nu_2$. Then $|\vec{\nu}_\sigma|^2 = \nu_3^2 + \nu\bar{\nu}$, and the Fayet-Iliopoulos terms are given in Eq. (19). One readily checks that all terms are gauge invariant. Varying the contribution of the bifundamental matter to the gauge theory action, $S_b = S_{b,1} + S_{b,2} + S_{FI}$, with respect to the auxiliary fields yields

$$\frac{\delta S_b}{\delta D} = \tfrac{\mathrm{i}}{2} \sum_{\sigma=1}^{k} \left(\bar{B}_{1\sigma} B_{1\sigma} - B_{2\sigma} \bar{B}_{2\sigma} + \nu_{3\sigma} \right) \delta(s - p^L_\sigma) - \left(B_{1\sigma} \bar{B}_{1\sigma} - \bar{B}_{2\sigma} B_{2\sigma} + \nu_{3\sigma} \right) \delta(s - p^R_{\sigma-1}) \ , \tag{27}$$

$$\frac{\delta S_b}{\delta F_1} = \tfrac{1}{2} \sum_{\sigma=1}^{k} (B_{1\sigma} B_{2\sigma} + \nu_\sigma \mathbb{I})\, \delta(s - p^R_{\sigma-1}) - (B_{2\sigma} B_{1\sigma} + \nu_\sigma \mathbb{I})\, \delta(s - p^L_\sigma) \ , \tag{28}$$

$$\frac{\delta S_f}{\delta F_2} = 0 \ , \tag{29}$$

$$\frac{\delta S_b}{\delta L_{1\sigma}} = \tfrac{1}{2} \left(\bar{L}_{1\sigma} - Z(p^L_\sigma) B_{2\sigma} + B_{2\sigma} Z(p^R_{\sigma-1}) \right) \ , \tag{30}$$

$$\frac{\delta S_b}{\delta L_{2\sigma}} = \tfrac{1}{2} \left(\bar{L}_{2\sigma} - B_{1\sigma} Z(p^L_\sigma) + Z(p^R_{\sigma-1}) B_{1\sigma} \right) \ . \tag{31}$$

3.6 Chern-Simons boundary terms and from (p, q)-branes

As a side remark to the main thread of our discussion, let us briefly consider another type of defect in the CHW configuration: the (p, q)-branes. Their contribution to the field theory on the D3-branes is a Chern-Simons term with Chern-Simons level $k = \frac{p}{q}$, cf. [30] and [31]. Intuitively speaking, the type IIB supergravity background contains an RR-scalar (axion), which gives rise to a θ-term in the gauge theory on the $D3$-branes. This θ-term can be turned into a Chern-Simons term at the codimension one boundary given by the (p, q)-brane.

In terms of superfields, the contribution of a (p, q)-brane to the effective description of the CHW configuration should read as

$$S_{(p,q)} = \frac{p}{8\pi q} \int \mathrm{d}^3x \, \mathrm{d}^4\theta \int_0^1 \mathrm{d}u \, \mathrm{tr} \left(\bar{D}^\alpha \left(e^{-2\mathrm{i}\mathscr{V}(u)} D_\alpha e^{2\mathrm{i}\mathscr{V}(u)} \right) e^{-2\mathrm{i}\mathscr{V}(u)} \partial_u e^{2\mathrm{i}\mathscr{V}(u)} \right) \ , \tag{32}$$

where $\mathscr{V}(u)$ is a function on the interval $[0, 1]$ satisfying the boundary conditions $\mathscr{V}(0) = 0$ and $\mathscr{V}(1) = \mathcal{V}$. The integral over u is inserted to have the action manifestly gauge invariant, cf. [32, 33]. The first factor under the trace is a u-dependent generalization $\mathit{\Sigma}(u)$ of the complex linear superfield Σ. In the following, we choose $\mathscr{V}(u) = u\mathcal{V}$.

After going to Wess-Zumino gauge, the action simplifies considerably and using the boundary conditions, one can perform the u-integration. In components, the action (32) reads as

$$\frac{\mathrm{i}p}{4\pi q}\mathrm{tr}\int \mathrm{d}^3x\big(\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \tfrac{2}{3}A_\mu A_\nu A_\rho\right) + 16\mathrm{i}\lambda^\alpha\bar{\lambda}_\alpha - 2DZ_3\big), \tag{33}$$

and it therefore gives a new contribution to the D-flatness condition:

$$\frac{\delta S_{(p,q)}}{\delta D} = -\frac{\mathrm{i}p}{2\pi q}Z_3 \ . \tag{34}$$

After integrating out the D-field, one obtains among others the following terms in the component action:

$$\int \mathrm{d}^3x\,\mathrm{d}s\,\delta(s-s^{(p,q)})\mathrm{tr}\left(-\frac{p^2}{8\pi^2q^2}Z_3^2 + \frac{\mathrm{i}p}{2\pi q}Z_3\nabla_sY_1 + \frac{p}{4\pi q}Z_3([Z,\bar{Z}]+[Y,\bar{Y}])\right)\ , \tag{35}$$

where we abbreviated $\nabla_s := \frac{\partial}{\partial s} + v_6$. The last term is the same as the one obtained in a related discussion in [34], where an additional θ-term - corresponding to our axion background - was added to a bulk theory with a supersymmetric boundary.

Assuming that the (p,q)-brane is oriented such that it preserves $\mathcal{N}_{3\mathrm{d}} = 3$ supersymmetry, we expect the action to be invariant under the diagonal subgroup $SU(2)_D$ of the R-symmetry group. This leads us to add the following term to the action:

$$S_{(p,q),\mathcal{N}_{3\mathrm{d}}=3} = \frac{p}{4\pi q}\int \mathrm{d}^3x\,\mathrm{d}s\,\delta(s-s^{(p,q)})\left(\int \mathrm{d}^2\theta\mathcal{Z}^2 + \int \mathrm{d}^2\bar{\theta}\bar{\mathcal{Z}}^2\right)\ , \tag{36}$$

which implies the following contribution to the F_1-flatness condition:

$$\frac{\delta S_{(p,q),\mathcal{N}_{3\mathrm{d}}=3}}{\delta F_1} = \frac{p}{4\pi q}Z \ . \tag{37}$$

4 Vacuum conditions and instanton moduli spaces

4.1 D- and F-flatness conditions

The vacuum conditions determining the Higgs branch of our gauge theory are the flatness conditions for the auxiliary fields contained in the superfields $\mathcal{V}$ and $\mathcal{Z}$ of the

$\mathcal{N}_{3d} = 4$ vector superfield[5] $\tilde{\Psi}$. Since the auxiliary fields enter the action algebraically, they can be integrated out using their equations of motion $\delta S/\delta D = 0$ and $\delta S/\delta F_1 = 0$. Finding F and D from these equations and substituting back into the action leads to potential terms of the form D^2 and F^2 thus the vacuum condition is $D = 0$ and $F_1 = 0$. Combining Eqs. (11, 22, 27) the D-flatness condition reads[6]

$$\begin{aligned}\nabla_s Y_1 + \tfrac{\mathrm{i}}{2}[Z,\bar{Z}] + \tfrac{\mathrm{i}}{2}[Y,\bar{Y}] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n}(Q_{1j}\bar{Q}_{1j} - \bar{Q}_{2j}Q_{2j})\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{2\sigma}\bar{B}_{2\sigma} - \bar{B}_{1\sigma}B_{1\sigma} - \nu_{3\sigma}\mathbb{I}\right)\delta(s-p^L_\sigma) \\ + \left(B_{1\sigma}\bar{B}_{1\sigma} - \bar{B}_{2\sigma}B_{2\sigma} + \nu_{3\sigma}\mathbb{I}\right)\delta(s-p^R_{\sigma-1}) = 0 \ ,\end{aligned} \quad (38)$$

and Eqs (13, 22, 28) lead to the F-flatness condition

$$\begin{aligned}\nabla_s Y + \mathrm{i}[Y_1, Y] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n} Q_{1j}Q_{2j}\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{1\sigma}B_{2\sigma} + \nu_\sigma\mathbb{I}\right)\delta(s-p^R_{\sigma-1}) - \left(B_{2\sigma}B_{1\sigma} + \nu_\sigma\mathbb{I}\right)\delta(s-p^L_\sigma) = 0 \ ,\end{aligned} \quad (39)$$

where again $\nabla_s := \frac{\partial}{\partial s} + v_6$.

The F-flatness conditions from the superfields $\mathcal{Y}$ and $\mathcal{X}$ contained in the $\mathcal{N}_{3d} = 4$ hypermultiplet $\tilde{\Upsilon}$ are $G = F_2 = 0$, which, amount to

$$\nabla_s Z + \mathrm{i}[Y_1, Z] = 0 \ , \quad (40)$$

and

$$[Z, Y] = 0 \ . \quad (41)$$

At a position λ_j, where a fundamental impurity wall is located, we have the following additional equations for the component fields of $\mathcal{Q}_{1j}$ and $\mathcal{Q}_{2j}$:

$$(\bar{J}_{1j} + Q_{2j}Z) = 0 \ , \quad (\bar{J}_{2j} + ZQ_{1j}) \ = 0 \ . \quad (42)$$

[5]See the appendix for the precise definition of $\tilde{\Psi}$.

[6]Inserting a (p,q)-brane at $s^{p,q}$ into our CHW configuration adds terms $-\frac{\mathrm{i}p}{2\pi q}Z_3\delta(s-s^{p,q})$ and $\frac{p}{4\pi q}\delta(s-s^{p,q})Z$ to the left hand side of Eqs. (38) and (39) respectively.

A bifundamental wall at p_σ gives rise to two conditions on its two sides arising from the auxiliary fields of $\mathcal{B}_{1\sigma}$ and $\mathcal{B}_{2\sigma}$:

$$\begin{aligned} \bar{L}_{1\sigma} - Z(p_\sigma^L)B_{2\sigma} + B_{2\sigma}Z(p_{\sigma-1}^R) = 0 \ , \\ \bar{L}_{2\sigma} - B_{1\sigma}Z(p_\sigma^L) + Z(p_{\sigma-1}^R)B_{1\sigma} = 0 \ . \end{aligned} \tag{43}$$

4.2 R-symmetry consequences

The equations derived above yield vacua preserving the $\mathcal{N}_{3d} = 2$ supersymmetries which are manifest in our superspace formulation. These supersymmetries are a subset of the $\mathcal{N}_{3d} = 4$ supersymmetries actually preserved by CHW configurations. To obtain the equations describing $\mathcal{N}_{3d} = 4$ supersymmetric vacua, one needs to consider the set of flatness conditions obtained by all possible rewritings of the $\mathcal{N}_{3d} = 4$ supersymmetries in $\mathcal{N}_{3d} = 2$ language. This can be achieved by complementing the above equations such that the new set of equations is invariant under the R-symmetry group $SU(2)_Z$ and is equivariant under $SU(2)_Y$.

First, equations (40) and (41) have to be replaced by

$$[Z, Y] = [Z_3, Y] = [Z, Y_1] = [Z_3, Y_1] = 0 \quad \text{and} \quad \nabla_s Z = \nabla_s Z_3 = 0 \ , \tag{44}$$

and Z and Z_3 ought to be covariantly constant in the bulk along the s-direction. On a branch with nontrivial vacuum expectation values of $Y_1(s), Y_2(s)$, and $Y_3(s)$, i.e. on the Higgs branch, Eqs. (44) imply that Z and Z_3 vanish. Correspondingly, (38) and (39) become

$$\begin{aligned} \nabla_s Y_1 + \tfrac{\mathrm{i}}{2}[Y, \bar{Y}] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n}(Q_{1j}\bar{Q}_{1j} - \bar{Q}_{2j}Q_{2j})\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{2\sigma}\bar{B}_{2\sigma} - \bar{B}_{1\sigma}B_{1\sigma} - \nu_{3\sigma}\mathbb{I}\right)\delta(s-p_\sigma^L) \\ + \left(B_{1\sigma}\bar{B}_{1\sigma} - \bar{B}_{2\sigma}B_{2\sigma} + \nu_{3\sigma}\mathbb{I}\right)\delta(s-p_{\sigma-1}^R) = 0 \ , \end{aligned} \tag{45}$$

$$\begin{aligned} \nabla_s Y + \mathrm{i}[Y_1, Y] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n} Q_{1j}Q_{2j}\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}(B_{1\sigma}B_{2\sigma} + \nu_\sigma\mathbb{I})\,\delta(s-p_{\sigma-1}^R) - (B_{2\sigma}B_{1\sigma} + \nu_\sigma\mathbb{I})\,\delta(s-p_\sigma^L) = 0 \ . \end{aligned} \tag{46}$$

The R-symmetry completions of equations (42) and (43) are simple to write down, however, they are not relevant for our discussion.

4.3 Masses

If a Z-branch were present, the D3-branes would be able to break up on the NS5-brane. This would give masses to the bifundamentals located at the intersection of the stack of D3-branes with the NS5-brane which would be proportional to the distance between the endpoints of the broken D3-branes. As in the case of the D5-branes, this distance is split into the complex $Z^L_{3aa} - Z^R_{3bb}$ and real $Z^L_{aa} - Z^R_{bb}$ components. The mass contribution of the former appears directly in the action (24), while the contribution of the latter arises after integrating out the auxiliary fields L_{ip}. Altogether, we have the expected result

$$m_B^2 = |Z^L_{aa} - Z^R_{bb}|^2 + (Z^L_{3aa} - Z^R_{3bb})^2 \ . \tag{47}$$

Mass terms for the fundamental hypermultiplet located at D5-brane positions arise from a finite distance between the D3-branes and the D5-branes in $\mathbb{R}^3_Z$. Although our D5-branes will be located at $\vec{z} = \vec{0}$, let us briefly comment on the more general situation of a D5-brane at $s = \lambda_j$ and $\vec{z}^{D5} = (z^{D5}, z^{D5}_3) \neq \vec{0}$. This situation is described by the following gauge theory action:

$$\begin{aligned} S_{f,1} &= \tfrac{1}{2} \int \mathrm{d}s\, \mathrm{d}^3x \int \mathrm{d}^4\theta \, (\bar{\mathcal{Q}}_{1j} e^{-2\mathrm{i}(\mathcal{V} - \mathrm{i}\theta\bar{\theta} z^{D5}_{3,j}\mathbb{I})} \mathcal{Q}_{1j} + \mathcal{Q}_{2j} e^{2\mathrm{i}(\mathcal{V} - \mathrm{i}\theta\bar{\theta} z^{D5}_{3,j}\mathbb{I})} \bar{\mathcal{Q}}_{2j}) \delta(s - \lambda_j) \ , \\ S_{f,2} &= \tfrac{1}{2} \int \mathrm{d}s\, \mathrm{d}^3x \left(\int \mathrm{d}^2\theta \, \mathcal{Q}_{2j} (\mathcal{Z} - z^{D5}_j) \mathcal{Q}_{1j} + c.c. \right) \delta(s - \lambda_j) \ . \end{aligned}$$

The modified terms preserve both supersymmetry and gauge invariance as they can be viewed as constant shifts of scalar fields.

The eigenvalues of Z and Z_3 correspond to the positions of the various D3-branes in the stack. The distances of the D3-branes to the D5-branes at $\vec{z}^{D5}$ are then given by the eigenvalues of the matrices $Z - z^{D5}\mathbb{I}$ and $Z_3 - z^{D5}_3 \mathrm{i}\mathbb{I}$. After integrating out the auxiliary fields $J_{1,2j}$ and diagonalizing Z and Z_3, one obtains the following mass terms for the fundamental hypermultiplets:

$$m^2_{\mathcal{Q}_p} = \left| Z_{aa} - z^{D5}_p \right|^2 + \left(Z_{3aa} - z^{D5}_{3,p} \mathrm{i}\mathbb{I} \right)^2 \ , \tag{48}$$

which is the expected result.

4.4 Interpretation of the Y-branch equations

The moduli space equations (38), (39), (40) and (41) reflect many of the interesting phenomena in CHW configurations. Besides the generation of masses from moving the various five-branes as discussed above, the breaking of D3-branes on both the NS5-branes and the D5-branes can be seen in the gauge theory. In the following, however, we are interested in using these equations to describe instantons on multi Taub-NUT spaces. In particular, we are about to demonstrate that the moduli space of the latter is the Y-branch of the gauge theory we are studying here.

Yang-Mills instantons on multi Taub-NUT spaces can be described in terms of bows [17]. Bows generalize quivers, and as for a quiver, one can define representations of a bow. Detailed explanations of these and other terms related to bows can be found in [17]. Each bow representation can be viewed in two ways: as a description of instantons of given charges on multi Taub-NUT spaces or as defining a gauge theory with impurities we have considered above. In the latter interpretation, each edge in the bow corresponds to an impurity wall with a bifundamental multiplet, while each marked point corresponds to an impurity wall with a fundamental multiplet on it. The representation ranks determine the ranks of the unitary gauge groups in the bulk between the impurity walls.

A representation $\mathfrak{R}$ of a bow determined an affine space $\mathrm{Dat}(\mathfrak{R})$, the bow data. In the gauge theory, $\mathrm{Dat}(\mathfrak{R})$ can be thought of as the configuration space of the scalar fields in the chiral superfields which parameterize the Y-branch. The space $\mathrm{Dat}(\mathfrak{R})$ is in fact a hyperkähler manifold, and there is a natural action of a gauge group $\mathcal{G}$ on $\mathrm{Dat}(\mathfrak{R})$ which preserves the hyperkähler structure. Therefore, one can construct the hyperkähler quotient $\mathcal{M} = \mathrm{Dat}(\mathfrak{R})///\mathcal{G}$. The corresponding moment map was given in [17]. The gauge theory counterpart of the hyperkähler quotient procedure amounts to imposing the D- and F-flatness conditions and dividing by the action of the gauge group.

In their complex form, the bow equations of [17], which are the moment map conditions, exactly agree with the vacuum equations (45) and (46), which define the Y-branch of the space of vacua of our gauge theory. We thus conclude that the Y-branch coincides with the moduli space of instantons on multi-Taub-NUT space. The dictionary establishing the correspondence of the quantities of the gauge theory with impurities with those of an instanton on the multi-Taub-NUT space is in Table 2 below. This is in fact expected, as a CHW configuration is T-dual to a configuration of D6-branes wrapping multi Taub-NUT space with a D2-branes within their world-volumes. At low energies, the latter brane configuration is described by Yang-Mills instantons on the wrapped space. Here, we obtained an independent gauge theoretic verification of this correspondence. In the process, we also gained some insight of how one might approach the other branches. We also have a gauge theoretic interpretation of the bow reciprocity, which manifests itself as electric-magnetic duality of the gauge theory with impurities. We are about to use electric-magnetic duality to extract the asymptotic of the Y-branch in the directions of maximally broken gauge group.

Gauge Theory with Impurity Walls	Instantons on Multi-Taub-NUT
Gauge group ranks	Instanton number + its monopole charges
Number of fundamental walls, n	Rank of the structure group, $U(n)$
Number of bifundamental walls, k	Number of Taub-NUT centers, k
Periodicity of the transverse coordinate s	Taub-NUT mass parameter
Positions of the fundamental walls	Conjugacy class of the holonomy around the TN circle at infinity
Positions of the bifundamental walls	Self-dual noncommutativity parameters of the multi-Taub-NUT
Fayet-Iliopoulos parameters	Positions of the Taub-NUT centers

Table 2: Correspondence between the gauge theory and instanton parameters.

5 Asymptotic of the Y-branch

Until this moment we identified the Y-branch with the Higgs branch of the gauge theory with impurity walls. The advantage of this consideration was that it produced an exact description of the metric on the Y-branch in terms of Eqs. (44) and (45). If one is interested in the asymptotic behavior of these metrics one can either use twistorial techniques developed by Bielawski [35, 36, 37] or apply the monopole dynamics techniques of Manton and Gibbons [38, 39]. Here we have yet another approach, which is entirely in the domain of the gauge theory.

A different description of the Y-brach emerges after applying electric-magnetic duality to the gauge theory we considered so far. If the original gauge theory had n fundamental impurity walls positioned at $s = \lambda_j$ and k bifundamental impurity walls at $s = p_\sigma$, then the dual gauge theory has k fundamental impurity walls at $s = p_\sigma$ and n bifundamental impurity walls positioned at $s = \lambda_j$. The Y-branch is the Coulomb brach of the latter theory and the metric on it receives both perturbative and nonperturbative corrections. At a generic point on the Coulomb branch the gauge symmetry is maximally broken. The metric on the Y-branch is given by quantum corrected gauge couplings of the surviving gauge theory. Eigenvalues of the Y_1, Y_2 and Y_3 Higgs fields provide good asymptotic coordinates on the Y-branch.

We are interested in finding the metric in the asymptotic directions in which the difference of any two eigenvalues of Y_j becomes large. Via a one loop computation we obtain the leading metric behavior.

On the Coulomb branch the G and F_2 flatness conditions augmented by R-symmetry imply that the nonvanishing Higgs fields parameterizing the Coulomb branch are covariantly constant in s, see Eq. (44). As a result, in extreme infrared the theory is effectively three-dimensional and we can perform our one loop computation in a three dimensional theory with the gauge group $\times_{j=1}^{n} U(R_j)$. The gauge coupling of the component $U(R_j)$ is $1/\sqrt{\lambda_{j+1} - \lambda_j}$, so that

$$\frac{1}{g_{3d,j}^2} = \frac{\lambda_{j+1} - \lambda_j}{g_{4d}^2}. \tag{49}$$

The ultraviolet spectrum of this theory is comprised of bifundamental supermultiplets in $(N_{j-1}, \bar{N}_j)$ and $(\bar{N}_{j-1}, N_j)$ representations and some fundamental multiplets. The number of the fundamental multiplets in N_j and $\bar{N}_j$ of $U(N_j)$ equals to the number of p_σ points between λ_j and λ_{j+1}. The mass of the fundamental multiplet associated to the point p_σ equals $\vec{\nu}_\sigma$.

The computation itself uses the background field method quite literally as discussed in [40], section 16.6. This background field calculation was done for pure $\mathcal{N}_{3d} = 4$ super Yang-Mills theory with gauge group $SU(2)$ in [14] and it was extended to fundamental matter in [15]. For our discussion, however, we need the corresponding result for arbitrary gauge group and both fundamental and bifundamental matter, which we derive in some detail below.

5.1 The Coulomb branch of 3d super Yang-Mills theory

It is sufficient to focus on the $\mathcal{N}_{3d} = 4$ euclidean super Yang-Mills theory with gauge group $U(N_L) \times U(N_R)$. The field content consists of an $\mathcal{N}_{3d} = 4$ vector multiplet consisting of a gauge potential $A_\mu = A_\mu^L + A_\mu^R$, two 3d Majorana spinors $\lambda^{L,R}$ and $\chi^{L,R}$ and three real scalars $Z_{L,R}^i$. We will also allow for hypermultiplets in the bifundamental, the adjoint and the fundamental representations of the gauge group. Their component fields will be labeled by h_k, $k = 1, 2$ for the complex scalars and ψ_k for the spinors of $SO(1,2)$. The kinetic terms in the action are given by

$$\begin{aligned} S^{L,R}_{\text{kin, gauge}} &= \frac{1}{g^2_{3d\,L,R}} \int \mathrm{d}^3x\ \mathrm{tr}\Big(\tfrac{1}{4}F^{L,R}_{\mu\nu}F^{\mu\nu}_{L,R} + \tfrac{1}{2}\nabla^{L,R}_\mu Z^{L,R}_i \nabla^{L,R}_\mu Z^{L,R}_i \\ &\qquad\qquad + \mathrm{i}\bar{\lambda}^{L,R}\bar{\nabla\!\!\!/}^{L,R}\lambda^{L,R} + \mathrm{i}\bar{\chi}^{L,R}\bar{\nabla\!\!\!/}^{L,R}\chi^{L,R}\Big)\ , \qquad (50) \\ S^{L,R}_{\text{kin, hyper}} &= \frac{1}{g^2_{3d\,L,R}} \int \mathrm{d}^3x\ \mathrm{tr}\left(\nabla_\mu h^\dagger_k \nabla^\mu h_k + \mathrm{i}\bar{\vartheta}_k \bar{\nabla\!\!\!/}\vartheta_k\right)\ . \end{aligned}$$

Here, $\bar{\nabla\!\!\!/} := \bar{\sigma}^\mu\nabla_\mu$. The covariant derivatives of the hypermultiplets are determined by their representation. We did not write down any potential terms, as we will not need them.

As generators for $U(N)$, we will use antihermitian linear combinations of the matrices $(\tau_{ab})_{ij} = \delta_{ai}\delta_{bj}$, $a,b = 1,\ldots,N$, which satisfy the normalization condition $\mathrm{tr}(\tau^T_{ab}\tau_{ab}) = 1$. In the latter equation, there is no sum implied. For indices of the kind of a,b, we will always make the sums explicit in all formulas. The Cartan subalgebra of the gauge group is generated by the elements τ_{aa}. Note that $\sum_a \tau_{aa}$ gives the $\mathfrak{u}(1)$-part of $\mathfrak{u}(N)$, while the linear combinations of generators $\tau_{aa} - \tau_{bb}$ for $a \neq b$ span the Cartan subalgebra of $\mathfrak{su}(N) \subset \mathfrak{u}(N)$. For the generators τ_{ab}, the following list of identities holds:

$$\begin{aligned} \tau_{ab}\tau_{cd} = \delta_{bc}\tau_{ad}\ , \quad \mathrm{ad}_{\tau_{ab}}(\tau_{cd}) &= [\tau_{ab},\tau_{cd}] = \tau_{ad}\delta_{bc} - \tau_{cb}\delta_{da}\ , \\ [\tau_{aa},\tau_{bb}] = 0\ , &\quad [\mathrm{ad}_{\tau_{aa}}, \mathrm{ad}_{\tau_{bb}}] = 0\ . \end{aligned} \qquad (51)$$

To distinguish the generators of $U(N_L)$ from those of $U(N_R)$, we will write τ^L_{ab} and τ^R_{ab}, where necessary.

We are interested in a generic point on the Coulomb branch of the theory, where the gauge groups $U(N_L) \times U(N_R)$ are maximally Higgsed to $U(1)^{N_L} \times U(1)^{N_R}$ due to the Higgs scalars in the vector multiplet acquiring a generic vacuum expectation value (vev). As the gauge group is abelian, we can dualize the $N_L + N_R$ resulting photons into periodic scalars σ^a_L, $a = 1,\ldots,N_L$, and σ^a_R, $a = 1,\ldots,N_R$, parameterizing $T^{N_L+N_R}$.

What are the periods?

More explicitly, we add the following surface term to the action:

$$S_\theta = \frac{\mathrm{i}}{8\pi}\int \mathrm{d}^3x\ \epsilon^{\mu\nu\rho}\sum_{a=1}^{N_L}\sigma^a_L\partial_\mu F^{L,a}_{\nu\rho} + \epsilon^{\mu\nu\rho}\sum_{a=1}^{N_R}\sigma^a_R\partial_\mu F^{R,a}_{\nu\rho}\ , \qquad (52)$$

which, after integrating out the abelian field strength, yields the kinetic term for the dual photons σ^a:

$$S_{\text{kin, dual}} = \frac{4g_{3d\,L}^2}{(8\pi)^2} \int \mathrm{d}^3x \sum_{a=1}^{N_L} \tfrac{1}{2}\partial_\mu \sigma_L^a \partial^\mu \sigma_L^a + \frac{4g_{3d\,R}^2}{(8\pi)^2} \int \mathrm{d}^3x \sum_{a=1}^{N_R} \tfrac{1}{2}\partial_\mu \sigma_R^a \partial^\mu \sigma_R^a \ . \tag{53}$$

The open part of the classical moduli space $\mathcal{M}_{cl}$ of this theory is a subspace of the space $\mathbb{R}^{3(N_L+N_R)}$ parameterized by mutually commuting scalar fields $Z^i = \sum_{a=1}^{N_L+N_R} z^{i,a}\tau_{aa}$ times the torus parameterized by the vev of the dual photons $T^{N_L+N_R}$. At the set Δ of points in $\mathbb{R}^{3(N_L+N_R)}$ with $z^{i,a} = z^{i,b}$ for any a and $b \neq a$, gauge symmetry is enhanced. Correspondingly, we remove this set from $\mathbb{R}^{3(N_L+N_R)}$. The resulting space still has to be factored by the symmetric group $S_{N_L} \times S_{N_R}$ to eliminate permutations of the eigenvalues $z^{i,a}$. Far away from Δ, the classical moduli space thus has the form

$$\frac{(\mathbb{R}^{3(N_L+N_R)}\backslash\Delta) \times T^{N_L+N_R}}{S_{N_L} \times S_{N_R}} \ , \tag{54}$$

with its flat metric. In the next section, we compute the one-loop corrections to the Kähler metric on this moduli space.

5.2 One-loop correction to the gauge couplings

To perform the one-loop background field computation, we split the Yang-Mills fields into a low-momentum background component[7] and a high-momentum part and integrate out the latter. Explicitly, we rewrite the Yang-Mills action using this splitting and keep only terms up to second order in the high-momentum parts. The functional integrals over the high-momentum fields are Gaußian and can be trivially performed. After re-exponentiating the resulting determinants, we can read off their contributions to the effective action.

In terms of ordinary perturbation theory, this means that we compute Feynman diagrams with one loop which have low-momentum parts as external legs and high-momentum parts in the loop. We then replace these diagrams by effective vertices.

For convenience, we use R-symmetry to rotate the vev of the scalar fields into the scalar field Z^3. Furthermore, as done in [14], we combine the gauge field A_μ and the three scalars Z^i into a six-dimensional gauge field $\mathcal{A}_M$, $M = 0, \ldots, 5$ so that $\mathcal{A}_\mu = A_\mu$ and $\mathcal{A}_{i+2} = Z^i$. Similarly, we combine the two Majorana spinors of $SO(1,2)$ λ and χ into a Weyl spinor η of $SO(1,5)$. As the gauge group is maximally broken, we have $\mathcal{A}_M$ aligned in the direction of Cartan generators: $\mathcal{A}_M = \sum_a \mathcal{A}_M^a \tau_{aa}$.

[7] The background component is supersymmetric and thus satisfied the equations of motion.

As usual in the background field method, we now split our fields φ into slowly oscillating background fields $\mathring{\varphi}$ and a high-momentum part[8] $\tilde{\varphi}$:

$$\mathcal{A}_M = \begin{cases} \mathring{\mathcal{A}}_M + \tilde{\mathcal{A}}_M & , \quad M \leq 4 \ , \\ \mathring{\mathcal{A}}_M + \sum_a z^{3,a}\tau_{aa} + \tilde{\mathcal{A}}_M & , \quad M = 5 \ , \end{cases}$$
$$\eta = \mathring{\eta} + \tilde{\eta} \ , \quad h_i = \mathring{h}_i + \tilde{h}_i \ , \quad \vartheta_i = \mathring{\vartheta}_i + \tilde{\vartheta}_i \ . \tag{55}$$

To integrate out the high-momentum fields, we plug this expansion into the action. We gauge fix the action and introduce ghosts, which is done completely analogously to [14]. One can drop linear terms in any of the high-momentum fields, as they multiply terms proportional to equations of motion of the background fields. We also drop terms of higher order than two, as these do not contribute to the renormalization of purely low-momentum vertices at one loop. The key observation is that the remaining terms in the action are all of the form

$$\int \mathrm{d}^3 x \, \mathrm{tr}\left(\tilde{\varphi}^\dagger \left(\Delta_\varphi^{\kappa_\varphi}\right)\tilde{\varphi}\right) \ , \tag{56}$$

where φ denotes an arbitrary field and $\kappa_\varphi = 1$ for bosons and $\kappa_\varphi = \frac{1}{2}$ for spinor fields. The ghost contribution to the action is also of this form with $\kappa_{\mathrm{gh}} = 1$.

The resulting functional integrals are Gaußian and can be easily performed: They lead to determinants of the Δ_φ raised to a certain power. Re-exponentiating them to read off the one-loop corrections to the action yields

$$\delta S_{1\ell} = \sum_\varphi \pi_\varphi \mathrm{tr}(\log\Delta_\varphi) \ , \quad \pi_{\mathcal{A}} = -\tfrac{1}{2} \ , \ \ \pi_\eta = \tfrac{1}{2} \ , \ \ \pi_{\mathrm{gh}} = 1 \ , \ \ \pi_h = -1 \ , \ \ \pi_\vartheta = \tfrac{1}{2} \ , \tag{57}$$

where the π_φ are the powers of the determinants appearing from the Gaußian functional integral. The trace symbol here denotes a trace over gauge and spinor indices as well as all necessarily implied integrals over momentum spaces[9]. Summarizing, we perform the following approximation of the functional integral over the higher momentum modes:

$$\begin{aligned}\mathcal{Z} &= \int\left(\prod_\varphi \mathscr{D}\varphi\right) e^{-\frac{\mathrm{i}}{\hbar}S[\varphi]} \int\left(\prod_{\tilde{\varphi}} \mathscr{D}\tilde{\varphi}\right) e^{-\frac{\mathrm{i}}{\hbar}\sum_{\tilde{\varphi}}\int \mathrm{d}^3x\,\mathrm{tr}\left(\tilde{\varphi}^\dagger\left(\Delta_\varphi^{\kappa_\varphi}\right)\tilde{\varphi}\right)} \\ &\approx \int\left(\prod_\varphi \mathscr{D}\varphi\right) e^{-\frac{\mathrm{i}}{\hbar}\left(S[\varphi]+\sum_\varphi \pi_\varphi \mathrm{tr}(\log\Delta_\varphi)\right)} \ .\end{aligned} \tag{58}$$

[8]For simplicity, we drop the labels L, R when no confusion can arise.

[9]There are n momentum space integrals at $\mathcal{O}(\Delta_\varphi^n)$ in the expansion of the logarithm.

In order to remain consistent with our 1-loop approximation scheme, we should compute $\delta S_{1\ell}$ only up to second order in the high-momentum fields. This is what we will do below.

The kernel Δ_φ depends exclusively on the spin of the field φ and its gauge representation. Its explicit form is easily obtained by adapting the formulas in [14] or [40]:

$$\begin{aligned}
\Delta_\varphi &= -\partial^2 + \Delta_\varphi^{(1)} + \Delta_\varphi^{(2)} + \Delta_\varphi^{(J)} \ , \\
\Delta_\varphi^{(1)} &= \mathrm{i}\left\{\partial^\mu, \sum_{a=1}^{N_L} \mathring{\mathcal{A}}_\mu^{L,a}\tau_{aa}^L + \sum_{a=1}^{N_R} \mathring{\mathcal{A}}_\mu^{R,a}\tau_{aa}^R\right\} \ , \\
\Delta_\varphi^{(2)} &= \sum_{a,b,L/R} \left(\mathring{\mathcal{A}}_M^{L/R,a}\mathring{\mathcal{A}}_M^{L/R,b} - 2z_{L/R}^{3,a}\mathring{\mathcal{A}}_5^{L/R,b} - z_{L/R}^{3,a}z_{L/R}^{3,b}\right)\tau_{aa}^{L/R}\tau_{bb}^{L/R} \ , \\
\Delta_\varphi^{(J)} &= \sum_{a=1}^{N_L} \mathring{\mathcal{F}}_{MN}^{L,a}\tau_{aa}^L J_\varphi^{MN} + \sum_{a=1}^{N_R} \mathring{\mathcal{F}}_{MN}^{R,a}\tau_{aa}^R J_\varphi^{MN} \ .
\end{aligned} \tag{59}$$

Here, the sum in $\Delta_\varphi^{(2)}$ runs over all possible combinations of indices a, b and gauge potentials and Cartan generators of $U(N_L)$ and $U(N_R)$. For fields $\varphi_{\mathrm{f,L}}$, $\varphi_{\mathrm{ad,L}}$ and $\varphi_{\mathrm{bf,LR}}$, in the fundamental representation of $U(N_L)$, the adjoint representation of $U(N_L)$ and the bifundamental representation of $U(N_L) \times U(N_R)$, we have

$$\begin{aligned}
\Delta_\varphi^{(1)}\varphi_{\mathrm{f,L}} &= \mathrm{i}\left\{\partial^\mu, \sum_{a=1}^{N_L} \mathring{\mathcal{A}}_\mu^{L,a}\tau_{aa}^L\right\}\varphi_{\mathrm{f,L}} \ , \\
\Delta_\varphi^{(2)}\varphi_{\mathrm{ad,L}} &= \sum_{a,b=1}^{N_L} \left(\mathring{\mathcal{A}}_M^{L,a}\mathring{\mathcal{A}}_M^{L,b} - 2z_L^{3,a}\mathring{\mathcal{A}}_5^{L,b} - z_L^{3,a}z_L^{3,b}\right)[\tau_{aa}^L, [\tau_{bb}^L, \varphi_{\mathrm{ad,L}}]] \\
\Delta_\varphi^{(2)}\varphi_{\mathrm{bf,LR}} &= \sum_{a,b=1}^{N_L} \left(\mathring{\mathcal{A}}_M^{L,a}\mathring{\mathcal{A}}_M^{L,b} - 2z_L^{3,a}\mathring{\mathcal{A}}_5^{L,b} - z_L^{3,a}z_L^{3,b}\right)\tau_{aa}^L\tau_{bb}^L\varphi_{\mathrm{bf,LR}} \\
&\quad - \sum_{a=1}^{N_L}\sum_{b=1}^{N_R} \left(\mathring{\mathcal{A}}_M^{L,a}\mathring{\mathcal{A}}_M^{R,b} - 2z_L^{3,a}\mathring{\mathcal{A}}_5^{R,b} - z_L^{3,a}z_R^{3,b}\right)\tau_{aa}^L\varphi_{\mathrm{bf,LR}}\tau_{bb}^R \\
&\quad - \sum_{a=1}^{N_R}\sum_{b=1}^{N_L} \left(\mathring{\mathcal{A}}_M^{R,a}\mathring{\mathcal{A}}_M^{L,b} - 2z_R^{3,a}\mathring{\mathcal{A}}_5^{L,b} - z_R^{3,a}z_L^{3,b}\right)\tau_{bb}^L\varphi_{\mathrm{bf,LR}}\tau_{aa}^R \\
&\quad + \sum_{a,b=1}^{N_R} \left(\mathring{\mathcal{A}}_M^{R,a}\mathring{\mathcal{A}}_M^{R,b} - 2z_R^{3,a}\mathring{\mathcal{A}}_5^{R,b} - z_R^{3,a}z_R^{3,b}\right)\varphi_{\mathrm{bf,LR}}\tau_{aa}^R\tau_{bb}^R \ .
\end{aligned} \tag{60}$$

The generators J_φ^{MN} are the generators of the Lorentz group in six dimensional Minkowski space in the representation given by the field φ. We also recall that all background fields live in the Cartan subalgebra and thus

$$\mathring{\mathcal{F}}^a_{MN} := \partial_M \mathring{A}^a_N - \partial_N \mathring{A}^a_M \ . \tag{61}$$

To evaluate the effective action $\delta S_{1\ell}$ to quadratic order in the background fields, we Taylor expand the logarithm around

$$-\partial^2 - \sum_{a,b,L/R} z^{3,a}_{L/R} z^{3,b}_{L/R} \tau^{L/R}_{aa} \tau^{L/R}_{bb} =: \mathcal{G}^{-1} \tag{62}$$

and drop terms of higher than quadratic order in $\mathring{\mathcal{A}}_M$. Up to an irrelevant constant, we obtain:

$$\delta S_{1\ell} = \sum_\varphi \pi_\varphi \mathrm{tr}\left(\mathcal{G}(\Delta^{(\delta)}_\varphi) - \tfrac{1}{2}\mathcal{G}(\Delta^{(\delta)}_\varphi)\mathcal{G}(\Delta^{(\delta)}_\varphi)\right) \ , \tag{63}$$

where $\Delta^{(\delta)}_\varphi$ is given by

$$\Delta^{(\delta)}_\varphi := \Delta^{(1)}_\varphi + \Delta^{(2)}_\varphi + \Delta^{(J)}_\varphi - \sum_{a,b,L/R} z^{3,a}_{L/R} z^{3,b}_{L/R} \tau^{L/R}_{aa} \tau^{L/R}_{bb} \ . \tag{64}$$

The form of the propagator $\mathcal{G}$ now depends on the representation of the field φ. The nonvanishing components of $\mathcal{G}$ for a field φ in the various representations are:

$$\mathcal{G} = \begin{cases} \mathcal{G}^{LL,LL}_{ab,cd} = \dfrac{\delta_{ac}\delta_{bd}}{p^2 - (z^{3,c} - z^{3,d})^2} \ , & \varphi \text{ in the adjoint of } U(N_L) \ , \\ \mathcal{G}^{RR,RR}_{ab,cd} = \dfrac{\delta_{ac}\delta_{bd}}{p^2 - (z^{3,c} - z^{3,d})^2} \ , & \varphi \text{ in the adjoint of } U(N_R) \ , \\ \mathcal{G}^{LR,LR}_{ab,cd} = \dfrac{\delta_{ac}\delta_{bd}}{p^2 - (z^{3,c} - z^{3,d})^2} \ , & \varphi \text{ in the } (\overline{N}_R, N_L) \text{ of } U(N_R)\times U(N_L) \ , \\ \mathcal{G}_{ab} = \dfrac{\delta_{ab}}{p^2 - (z^{3,a})^2} \ , & \varphi \text{ in the fundamental representation} \ , \end{cases} \tag{65}$$

where the superscripts L/R indicate to which gauge group the respective indices belong. Symmetry considerations now allow us to reduce (63) significantly. First, the term linear in $\Delta^{(J)}_\varphi$ vanishes, since $\mathrm{tr}(J^{MN}_\varphi) = 0$. Most important, however, is supersymmetry: Those contributions to $\delta S_{1\ell}$ which do not contain $\Delta^{(J)}_\varphi$ are up to the factors of π_φ are identical for all fields. They therefore cancel due to the relations

$$6\pi_{\mathcal{A}} + 4\pi_\psi + \pi_{gh} = 0 \quad \text{and} \quad \pi_h + 2\pi_\chi = 0 \ . \tag{66}$$

This reduces (63) to

$$\delta S_{1\ell} = \sum_{\varphi} \pi_\varphi \mathrm{tr}\left(-\tfrac{1}{2}\mathcal{G}\Delta_\varphi^{(J)}\mathcal{G}\Delta_\varphi^{(J)}\right) \ . \tag{67}$$

For a field φ in the fundamental representation[10] of $U(N_L)$, we arrive at the following expression in the momentum space:

$$\begin{aligned}\delta S_{1\ell,\varphi} = &-\tfrac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a,b=1}^{N_L} \mathring{\mathcal{F}}^{L,a}_{MN}(k)\mathring{\mathcal{F}}^{L,b}_{RS}(-k)\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)\\ &\times \mathrm{tr}\left(\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\delta_{ba}}{p^2-(z^{3,a})^2}\tau_{aa}\frac{\delta_{ab}}{(p+k)^2-(z^{3,b})^2}\tau_{bb}\right)\\ = &-\tfrac{1}{2}\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a=1}^{N_L}\left(\frac{\mathrm{i}}{8\pi|z^{3,a}|}+\mathcal{O}(k^2)\right)\mathring{\mathcal{F}}^{a,L}_{MN}(k)\mathring{\mathcal{F}}^{a,L}_{RS}(-k) \ ,\end{aligned} \tag{68}$$

where

$$\begin{aligned}&\mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi) = (g^{MR}g^{NS}-g^{MS}g^{NR})C_\varphi \ ,\\ &C_{\mathcal{A}} = 2 \ , \quad C_\psi = C_\chi = 1 \ , \quad C_{gh} = C_h = 0 \ .\end{aligned} \tag{69}$$

Fields in the adjoint representation of $U(N_L)$ yield the following contribution:

$$\begin{aligned}\delta S_{1\ell,\varphi} = &-\tfrac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a,b=1}^{N_L} \mathring{\mathcal{F}}^{L,a}_{MN}(k)\mathring{\mathcal{F}}^{L,b}_{RS}(-k)\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)\\ &\times \sum_{c,d}\left(\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\delta_{ac}-\delta_{ad}}{p^2-(z^{3,c}-z^{3,d})^2}\frac{\delta_{bc}-\delta_{bd}}{(p+k)^2-(z^{3,c}-z^{3,d})^2}\right)\\ = &-\tfrac{1}{2}\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a,b=1,a\neq b}^{N_L}\left(\frac{\mathrm{i}}{8\pi|z^{3,a}-z^{3,b}|}+\mathcal{O}(k^2)\right)\\ &\times(\mathring{\mathcal{F}}^{L,a}_{MN}(k)-\mathring{\mathcal{F}}^{L,b}_{MN}(k))(\mathring{\mathcal{F}}^{L,a}_{RS}(-k)-\mathring{\mathcal{F}}^{L,b}_{RS}(-k)) \ .\end{aligned} \tag{70}$$

[10] The contribution for a field in the fundamental representation of $U(N_R)$ is the same up to replacing L with R everywhere.

From (65) it is obvious that the contribution of a bifundamental field is

$$\delta S_{1\ell,\varphi} = -\tfrac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a=1}^{N_L}\sum_{b=1}^{N_R}\mathring{\mathcal{F}}^{L,a}_{MN}(k)\mathring{\mathcal{F}}^{R,b}_{RS}(-k)\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)$$
$$\times\sum_{c=1}^{N_L}\sum_{d=1}^{N_R}\left(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\delta_{ac}}{p^2-(z^{3,c}-z^{3,d})^2}\frac{-\delta_{bd}}{(p+k)^2-(z^{3,c}-z^{3,d})^2}\right)$$
$$= \tfrac{1}{2}\pi_\varphi \mathrm{tr}(J^{MN}_\varphi J^{RS}_\varphi)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{a=1}^{N_L}\sum_{b=1}^{N_R}\left(\frac{\mathrm{i}}{8\pi|z^{3,a}-z^{3,b}|}+\mathcal{O}(k^2)\right)\mathring{\mathcal{F}}^{L,a}_{MN}(k)\mathring{\mathcal{F}}^{R,b}_{RS}(-k)\ . \tag{71}$$

We have not explicitly taken the modification to the kernel of $\mathring{\mathcal{A}}$ into account. However, we know that the total result is gauge invariant, and therefore these modifications just correct the action appropriately, so that the shift in the gauge coupling which we observed in (68) and (70) is also reflected in the matter terms.

Let us now consider the case of nonvanishing mass terms for the hypermultiplets. First, note that the cancellations we observed above are independent of the masses. (The supersymmetry cancellation is due to $\mathcal{N}_{3d}=2$ SUSY, which is compatible with the introduction of mass terms.) Second, the one-loop corrections due to hypermultiplets are due to the spinors χ, since the generators J^{MN}_φ for scalars vanish. In our discussion of the superspace action in previous sections we introduced an $SO(3)$-multiplet of mass terms: a complex mass from a mass term in the superpotential, and a third real mass from adding terms to the vector superfield which is necessary for the $so(3)_Z$ R-symmetry. Their effect on our formulas is the simple shift

$$|z^{3,a}| \to \sqrt{(\nu_3+z^{3,a})^2+(\nu_1)^2+(\nu_2)^2} = |\vec{z}^a-\vec{\nu}|\ . \tag{72}$$

Similarly, we could introduce mass terms for adjoint and bifundamental hypermultiplets. Their effects are slightly more complicated. As we will not need them in the subsequent discussion, we refrain from presenting them in detail.

5.3 Special cases

We now verify our results by comparing them to those obtained in [14, 15]. That is, we restrict ourselves to gauge group $SU(2)$. A Cartan subalgebra of $\mathfrak{su}(2)$ can be spanned by σ^3, the third Pauli matrix, and we have

$$\mathring{\mathcal{F}}_{MN} = \begin{pmatrix}\mathring{\mathcal{F}}^1_{MN} & 0\\ 0 & \mathring{\mathcal{F}}^2_{MN}\end{pmatrix} = (\mathring{\mathcal{F}}^1_{MN}+\mathring{\mathcal{F}}^2_{MN})\frac{\mathbb{I}}{2}+(\mathring{\mathcal{F}}^1_{MN}-\mathring{\mathcal{F}}^2_{MN})\frac{\sigma^3}{2} = \mathring{\mathcal{F}}^{\mathbb{I}}_{MN}\frac{\mathbb{I}}{2}+\mathring{\mathcal{F}}^{\sigma^3}_{MN}\frac{\sigma^3}{2}\ . \tag{73}$$

Thus, to restrict to gauge group $SU(2)$, we require that $\mathcal{F}^1_{MN} = -\mathcal{F}^2_{MN}$ and $z^{3,1} = -z^{3,2}$.

First of all, we consider the pure gauge theory without matter fields. To make contact with [14], we also introduce $M_W := \frac{1}{2}z^{3,1} = -\frac{1}{2}z^{3,2}$. The one-loop corrections of the fields $\mathcal{A}$ and η then read as

$$\begin{aligned} \delta S_{1\ell,\mathcal{A}} &= 2\frac{\mathrm{i}}{8\pi M_W}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathring{\mathcal{F}}^{\sigma^3}_{MN}\mathring{\mathcal{F}}^{\sigma^3,MN}\ , \\ \delta S_{1\ell,\eta} &= -\frac{\mathrm{i}}{8\pi M_W}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathring{\mathcal{F}}^{\sigma^3}_{MN}\mathring{\mathcal{F}}^{\sigma^3,MN}\ , \end{aligned} \tag{74}$$

which is the same result as in [14]. For an adjoint hypermultiplet with mass m_{ad}, we have

$$\delta S_{1\ell,\mathrm{hyper,ad}} = -\frac{1}{2}\left(\frac{\mathrm{i}}{8\pi|m_{\mathrm{ad}}+M_W|}+\frac{\mathrm{i}}{8\pi|m_{\mathrm{ad}}-M_W|}\right)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathring{\mathcal{F}}^{\sigma^3}_{MN}\mathring{\mathcal{F}}^{\sigma^3,MN}\ , \tag{75}$$

and for a fundamental multiplet with mass ν_{f}, we obtain

$$\delta S_{1\ell,\mathrm{hyper,f}} = -\frac{1}{8}\left(\frac{\mathrm{i}}{8\pi|\nu_{\mathrm{f}}+M_W|}+\frac{\mathrm{i}}{8\pi|\nu_{\mathrm{f}}-M_W|}\right)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathring{\mathcal{F}}^{\sigma^3}_{MN}\mathring{\mathcal{F}}^{\sigma^3,MN}\ , \tag{76}$$

in agreement with [15]. As observed there, the $\mathcal{N}_{3d} = 8$ theory corresponds to one massless hypermultiplet in the adjoint representation. The hypermultiplet's contribution cancels exactly the contribution from the vector multiplet and the one-loop corrections vanish.

5.4 The asymptotic metric on the moduli spaces

The mirror gauge theory we study has maximally supersymmetric Yang-Mills in four-dimensional bulk with n bifundamental walls positioned at $s = \lambda_j$ and with k fundamental walls positioned at $s = p_\sigma$. Each fundamental multiplet confined to the wall at $s = p_\sigma$ has a mass $\vec{\nu}_\sigma$. The gauge group on the i^{th} interval $[\lambda_i, \lambda_{i+1}]$ is $U(N_i)$. As discussed in Section **??** at point λ_i the space-time is cut into two halves with one gauge group $U(N_{i-1})$ acting on the boundary of the left half and with $U(N_i)$ acting on the boundary of the right half. So far we focussed on a product $U(N_i)\times U(N_{i+1})$ of two neighboring groups. Here we assemble all of the contributions to extract the asymptotic metric on the Coulomb branch of this gauge theory. To be exact, we have computed one-loop-corrected gauge gouplings which, via supersymmetry, or equivalently via hyperkählerity, completely fix the rest of the metric.

We can now read off the desired one-loop corrections from the results obtained in the previous sections. We have $U(N_i)$ gauge group in the i^{th} interval between $s = \lambda_i$ and $s = \lambda_{i+1}$ and we have n intervals altogether. This yields $N_1 + \ldots + N_n$ Cartan generators in total, and we label them by consecutive integers. The Cartan generators of the i^{th} interval correspond to the integers $\mathfrak{l}_i, \ldots, \mathfrak{r}_i$, where $\mathfrak{l}_i = \sum_{j=1}^{i-1} N_j$ and $\mathfrak{r}_i = -1 + \sum_{j=1}^{i} N_j$. The tree-level action is

$$-\frac{\mathrm{i}}{4} \sum_{a=1}^{N_1+\ldots+N_L} \frac{\lambda_{i+1} - \lambda_i}{g^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathring{\mathcal{F}}^a_{MN}(k) \mathring{\mathcal{F}}^{a,MN}(-k) \ . \tag{77}$$

For the vector multiplet in the interval i, we obtain a correction

$$\begin{aligned} \delta S_{1\ell,\text{vector}} = \frac{\mathrm{i}}{16\pi} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{a,b=\mathfrak{l}_i, a\neq b}^{\mathfrak{r}_i} \frac{1}{|z^{3,a} - z^{3,b}|} \times \\ \times \left(\mathring{\mathcal{F}}^a_{MN} \mathring{\mathcal{F}}^{a,MN} - 2 \mathring{\mathcal{F}}^a_{MN} \mathring{\mathcal{F}}^{b,MN} + \mathring{\mathcal{F}}^b_{MN} \mathring{\mathcal{F}}^{b,MN} \right) \ . \end{aligned} \tag{78}$$

The bifundamental fields transforming nontrivially under the $U(N_i)$ factor of the gauge group are positioned at λ_j with $j = i+1$ or $j = i$ and yield a contribution

$$\delta S_{1\ell,\text{bifund.}} = -\frac{\mathrm{i}}{16\pi} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{a=\mathfrak{l}_i}^{\mathfrak{r}_i} \sum_{b=\mathfrak{l}_j}^{\mathfrak{r}_j} \frac{1}{|z^{3,a} - z^{3,b}|} \left(\mathring{\mathcal{F}}^a_{MN} \mathring{\mathcal{F}}^{b,MN} \right) \ , \tag{79}$$

and each massive hypermultiplet at $s = p_\sigma$ with $p_\sigma \in (\lambda_i, \lambda_{i+1})$ adds

$$\delta S_{1\ell,\text{hyper}} = -\frac{\mathrm{i}}{16\pi} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{a=l_1}^{r_1} \frac{1}{|\vec{z}^a - \vec{\nu}_\sigma|} \mathring{\mathcal{F}}^a_{MN} \mathring{\mathcal{F}}^{a,MN} \ . \tag{80}$$

Altogether, we get the following one-loop-corrected coupling constant for any pair of the Cartan generators:

$$\left(\frac{1}{g^2_{3d}} \right)_{ab} = \underbrace{\frac{\lambda_{i+1} - \lambda_i}{g^2} \delta_{ab}}_{\text{tree level}} + \underbrace{s_{ab}}_{\text{vector mplt.}} + \underbrace{n_{ab}}_{\text{bifundamentals}} + \underbrace{d_{ab}}_{\text{fundamentals}} \ , \tag{81}$$

where

$$s_{ab} = \begin{cases} -\sum_{c=l_i}^{r_i} \frac{1}{2\pi|z^{3,a}-z^{3,c}|} & \text{for } a=b\ ,\ a\in[l_i,r_i]\ , \\ \frac{1}{2\pi|z^{3,a}-z^{3,b}|} & \text{for } a\neq b\ ,\ a,b\in[l_i,r_i]\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

$$n_{ab} = \begin{cases} \frac{1}{4\pi|z^{3,a}-z^{3,b}|} & \text{for } a\in[l_i,r_i]\ ,\ b\in[l_j,r_j]\ ,\ i=j\pm1\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{82}$$

$$d_{ab} = \begin{cases} \sum_{\sigma|\lambda_i<p_\sigma<\lambda_{i+1}} \frac{1}{4\pi|\vec{z}^a-\vec{m}|} & \text{for } a=b\ ,\ a\in[l_1,r_1]\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

We can undo our choice of orientation of the Higgs vev by putting the components $z^{3,a}$ into the vector $\vec{z}^a$. Our result agrees with the asymptotic metric for balanced representations of A_k bows as given in [17], equation (128)-(130).

6 Conclusions

While gauge theories with impurities are interesting in their own right, here we employed them as a tool for studying the moduli spaces of instantons on multi Taub-NUT spaces. To this end we used maximally supersymmetric four-dimensional Yang-Mills theory coupled to both fundamental and bifundamental $\mathcal{N}_{4d}=2$ matter confined to three-dimensional impurity walls. Just as in [8, 9], we worked in $\mathcal{N}_{3d}=2$ superspace language. This allowed us to study the moduli space of vacua of the gauge theory by considering D- and F-flatness conditions.

A string theory realizations of this theory is given by the Chalmers-Hanany-Witten configuration of branes in type IIB string theory. Via T-duality, this configuration is related to another string theory background which can be effectively described by Yang-Mills instantons on multi-Taub-NUT space TN_k. Here, the number k of the Taub-NUT centers corresponds to the number of the NS5 branes in the CHW configuration. We showed that it is the Higgs branch of the impurity gauge theory that is identified via T-duality with the moduli space of instanton on TN_k.

To identify the Higgs branch, we derived conditions on supersymmetric vacuum configurations. We found that the resulting equations are exactly the moment map conditions appearing in the bow construction of [10] and [17]. This independently verifies the string duality statement.

We then used this relation to compute the asymptotic metric of the moduli space of instantons on TN_k using the gauge theory as follows: Applying electric-magnetic duality to our impurity theory, we obtain the same type of gauge theory with the two types of impurity walls interchanged. The resulting mirror theory has the moduli space of instantons on TN_k as its Coulomb branch. Its metric is determined by the kinetic term couplings of the effective theory. We performed a one-loop background field computation, which is an extension of the calculation presented in [14, 15]. The resulting asymptotic metric on the Coulomb branch is exactly the asymptotic metric on the moduli space of instantons found in [17].

We expect that the techniques we have used here can be fruitfully applied to other questions, as well. For example, the superfield action we have used defines a gauge theory with impurities for any bow representation. One could use this theory to verify that the moduli space of an E-type bow is insensitive to the interval lengths (i.e. to the three-dimensional couplings) and that it coincides with the moduli space of a quiver representation obtained by shrinking all of the bow intervals. Another intriguing direction for future research is to explore such impurity theories on curved space-time background.

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Appendix

We use the embedding of $\mathcal{N}_{3d} = 2$ superspace into $\mathcal{N}_{d=4} = 2$ superspace as presented in [9]. The coordinates $(x^{\hat{\mu}}, \theta^1, \bar{\theta}_1, \theta^2, \bar{\theta}_2)$ where $\hat{\mu} = 0, 1, 3$, parameterize the $\mathcal{N}_{4d} = 2$ superspace. Consider the linear combinations

$$\theta = \tfrac{1}{2}(\theta_1 + \bar{\theta}^1 - \theta_2 - \bar{\theta}^2) \text{ and } \not{\theta} = \tfrac{1}{2\mathrm{i}}(\theta_1 - \bar{\theta}^1 - \theta_2 + \bar{\theta}^2) \ , \tag{83}$$

with analogous linear combinations for the supercharges and the superspace covariant derivatives. The subspace given by $\not{\theta} = 0$ and $x^{\hat{2}} = s$, with some fixed value of the parameter s, is then preserved by the $\mathcal{N}_{3d} = 2$ supersymmetry algebra. Thus one can use the coordinates $(x^\mu, \theta, \bar{\theta}), \mu = 0, 1, 2$, to parameterize the three-dimensional superspace, with s playing the role of a parameter external to this superspace.

We take our fields to be antihermitian so that $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and the covariant derivatives are of the form $\nabla_\mu B = \partial_\mu B + [A_\mu, B]$ or $\nabla_\mu \mathcal{Q} = \partial_\mu \mathcal{Q} + A_\mu \mathcal{Q}$. A bar denotes hermitian conjugation.

Spinor Conventions

We use the standard superfield conventions as given, e.g., in [26]. The metric is $\eta_{\mu\nu}$ =diag$(-1,1,1,1)$. We use $\epsilon^{12} = -\epsilon_{12} = 1$ for raising and lowering spinor indices. So $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, where $\epsilon^{\alpha\beta} = i\sigma^2$. The Pauli matrices are: $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\sigma_0 = -1_{2\times 2}$. The spinor summation conventions are:

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \tag{84}$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_\alpha\bar{\chi}^\alpha = -\bar{\psi}^\alpha\bar{\chi}_\alpha = \bar{\chi}_\alpha\bar{\psi}^\alpha = \bar{\chi}\bar{\psi}, \tag{85}$$

$$\psi\bar{\chi} = \psi^\alpha\bar{\chi}_\alpha = -\psi_\alpha\bar{\chi}^\alpha = \bar{\chi}^\alpha\psi_\alpha = \bar{\chi}\psi. \tag{86}$$

Some useful spinor relations are:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \qquad \bar{\theta}^\alpha\bar{\theta}^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\bar{\theta}^2, \tag{87}$$

$$\theta_\alpha\theta^\beta = -\frac{1}{2}\delta^\beta_\alpha\theta^2, \qquad (\theta\sigma^{\hat{\mu}}\bar{\theta})(\theta\sigma^{\hat{\mu}}\bar{\theta}) = -\frac{1}{2}\theta^2\bar{\theta}^2\eta^{\hat{\mu}\hat{\nu}}. \tag{88}$$

In particular for Equation (4) in Section 3.1 we have

$$\begin{aligned} (\bar{\theta}_\alpha\theta^\alpha)^2 =& \bar{\theta}_\alpha\theta^\alpha\bar{\theta}_\beta\theta^\beta = -\theta^\alpha\bar{\theta}_\alpha\bar{\theta}_\beta\theta^\beta \\ =& \frac{1}{2}\bar{\theta}^2\theta^\alpha\epsilon_{\alpha\beta}\theta^\beta = \frac{1}{2}\bar{\theta}^2\theta^\alpha\theta_\alpha = \frac{1}{2}\bar{\theta}^2\theta^2, \end{aligned} \tag{89}$$

where we have used Equation (86) in the first line and Equation (87) in going from the first to the second line.

Integration in superspace has the following properties: $\int d\theta = 0$, $\int d\theta\ \theta = 1$, so that $\int d\theta_\alpha\theta^\beta = \partial_\alpha\theta^\beta = \delta^\beta_\alpha$.

Superfields

The $\mathcal{N}_{3d} = 2$ chiral superfields read in chiral coordinates $y^{\hat{\mu}} = x^{\hat{\mu}} + \mathrm{i}\theta\sigma^{\hat{\mu}}\bar{\theta}$, $\hat{\mu} = 0,1,3$, as follows:

$$\begin{aligned} \mathcal{X} &= v_6(y) + iY_1(y) + \sqrt{2}\theta\psi(y) + \theta^2 G(y)\ , \\ \mathcal{U}^1 &= \mathcal{Z} = Z(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^i(y)\ , \\ \mathcal{U}^2 &= \mathcal{Y} = Y(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^i(y)\ . \end{aligned} \tag{90}$$

while the $\mathcal{N}_{3d} = 2$ vector superfield is given in Wess-Zumino gauge by

$$\mathcal{V} = -\theta\sigma^2\bar{\theta}Z_3 - \theta\sigma^\mu\bar{\theta}v_\mu + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D\ . \tag{91}$$

The vector superfield $\mathcal{V}$ together with the chiral superfield $\mathcal{X}$ form a $\mathcal{N}_{4d} = 2$ vector supermultiplet:

$$\Psi(y) = \mathcal{X}(y,\theta_1) + \mathrm{i}\sqrt{2}\theta_2 W(y,\theta_1) + \theta_2^2 G(Y,\theta_1)\ . \tag{92}$$

Here, W is the chiral fermionic field strength of the superfield $\mathcal{V}$. Performing now the coordinate transformation (83), the $\mathcal{N}_{4d} = 2$ vector superfield splits at $\not\theta = 0$ into an $\mathcal{N}_{3d} = 4$ complex linear superfield, which is given by the $\mathcal{N}_{3d} = 2$ chiral superfield $\mathcal{X}$ and the $\mathcal{N}_{3d} = 2$ complex linear superfield Σ.

The Linear Multiplet

The vector multiplet gives rise to the linear multiplet Σ defined by

$$\Sigma = \epsilon^{\alpha\beta}\bar{D}_\alpha(e^{2iV}D_\beta e^{-2iV}). \tag{93}$$

The calculation is made simpler by writing Σ as a function of $y^{\hat{\mu}} = x^{\hat{\mu}} + i\theta\sigma^{\hat{\mu}}\bar{\theta}$:

$$\mathcal{V}(y) = -i\theta\bar{\theta}Z_3 - \theta\sigma^{\hat{\mu}}\bar{\theta}v_{\hat{\mu}} + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2(D - i\partial_{\hat{\mu}}v^{\hat{\mu}}). \tag{94}$$

We note that

$$e^{2i\mathcal{V}}D_\beta e^{-2i\mathcal{V}} = -2iD_\beta\mathcal{V} + 2[\mathcal{V}, D_\beta\mathcal{V}], \tag{95}$$

since powers of V higher than V^2 vanish due to the properties of the Grassman variables θ and $\bar{\theta}$. Writing $D_\beta\mathcal{V}$ in components:

$$\begin{aligned} D_\beta\mathcal{V} =&(\partial_\beta + 2i\sigma^{\hat{\mu}}_{\beta\gamma}\bar{\theta}^\gamma\partial_{\hat{\mu}})\mathcal{V}(y) \\ =& - i\bar{\theta}_\beta Z_3 - \sigma^{\hat{\mu}}_{\beta\gamma}\bar{\theta}^\gamma v_{\hat{\mu}} + 2i\theta_\beta\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\lambda_\beta + \theta_\beta\bar{\theta}^2 D \\ &+ \sigma^{\hat{\mu}}_{\beta\gamma}\theta^\gamma\bar{\theta}^2\partial_{\hat{\mu}}Z_3 - i\bar{\theta}^2\sigma_\beta^{\hat{\mu}\hat{\nu}\gamma}\theta_\gamma(\partial_{\hat{\mu}}v_{\hat{\nu}} - \partial_{\hat{\nu}}v_{\hat{\mu}}) + \theta^2\bar{\theta}^2\sigma^{\hat{\mu}}_{\beta\gamma}\partial_{\hat{\mu}}\bar{\lambda}^\gamma, \end{aligned} \tag{96}$$

we can now write an expression for the commutator:

$$[\mathcal{V}, D_\beta\mathcal{V}] = - \bar{\theta}^2\sigma_\beta^{\hat{\mu}\hat{\nu}\gamma}\theta_\gamma[v_{\hat{\mu}}, v_{\hat{\nu}}] - i\theta^2\bar{\theta}^2\sigma^{\hat{\mu}}_{\beta\gamma}[v_{\hat{\mu}}, \bar{\lambda}^\gamma]. \tag{97}$$

Now we have Equation (95) in component form and since Σ is a function of y, the covariant derivative $\bar{D}_\alpha$ reduces to $-\bar{\partial}_\alpha$. Using the Taylor expansion $f(x) = f(y) - i\theta\sigma^m\bar{\theta}\partial_m f(y) + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box f(y)$ we retrieve the component expansion fo the linear multiplet $\Sigma(x)$:

$$\begin{aligned} \Sigma(x) =&4Z_3 - 4\theta\bar{\lambda} - 4\bar{\theta}\lambda - 4i\theta\sigma\bar{\theta}D - 2\theta\sigma_\mu^{3d}\bar{\theta}^{\mu\nu\kappa}F_{\nu\kappa} - 2i\bar{\theta}^2\theta\sigma^\mu_{3d}\nabla_\mu\lambda \\ &+ 2i\theta^2\bar{\theta}\bar{\sigma}^\mu\nabla_\mu\bar{\lambda} - \theta^2\bar{\theta}^2\Box Z_3. \end{aligned} \tag{98}$$

$\mathcal{N}_{3d} = 2$ **matter supermultiplets on impurity walls**

Each fundamental defect walls is carrying a fundamental chiral supermultiplet

$$\mathcal{Q}_{1j} = Q_{1j} + \sqrt{2}\theta\zeta_{1j} + \theta^2 J_{1j}, \tag{99}$$

and an anti-fundamental chiral supermultiplet

$$\mathcal{Q}_{2j} = Q_{2j} + \sqrt{2}\theta\zeta_{2j} + \theta^2 J_{2j}. \tag{100}$$

While each bifundamental defect wall carries one chiral supermultiplet in the $(N_{\sigma-1}, \bar{N}_\sigma)$ representation of $U(N_{\sigma-1}) \times U(N_\sigma)$

$$\mathcal{B}_{1j} = B_{1j} + \sqrt{2}\theta\xi_{1j} + \theta^2 L_{1j}, \tag{101}$$

and one chiral supermultiplet in the $(\bar{N}_{\sigma-1}, N_\sigma)$ representation

$$\mathcal{B}_{2j} = B_{2j} + \sqrt{2}\theta\xi_{2j} + \theta^2 L_{2j}. \tag{102}$$

References

[1] S. Elitzur, A. Giveon, and D. Kutasov, *Branes and N = 1 duality in string theory,* Phys. Lett. B **400** (1997) 269 [hep-th/9702014].

[2] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. P. Warner, *Self-Dual Strings and N=2 Supersymmetric Field Theory,* Nucl. Phys. B **477** (1996) 746 [hep-th/9604034].

[3] E. Witten, *Solutions of four-dimensional field theories via M-theory,* Nucl. Phys. B **500** (1997) 3 [hep-th/9703166].

[4] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity,* Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[5] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions,* hep-th/9607163.

[6] G. Chalmers and A. Hanany, *Three dimensional gauge theories and monopoles,* Nucl. Phys. B **489** (1997) 223 [hep-th/9608105].

[7] S. A. Cherkis and A. Kapustin, *Singular monopoles and supersymmetric gauge theories in three dimensions,* Nucl. Phys. B **525** (1998) 215 [hep-th/9711145].

[8] O. DeWolfe, D. Z. Freedman, and H. Ooguri, *Holography and defect conformal field theories,* Phys. Rev. D **66** (2002) 025009 [hep-th/0111135].

[9] J. Erdmenger, Z. Guralnik, and I. Kirsch, *Four-dimensional superconformal theories with interacting boundaries or defects,* Phys. Rev. D **66** (2002) 025020 [hep-th/0203020].

[10] S. A. Cherkis, *Instantons on the Taub-NUT Space,* Adv. Theor. Math. Phys. **14** (2010) 609 [0902.4724 [hep-th]].

[11] C. Montonen and D. I. Olive, *Magnetic monopoles as gauge particles?,* Phys. Lett. B **72** (1977) 117.

[12] E. Witten and D. I. Olive, *Supersymmetry algebras that include topological charges,* Phys. Lett. B **78** (1978) 97.

[13] K. A. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories,* Phys. Lett. B **387** (1996) 513 [hep-th/9607207].

[14] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong, and S. Vandoren, *Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold,* Nucl. Phys. B **502** (1997) 59 [hep-th/9703228].

[15] N. Dorey, D. Tong, and S. Vandoren, *Instanton effects in three-dimensional supersymmetric gauge theories with matter,* JHEP **04** (1998) 005 [hep-th/9803065].

[16] K.-M. Lee, E. J. Weinberg, and P. Yi, *The Moduli space of many BPS monopoles for arbitrary gauge groups,* Phys.Rev. D **54** (1996) 1633 [hep-th/9602167 [hep-th]].

[17] S. A. Cherkis, *Instantons on gravitons,* dedicated to the memory of Israel Moiseevich Gelfand, 1007.0044 [hep-th].

[18] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,* Nucl. Phys. B **492** (1997) 152 [hep-th/9611230].

[19] S. A. Cherkis, *Moduli spaces of instantons on the Taub-NUT space,* Commun. Math. Phys. **290** (2009) 719 [0805.1245 [hep-th]].

[20] M. R. Douglas, *Branes within branes,* hep-th/9512077.

[21] M. R. Douglas and G. W. Moore, *D-branes, quivers, and ALE instantons,* hep-th/9603167.

[22] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions,* Nucl. Phys. B **499** (1997) 67 [hep-th/9703110].

[23] S. A. Cherkis and A. Kapustin, *Singular monopoles and gravitational instantons,* Commun. Math. Phys. **203** (1999) 713 [hep-th/9803160].

[24] C. D. Blair and S. A. Cherkis, *One monopole with k singularities,* JHEP **1011** (2010) 127 [`1009.5387 [hep-th]`].

[25] C. D. Blair and S. A. Cherkis, *Singular monopoles from Cheshire bows,* Nucl. Phys. B **845** (2011) 140 [`1010.0740 [hep-th]`].

[26] J. Wess and J. Bagger, *Supersymmetry and supergravity,* Princeton, USA: Univ. Pr. (1992).

[27] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems,* Phys. Lett. B **318** (1993) 469 [`hep-ph/9309335`].

[28] P. C. Argyres, M. R. Plesser, and N. Seiberg, *The Moduli Space of N=2 SUSY QCD and Duality in N=1 SUSY QCD,* Nucl. Phys. B **471** (1996) 159 [`hep-th/9603042`].

[29] A. Kapustin and S. Sethi, *The Higgs branch of impurity theories,* Adv. Theor. Math. Phys. **2** (1998) 571 [`hep-th/9804027`].

[30] T. Kitao, K. Ohta, and N. Ohta, *Three-dimensional gauge dynamics from brane configurations with (p,q)-fivebrane,* Nucl. Phys. B **539** (1999) 79 [`hep-th/9808111`].

[31] O. Bergman, A. Hanany, A. Karch, and B. Kol, *Branes and supersymmetry breaking in 3D gauge theories,* JHEP **10** (1999) 036 [`hep-th/9908075`].

[32] B. M. Zupnik and D. G. Pak, *Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities,* Theor. Math. Phys. **77** (1988) 1070.

[33] E. A. Ivanov, *Chern-Simons matter systems with manifest $\mathcal{N} = 2$ supersymmetry,* Phys. Lett. B **268** (1991) 203.

[34] D. Gaiotto and E. Witten, *Supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory,* `0804.2902 [hep-th]`.

[35] R. Bielawski, *Hyperkahler Structures and Group Actions,* J. London Math. Soc. **55** (1997) 400.

[36] R. Bielawski, *Asymptotic metrics for SU(N)-monopoles with maximal symmetry breaking,* Commun. Math. Phys. **199** (1998) 297 [`hep-th/9801092`].

[37] R. Bielawski, *Monopoles and the Gibbons-Manton metric,* Commun. Math. Phys. **194** (1998) 297 [`hep-th/9801091`].

[38] N. Manton, *Monopole Interactions at Long Range,* Phys.Lett. B **154** (1985) 397.

[39] G. W. Gibbons and N. S. Manton, *The Moduli space metric for well separated BPS monopoles,* Phys. Lett. B **356** (1995) 32 [`hep-th/9506052`].

[40] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory,* Reading, USA: Addison-Wesley (1995) 842 p.