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Singular Monopoles from Cheshire Bows

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Abstract

Singular monopoles are nonabelian monopoles with prescribed Dirac-type singularities. All of them are delivered by the Nahm's construction of monopoles. In practice, however, its effectiveness is limited to the cases of one or two singularities. We present an alternative construction of singular monopoles formulated in terms of Cheshire bows. To illustrate the advantages of this bow construction we obtain an explicit expression for one $U(2)$ gauge group monopole with any given number of singularities of Dirac type.

Contents

1 Introduction

We formulate a new construction of singular monopoles and illustrate its every step by explicitly computing one monopole with k Dirac-type singularities as an example. Until now the conventional techniques were limited to $k = 1$ and $k = 2$ cases. Our construction is equally effective for any number of singularities. The elements of our construction are conveniently organized in terms of bows, which are generalizations of quivers introduced in [1, 2, 3]. Originally bows were introduced in order to find Yang-Mills instantons on curved backgrounds of asymptotically locally flat gravitational instantons. As we argue here, by specifying to what we call Cheshire bow representations one obtains an alternative way of finding all singular monopoles.

1.1 The Use of Singular Monopoles

Singular monopoles play important role in a number of physical problems and have diverse mathematical applications. These classical Yang-Mills-Higgs configurations are directly related to

- the vacua and the low energy behavior of the super-Yang-Mills theory in three dimensions,
- the electric-magnetic duality of maximally supersymmetric Yang-Mills in four space-time dimensions,
- Yang-Mills instantons on curved backgrounds,
- string theory brane configurations, and
- gravitational instantons.

As fist suggested in [4] and explored in e.g. [5, 6, 7], the moduli spaces of vacua of the quantum $\mathcal{N} = 4$ three-dimensional gauge theories are given by the moduli spaces of singular monopoles. In particular the quantum moduli space of vacua of the $\mathcal{N} = 4$ $U(n)$ super Yang-Mills theory with k matter hypermultiplets in the fundamental representation is the classical moduli space of n $U(2)$ monopoles with k minimal singularities. In the exploration of the Montonen-Olive duality [8], or more exactly its supersymmetric version [9], in [10, 11] the Goddard-Nuyts-Olive (GNO) singularities [12] of the type we study here represent 't Hooft operators that are dual to the Wilson operators of the electric-magnetic dual super-Yang-Mills theory. In fact it is the study of [12] of the monopole singularities that prompted the discovery of the electric-magnetic duality in [8]. On the other hand, it was demonstrated in [13] that one of the consequences of the electric-magnetic duality of the maximally supersymmetric Yang-Mills theory is the geometric Langlands correspondence. As a result, singular monopoles are significant in the study of the geometric Langlands duality; in particular, in [13] the moduli spaces of singular monopoles were identified with the spaces of Hecke transformations. Such a close relationship was also emphasized in [14].

There is a very close connection between monopoles and instantons. For example an instanton on a space with a periodic direction, called a caloron, can be thought of as a nonlinear superposition of monopoles and antimonopoles [15, 16]. In a different view [17, 18] a caloron with a gauge group G can be thought as a monopole with the loop group of G as its structure group. If one wishes to extend these statements to instantons on a multi-Taub-NUT space TN_k, then the corresponding generalization of the former is that an instanton on TN_k is a nonlinear superposition of singular monopoles and antimonopoles. And the analogue of the latter statement is that an instanton on TN_k with a gauge group G is a monopole with the loop group of G.

Singular monopoles describe Chalmers-Hanany-Witten brane configurations of the type IIB string theory [4, 6] and are very useful in exploring their various properties. In [19] they were instrumental in obtaining the twistor spaces of Gravitational Instantons, metrics on which were found in [20].

The twistor theory and the moduli spaces of singular monopoles were first studied in [21]. In particular the moduli space of one $U(2)$ monopole with k minimal singularities, which is the configuration we explicitly obtain here, is the k-centered multi-Taub-NUT space [21]. The centered moduli space of two $U(2)$ monopoles with k singularities is the D_k ALF space [19, 20].

1.2 Singular Monopole Constructions

By a BPS monopole[1] we understand a pair (A, Φ) of a hermitian connection A and a hermitian Higgs field Φ satisfying the Bogomolny equation

$$F_{ab} + \sum_{c=1}^{3} \epsilon_{abc}[D_c, \Phi] = 0, \tag{1}$$

where F is the curvature of A. A singular monopole with singularities at points $\vec{\nu}_j \in \mathbb{R}^3$, $j = 1, \ldots, k$ is a BPS monopole with A and Φ regular everywhere except at points $\vec{\nu}_j$, where locally it is required to have the prescribed behavior

$$\Phi = \frac{(1 + \not{n})}{4\left|\vec{t} - \vec{\nu}_j\right|} + O\left(\left|\vec{t} - \vec{\nu}_j\right|^0\right), \qquad A = \frac{1 + \not{n}}{2}\omega_j + O\left(\left|\vec{t} - \vec{\nu}_j\right|^0\right), \tag{2}$$

Here $\omega_j = -\frac{(\vec{T}\times\vec{t}_j)\cdot d\vec{t}}{2t_j(Tt_j - \vec{T}\cdot\vec{t}_j)}$, we have a unit vector $\vec{n} = (n_1, n_2, n_3)$ and are using the notation $\not{n} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ with $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices. This is exactly the Dirac monopole at each $\vec{\nu}_j$ embedded into the gauge group $U(2)$.

The technique of constructing a general regular monopole was discovered by Nahm [22, 23]. For a $U(2)$ monopole with k singularities this technique was used in [19, 24] to study the metric on the moduli space of such

[1]Normally one requires a monopole to have finite energy $\int_{\mathbb{R}^3} \mathrm{tr}(F \wedge *F + D\Phi \wedge *D\Phi)$. For singular monopoles, however, this condition is relaxed. Instead one excises small balls B_j centered around the points ν_j and requires the energy outside $\int_{\mathbb{R}^3 \setminus \cup_j B_j} \mathrm{tr}(F \wedge *F + D\Phi \wedge *D\Phi)$ to be finite, while the singularity inside each ball B_j is prescribed.

monopoles. The starting point of this construction of singular monopoles is a solution of the Nahm equations either on a real line or on a semi-infinite interval. While being very efficient in the study of the moduli spaces, it would be difficult to apply this formulation if one is to find the monopole configurations themselves. For the case of one or two singularities this construction is tractable and was employed in [25, 26] producing explicit solutions. Unfortunately, for a more general case, the difficulty is that the Nahm data, which is the starting point of the construction, contains the rank k solution of the Nahm equations on a semi-infinite interval. For $k > 2$ such solutions are difficult to construct and to work with.

In order to circumvent this difficulty, we shall employ the novel technique of bow diagrams introduced in [3] and developed in [1, 2]. Bow diagrams were introduced in order to construct all instantons, i.e. solutions of the self-duality equation, on the multi-Taub-NUT space with the metric $ds^2 = V\left(\vec{t}\right)d\vec{t}^{\,2} + \frac{1}{V\left(\vec{t}\right)}(d\theta + \omega)^2$. All such connections with self-dual curvature of given charges are given by a bow representation of the, so called, A_k bow, such as in Figure 2. A representation is determined by a collection of points on a bow and the ranks of bundles over the intervals between these points. The positions of these points correspond to the eigenvalues of the Polyakov loop at infinity of the multi-Taub-NUT space, while the ranks determine the charges.

What does the bow construction for instantons has to do with the singular monopole problem we are considering here? In [21] Kronheimer observed that any self-dual connection on a k-centered multi-Taub-NUT space that is invariant under the triholomorphic isometry of the multi-Taub-NUT space can is equivalent to a solution of the Bogomolny equation $F = - * D\Phi$ on $\mathbb{R}^3$, with k singularities corresponding to the Taub-NUT centers. Thus our problem of singular monopoles with k singularities is equivalent to the problem of θ-independent instantons on TN_k. In terms of the bow representation the condition that guarantees the invariance of the resulting solution under the triholomorphic isometry is that one of the ranks determining the bow representation is zero. We call such a representation a *Cheshire representation.* This is exactly what one needs to find the singular monopole solutions we are after. As a matter of fact this representation provides a construction for singular monopoles of any charge.

In the following section we present the A_k bow and explain its relation to the multi-Taub-NUT space and abelian instantons on it. Section 4 identifies the relevant Cheshire representations of the bow and its data. In Section 5 we apply the Nahm transform of [2] to obtain one generic $U(2)$ monopole solution with k minimal singularities positioned at $\vec{\nu}_j,\ j = 1, 2, \ldots, k$. We find its Higgs field and connection to have a relatively simple form:

$$\Phi\left(\vec{t}\right) = \left(\left[\lambda + \sum_{j=1}^{k} \frac{1}{4t_j}\right] \coth 2(\lambda+\alpha)z - \frac{1}{2z}\right) \frac{\not{z}}{z} + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{\not{T}_{j\,\perp}}{2t_j\big((T_j+t_j)^2 - z^2\big)} + \sum_{j=1}^{k} \frac{1}{4t_j}, \tag{3}$$

$$\begin{aligned} A\left(\vec{t}\right) = &\left(\frac{1}{2z} - \frac{1}{\sinh 2(\lambda+\alpha)z}\left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2\big((T_j+t_j)^2 - z^2\big)}\right]\right) \frac{i[\not{z}, d\not{t}]}{2z} \\ &+ \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{i[\not{t}_j, d\not{t}]_\perp}{4t_j\big((T_j+t_j)^2 - z^2\big)} \\ &- \left(1 + \frac{\not{z}}{z} \coth 2(\lambda+\alpha)z\right) \sum_{j=1}^{k} \frac{(\vec{T}_j \times \vec{t}_j)\cdot d\vec{t}}{2t_j((T_j+t_j)^2 - z^2)}. \end{aligned} \tag{4}$$

The eigenvalues of the Higgs field at infinity are $\pm\lambda$ and $-\vec{T}$ determines the position of the nonabelian monopole.

We would like to emphasize that the Cheshire bow construction we formulate here delivers all singular monopoles. We focus on one singular monopole as an illustrative example where every detail can be worked out explicitly.

2 Cheshire Bow Construction

The core idea of this work combines the observation of Kronheimer relating singular monopoles with instantons on multi-Taub-NUT space with the bow construction of such instantons. Let us begin by formulating the conventional Nahm transform for singular monopoles, highlighting the technical difficulties one faces in its practical application. Then we proceed by presenting Kronheimer's relation and formulating our generalization of the Nahm transform. This gives an alternative construction of singular monopoles.

2.1 The Nahm Transform

In order to construct $U(2)$ monopole of nonabelian charge m with k singularities using the conventional Nahm transform one begins by finding the

Nahm data $(T_1(s), T_2(s), T_3(s))$ consisting of three hermitian matrix valued functions of one variable s that satisfy the Nahm equations

$$\begin{cases} \frac{d}{ds}T_1 = i[T_2, T_3], \\ \frac{d}{ds}T_2 = i[T_3, T_1], \\ \frac{d}{ds}T_3 = i[T_1, T_2]. \end{cases} \tag{5}$$

If the asymptotic eigenvalues of the monopole Higgs field we are constructing are λ_1 and λ_2 with $\lambda_1 < \lambda_2$, then the Nahm data is of rank m on the interval $[\lambda_1, \lambda_2]$ and rank k on the semi-infinite intervals $(\lambda_2, +\infty)$. For concreteness, let us presume that $k > m$, then at λ_2 the matching condition is such that for $s > \lambda_2$

$$T_a(s) = \begin{pmatrix} \frac{\rho_a}{s-\lambda_2} + O(1) & O\Big((s-\lambda_2)^{\frac{k-m-1}{2}}\Big) \\ O\Big((s-\lambda_2)^{\frac{k-m-1}{2}}\Big) & T_j(\lambda_2) + O(s-\lambda_2) \end{pmatrix}, \tag{6}$$

where the residues ρ_1, ρ_2, and ρ_3 satisfy $[\rho_a, \rho_b] = \sum_c \epsilon_{abc} i\rho_c$, forming an irreducible representation of $su(2)$. The condition at λ_1 is that

$$T_a(s) = \frac{\rho'_a}{s-\lambda_1} \tag{7}$$

with ρ'_a forming an m-dimensional irreducible representation of the $su(2)$ generators. If the positions of the monopole singularities are $\vec{\nu}_j$, then the conditions one imposes on the eigenvalues of the Nahm data at $s = \infty$ are

$$\lim_{s\to+\infty} \text{EigVal } T_a(s) = \text{diag}(\nu_1^a, \nu_2^a, \ldots, \nu_k^a). \tag{8}$$

Given any such solution (T_1, T_2, T_3) Nahm constructs a family of Dirac (or Weyl) operators parameterized by $\vec{t} \in \mathbb{R}^3$: $\not{D} = -\frac{d}{ds} - \not{T} - \not{t}$, and a family of conjugate operators

$$\not{D}^\dagger = \frac{d}{ds} - \not{T} - \not{t}. \tag{9}$$

These operators act on L^2 fundamental spinors over the interval $(\lambda_1, +\infty)$. All such spinors form a trivial bundle over the $\mathbb{R}^3$ parameterized by $\vec{t}$, and the kernel of $\not{D}^\dagger$ is a subbundle of this trivial bundle. For each value of $\vec{t}$ the kernel is two dimensional. If $\psi_1(s, \vec{t})$ and $\psi_2(s, \vec{t})$ form an orthonormal basis of this kernel, then one forms the Higgs field and the connection with the components

$$\Phi_{\alpha\beta}(\vec{t}) = \int_{\lambda_1}^{+\infty} s\psi_\alpha^\dagger \psi_\beta \, ds, \qquad A^a_{\alpha\beta}(\vec{t}) = i\int_{\lambda_1}^{+\infty} \psi_\alpha^\dagger \frac{\partial}{\partial t^a}\psi_\beta \, ds, \tag{10}$$

which together constitute a singular monopole. This is the conventional Nahm transform as formulated in [24]. For every gauge equivalence class of solutions of the Nahm equations with the boundary conditions specified above it produces a $U(2)$ singular monopole with minimal singularities at $\vec{\nu}_j$ and nonabelian charge m.

This transform was successfully applied to find singular monopoles with one [26] and two singularities [25]. As we already pointed out, it is substantially more difficult, though not impossible, to use for a larger number of singularities. This is one of the reasons we proceed to introduce an alternative construction of singular monopoles.

2.2 Kronheimer's Correspondence

The multi-Taub-NUT is a four-dimensional space with the metric

$$ds^2 = V d\vec{t}^{\,2} + \frac{d\theta + \omega}{V}, \tag{11}$$

with θ of period 2π, $V = l + \sum_{j=1}^{k} \frac{1}{2|\vec{t}-\vec{\nu}_j|}$, and $d\omega = - *_3 dV$. A Yang-Mills connection $\hat{A}$ on this space can be written as

$$\hat{A} = A - \Phi \frac{d\theta + \omega}{V}. \tag{12}$$

As observed in [21], if this connection satisfies the self-duality equation on the multi-Taub-NUT space and if there is a gauge transformation that makes A and Φ θ-independent, then we can understand the fields A and Φ as a connection and Higgs field on $\mathbb{R}^3$ satisfying the Bogomolny equation

$$F + [D_A, \Phi] = 0. \tag{13}$$

If before the gauge transformation the field $\hat{A}$ was smooth and had a finite action, then the resulting configuration (A, Φ) is a singular monopole with singularities at the positions of the Taub-NUT centers $\vec{\nu}_j$. It is the action of this gauge transformation at the points $\vec{\nu}_j$ that determines the charges of the singularities [21].

With this in mind, instead of searching for singular monopoles we can try to solve an equivalent though at first sight more complicated looking problem, that of finding instantons on the multi-Taub-NUT space that are θ-independent.

2.3 Bows and Instantons on multi-Taub-NUT

A multi-Taub-NUT space with k Taub-NUT centers is a close cousin of the A_{k-1} Asymptotically Locally Euclidean (ALE) space. This space is given by the metric (11) with the parameter $l = 0$. The asymptotic form of its metric approaches the flat metric on $\mathbb{R}^4/\mathbb{Z}_k$. The instantons on the A_{k-1} ALE space, and on all ALE spaces, were constructed by Kronheimer and Nakajima [27]. This construction is formulated in terms of quivers. The relevant quiver is the affine A_{k-1} quiver, such as the one in Figure 1.

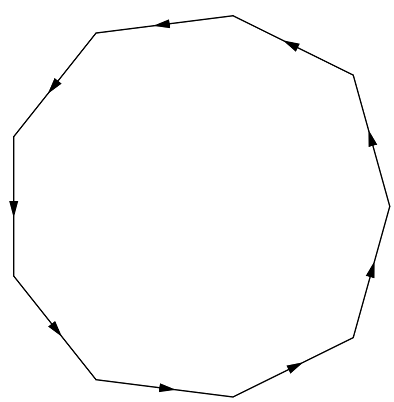

Figure 1: An example of the affine A_{k-1} quiver. This is an A_8 affine quiver giving the A_8 ALE space and instantons on it.

The recent construction of instantons on multi-Taub-NUT spaces [1, 2, 3] generalizes the notion of quivers to the notion of bows. If a quiver consists of points and oriented edges connecting them, a bow consists of intervals and oriented edges connecting them. We refer to [1] for the exact definitions. An A_k bow appears in Figure 2. It has various representations, each representation of a bow corresponding to a class of all instantons with given topological charges. A representation of a bow is a collection of points λ_j belonging to its intervals and a collection of vector bundles over the subintervals into which these intervals are divided by the λ-points. In particular some of these bundles can have rank zero, in which case their corresponding subintervals play no role and do not contribute to the final instanton connection. If this is indeed the case and a representation has at least one of its bundles of rank zero we call it a *Cheshire representation.*

Now, among all of the bow representations it remains to single out those that produce self-dual connections that are θ-independent. How does the θ

dependence arise?

On each of the bow intervals one considers the Nahm data consisting of the abelian $U(1)$ gauge field t_0 and three abelian Higgs fields t_1, t_2, t_3. The three Higgs fields give rise to the three of the multi-Taub-NUT coordinates assembled into a vector $\vec{t}$, while the coordinate θ is the logarithm of the Polyakov loop $\int t_0(s)ds$. Our construction is gauge invariant and therefore we can change the values of t_0, even gauging it away on some intervals completely. The only objects that remains invariant under the gauge transformations are the Polyakov loop and t_1, t_2, and t_3. Given some other bow representation we form a family of operators similar to the $\not{D}^\dagger$ operators of Eq. (9) that appeared in the conventional Nahm transform of Section 2.1. These operators depend only on the values of t_0 on the subintervals where the rank of the representation bundle is nonzero. Therefore, if all ranks of a representation are positive, then the resulting connection does depend on t_0 and therefore on θ. If one of the ranks is zero, however, then we can work in a gauge where t_0 is gauged away on all sub-intervals, except the one carrying the zero rank bundle. As a result the kernel of our operators will be independent of θ and so will be the resulting connection.

3 The Multi-Taub-NUT Space

A general definition of a bow, its representation, and its data can be found in [1]. Here we focus on the A_{k-1} bow, also called the TN_k bow, given in Figure 2. It consists of k intervals $I_j, j = 1, \ldots, k$ denoted by the wavy lines

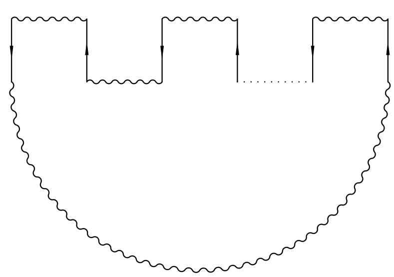

Figure 2: A_{k-1} Bow. It has multi-Taub-NUT space with k centers as the moduli space of its small representation. Any other representation of this bow delivers self-dual connections on this multi-Taub-NUT space.

and k oriented edges $e_j, j = 1, \ldots, k$ denoted by the arrows connecting the ends of the lines. We parameterize the intervals by the variable s, and for concreteness denote the left end of I_j by p_j^L and the right end by p_j^R so that $I_j = [p_j^L, p_j^R]$. In what follows we shal understand the variable s to be parameterizing a circle of circumference l. This circle is divided into intervals I_j, and even though in this picture any two neighboring intervals I_{j-1} and I_j share an endpoint, we still treat p_{j-1}^R and p_j^L as distinct points. One of the simplest representations of this bow has rank one bundles on each interval and no marked λ-points. We call this the Small Representation and denote the associated data by small letters t and b. Let us begin by discussing this representation in detail and by finding its moduli space.

Each interval I_j has an associated line bundle $e_j \to I_j$ with connection $\frac{d}{ds} - it_0(s)$ and three Higgs fields $t_1(s), t_2(s), t_3(s)$. Each edge, say the j^{th} edge, connects the intervals $j-1$ and j as in Figure 3, with the tail $t(j)$ being the right end of the $(j-1)^{\text{st}}$ interval, $p_j^L = h(j)$, and the head $h(j)$ being the left end of the j^{th} interval, $p_{j-1}^R = t(j)$. If $e_{t(j)}$ denotes the fiber of e_{j-1} at the right end of the interval I_{j-1} and $e_{h(j)}$ denotes the fiber of the bundle e_j at the left end of the interval I_j, then we consider the linear maps

$$b_j^{LR} : e_{t(j)} \to e_{h(j)} \qquad \text{and} \qquad b_j^{RL} : e_{h(j)} \to e_{t(j)}. \tag{14}$$

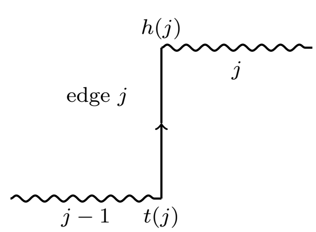

Figure 3: An edge

These are assembled into b_j^+ and b_j^- as

$$b_j^+ = \begin{pmatrix} \overline{b_j^{RL}} \\ -b_j^{LR} \end{pmatrix} \qquad \text{and} \qquad b_j^- = \begin{pmatrix} \overline{b_j^{LR}} \\ b_j^{RL} \end{pmatrix}. \tag{15}$$

Figure 4 assembles all this data into a decorated bow. The collection of the connection, the Higgs fields, and the linear maps is a point in the affine space of the small representation data.

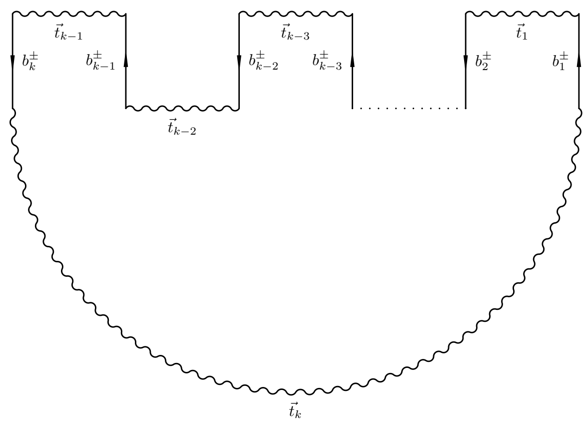

Figure 4: Small Bow Representation: This bow has k intervals. Assigning a line bundle to each defines a representation with k-centered Taub-NUT as its moduli space.

3.1 Moment Map Conditions

According to [1] the moduli space of the small bow representation is obtained by imposing the moment map conditions

$$\not\mu(t, b) = \sum_{j=1}^{k} (\delta(s - t(j)) - \delta(s - h(j)))\not\nu_j, \tag{16}$$

and dividing by the action of the gauge group. The moment map arises from considering the space of representation data, which is hyperkähler, and the natural action of the gauge group on it. The space being hyperkähler it has three symplectic structures and these are respected by the gauge transformations. It is the three Hamiltonians μ_1, μ_2, and μ_3 generating this action that form the moment map values

$$\not\mu(t, b) = -\frac{d}{ds}\not t + \sum_{j=1}^{k} \left(\delta(s - t(j)) b_j^- \left(b_j^-\right)^\dagger + \delta(s - h(j)) b_j^+ \left(b_j^+\right)^\dagger \right). \tag{17}$$

Within each interval this condition implies that the data satisfies the Nahm equations, which, since $t_\mu(s)$ is abelian read $\frac{d}{ds} t_i = 0$ for $i = 1, 2, 3$. Thus within each interval $\vec{t} = (t_1, t_2, t_3)^{\rm tr}$ is constant. At $t(j)$ Eqs. (16) and (17) read

$$b_j^+ (b_j^+)^\dagger = |\vec{t}(t(j)) - \vec{\nu}_j| + (\not t(t(j)) - \not\nu_j), \tag{18}$$

and at $h(j)$

$$b_j^- \left(b_j^-\right)^\dagger = |\vec{t}(h(j)) - \vec{\nu}_j| - \Big(\rotatebox{180}{t}(h(j)) - \rotatebox{180}{ν}_j\Big). \tag{19}$$

In particular these equations imply that $\rotatebox{180}{$t$}(t(j)) = \rotatebox{180}{t}(h(j))$ and thus $\vec{t}(s) = \vec{t}$ is not only constant within each interval, but, has the same value across all intervals for all values of s. Once this is established let us simplify our notation slightly by introducing

$$\vec{t}_j = \vec{t} - \vec{\nu}_j \qquad \text{and, accordingly,} \qquad \rotatebox{180}{t}_j = \rotatebox{180}{t} - \rotatebox{180}{ν}_j. \tag{20}$$

The remaining gauge freedom can be used to completely gauge away the connection component t_0 within each interval absorbing it into the phase factors of $b_j^\pm$. At this point the calculation reduces to that of [29].

As a result we obtain the moduli space of this small representation at level $\rotatebox{180}{$\nu$}$ that is four-real-dimensional. This space can be parameterized by $\vec{t}$ and the invariant combination of t_0 and complex phases of b_j, leading to the Gibbons-Hawking form of the metric

$$ds^2 = V d\vec{t}^{\,2} + \frac{1}{V}(d\theta + \omega)^2, \tag{21}$$

with $V = l + \sum_j \frac{1}{2|\vec{t}_j|}$, $\theta \sim \theta + 2\pi$, and the one-form ω satisfying $*dV = -d\omega$. Here l is the sum of the lengths l_j of the intervals I_j.

One can now see the significance of the values $\vec{\nu}_j$ of the moment map - these are the positions of the Taub-NUT centers. The size of the Taub-NUT circle at infinity on the other hand is determined by l, which is the total sum of lengths of all intervals in the bow.

Since this four-dimensional space is obtained as a moduli space of a bow representation it comes equipped with a family of self-dual connections parameterized by the union of all intervals of the bow. In our case all of these connections are abelian instantons. These abelian instantons are instrumental in our construction and we derive them now.

3.2 Natural Line Bundles and Self-dual Connections

The exact abelian instanton connection will depend on how we parameterize the intervals in the bow. Let us call the point at which $s = 0$ the *distinguished point.* We shall be interested in the connection associated to some point $s = s_0$. Let us call this point the *marked point.*

Let us consider a general position of the distinguished point on the kth interval, dividing it into left and right intervals on lengths u and $l_0 - u$. The marked point s_0 is in a general position belonging to the interval number $\mathrm{int}(s_0) : s_0 \in I_{\mathrm{int}(s_0)}$. The distinguished point and the marked point divide the TN_k bow into two parts. Let us call the part forming the path from the distinguished point to the marked point the *left* path, and the part forming the path from the marked point to the distinguished point the *right* path. The total length of the intervals belonging to the left path is s_0 and the total length of the intervals belonging to the right path is $l - s_0$, with $l = l_1 + \ldots + l_k$. We shall use the corresponding subscripts l and r to denote the quantities relating to these two parts. For example we denote the data of the left path by Dat_l and the data of the right path by Dat_r.

The data of the bow can be viewed as the direct product of the data of the left and right paths with zero-level hyperkähler reduction by the action of the gauge group G_{s_0} at the marked point. $\mathrm{Dat} = (\mathrm{Dat}_l \times \mathrm{Dat}_r)///G_{s_0}$. Moreover, if $\mathcal{G}_{s_0}$ is the group of gauge transformations that act trivially at the marked and at the distinguished point then it can be viewed as a direct product of similar groups $\mathcal{G}_l$ and $\mathcal{G}_r$ acting on the left and right path data respectively with trivial action at the marked and distinguished points.

The moduli space $\mathcal{M}$ of the small bow can thus be represented as a hyperkähler quotient in a number of ways:

$$\mathcal{M} = \mathrm{Dat}///\mathcal{G} = \mathrm{Dat}///(\mathcal{G}_{\int,} \times G_{s_0}) = \Big((\mathrm{Dat}_l///\mathcal{G}_l) \times (\mathrm{Dat}_r///\mathcal{G}_r)\Big)///G_{s_0}. \tag{22}$$

Here $///$ denotes the hyperkähler reduction [28]. Let us denote the moduli space of respectively the left and the right paths by $\mathcal{M}_l$ and $\mathcal{M}_r$ so that $\mathcal{M}_l = \mathrm{Dat}_l///\mathcal{G}_l$ and $\mathcal{M}_r = \mathrm{Dat}_r///\mathcal{G}_r$. Performing hyperkähler reduction within each interval reduces the Nahm data on each interval to $\mathbb{R}^3 \times S^1$. The remaining quotient by the gauge groups acting at the end of the intervals amounts to the quotient considered in [29] which results in a multi-Taub-NUT space. Thus $\mathcal{M}_l = \mathrm{TN}_{s_0}$ and $\mathcal{M}_r = \mathrm{TN}_{l-s_0}$ with metrics

$$ds_l^2 = V_l d\vec{t}_j^{\,2} + \frac{1}{V_l}(d\beta + \omega_l)^2, \qquad ds_r^2 = V_r d\vec{t'}_j^{\,2} + \frac{1}{V_r}(d\alpha + \omega_r)^2, \tag{23}$$

here α and β have period 2π and

$$V_l = s_0 + \sum_{j=1}^{\mathrm{int}(s_0)} \frac{1}{2t_j}, \qquad V_r = l - s_0 + \sum_{j=\mathrm{int}(s_0)+1}^{k} \frac{1}{2t_k}, \tag{24}$$

$$*_3\, d\omega_l = -dV_l, \qquad *_3\, d\omega_r = -dV_r. \tag{25}$$

The action of the $G_{s_0} = U(1)$ is by $(\alpha, \beta) \to (\alpha - \phi, \beta + \phi)$, the invariant of this action is $\theta = \alpha + \beta$ and the moment map is $\vec{t'}_{\mathrm{int}(s_0)} - \vec{t}_{\mathrm{int}(s_0)}$. Putting the moment map to zero we obtain the metric on the five-real-dimensional zero level set of G_{s_0}

$$ds^2 = V d\vec{t}^2 + \frac{1}{V}(d\theta + \omega)^2 + \frac{V}{V_l V_r}\left(d\beta + \omega_l - \frac{V_l}{V}(d\theta + \omega)\right)^2, \qquad (26)$$

where $V = V_l + V_r$ is the harmonic function of the k-centered Taub-NUT, $\omega = \omega_l + \omega_r$, $\vec{t} = \vec{t}_{\mathrm{int}(s_0)} = \vec{t'}_{\mathrm{int}(s_0)}$. Viewing this as a metric on the principal $U(1)_{s_0}$ bundle over $\mathcal{M}$ we have the natural connection a_{s_0} on this bundle

$$a_{s_0} = \omega_l - V_l \frac{(d\theta + \omega)}{V}. \qquad (27)$$

It is natural to associate the one-form connection $a^{(j)} = \omega_j - \frac{1}{2t_j}\frac{d\theta+\omega}{V}$, with $d\omega_j = -*_3 d\frac{1}{2t_j}$, to each of the Taub-NUT centers, then the above connection (27) in the chosen trivialization has the form

$$a_s = -s\frac{d\theta + \omega}{V} + \sum_{j=1}^{\mathrm{int}(s)} a^{(j)}, \qquad (28)$$

for $s = s_0$. This abelian connection is self-dual. Thus each point of a bow has an associated abelian self-dual connection given by Eq. (28).

4 Cheshire Representation

In order to obtain one singular monopole solution we begin with the large representation of Figure 5. For the sake of symmetry let us choose the distinguished point with $s = 0$ to be in the middle of the kth interval I_k. This representation has two λ-points at $s = \pm\lambda$.[2] All bundles have rank one, except for the bundle over a single subinterval which has rank zero. This subinterval has the λ-points as its ends. Since the rank zero bundle has no data associated to it, it is not drawn in Figure 5. This is a Cheshire representation, which ensures that the resulting instanton on the multi-Taub-NUT can be written in the form

$$\hat{A} = A - \Phi\frac{d\theta + \omega}{V}, \qquad (29)$$

with A and Φ independent of the variable θ. The fact that $\hat{A}$ has self-dual curvature in orientation $(dt_1, dt_2, dt_3, d\theta)$ is equivalent [21] to A and Φ

[2] This choice of λ-points makes it simpler to extract the $U(2)$ singular monopole expression from our answer. A priori any two points can be chosen as λ-points.

satisfying the Bogomolny equation $*_3 F = -[D_A, \Phi]$. One can see from the form of Eq. (29) that in such a reduction of a smooth self-dual connection to the monopole the resulting monopole can have $\frac{1}{t_j}$ type singularities at the positions of the Taub-NUT centers.

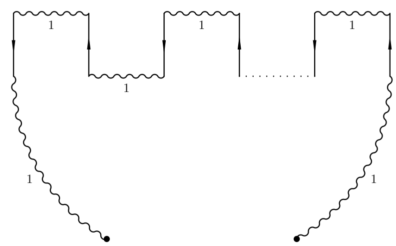

Figure 5: Large bow representation: This bow has k edges and $k+1$ intervals. Assigning a line bundle to each of the intervals. A solution of this bow determines a monopole on a k-centered Taub-NUT space.

For a charge m singular monopole the large bow is the same as in Figure 5 except that the ranks of all the bundles on the shown intervals are equal to m. In order to construct a monopole with the $U(n)$ gauge group, one considers a Cheshire bow with n λ-points with various bundle ranks equal to the nonabelian monopole charges.

Figure 5 illustrates the reason why our method has an advantage over the conventional Nahm transform. In the conventional Nahm data for a monopole with k singularities one has to work with the rank k Nahm data, which makes it into a highly nonlinear problem. In the Cheshire bow formulation, even though one still constructs a monopole with k singularities, only abelian rank 1 data appears on the intervals, which makes the whole construction tractable.

The data we associate to the large representation is denoted by capital letters T and B, as in Figure 6. The moment map conditions we impose for this data are

$$\not{\mu}(B,T) = -\sum_j \big(\delta(s-t(j)) - \delta(s-h(j))\big)\not{\nu}_j, \tag{30}$$

which are negative of those for the small bow of Eq. (16). Since the gauge group action on the large representation data (T, B) has the same form as on the small representation data the moment map is given by the same expression, which for an arbitrary rank bow data takes the form

$$\not{\mu}(T,B) = -\frac{d}{ds}\not{T} + \mathrm{vec}\,\not{T}\not{T} + \sum_{j=1}^{k}\left(\delta(s-t(j))B_j^-\left(B_j^-\right)^\dagger + \delta(s-h(j))B_j^+\left(B_j^+\right)^\dagger\right). \quad (31)$$

Here $\mathrm{vec}\,\not{T}\not{T} = i\epsilon_{abc}[T_a,T_b]\sigma_c$ and for the rank one large representation it vanishes. The gauge equivalence classes of solutions to the moment map equation (30) are in one-to-one correspondence with the $SU(2)$ singular monopoles with k minimal singularities. The positions of the singularities are fixed to be $\vec{\nu}_j$, while $-\vec{T}$ parameterizes the position of the nonabelian monopole. We have the abelian Nahm data $\vec{T}$ associated to each interval.

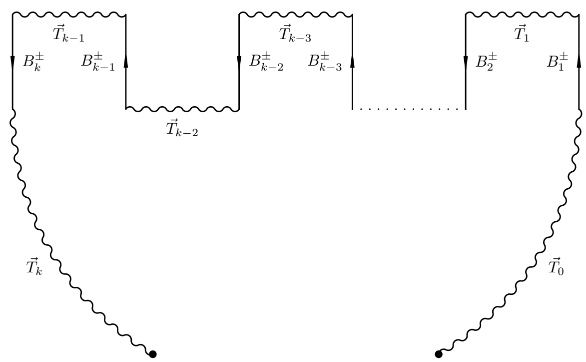

Figure 6: The large bow representation with its data. Black dots are the λ-points at $s = \pm\lambda$.

The Nahm equations imply that $\vec{T}$ is constant on each interval. To each edge we associate linear maps

$$B_j^{LR} : E_{t(j)} \to E_{h(j)} \qquad B_j^{RL} : E_{h(j)} \to E_{t(j)} \quad (32)$$

which we assemble into

$$B_j^+ = \begin{pmatrix} \overline{B_j^{RL}} \\ B_j^{LR} \end{pmatrix} \qquad B_j^- = \begin{pmatrix} \overline{B_j^{LR}} \\ -B_j^{RL} \end{pmatrix}. \quad (33)$$

The moment map condition of Eq. (30) reads

$$B_j^\pm B_j^{\pm\dagger} = |\vec{T} + \vec{\nu}_j| \pm (\not{T} + \not{\nu}_j), \quad (34)$$

which implies that $\vec{T}$ is not only constant within each interval but also has the same value on all intervals. To simplify our notation we introduce

$$\vec{T}_j = \vec{T} + \vec{\nu}_j, \qquad \text{so that} \qquad \not{T}_j = \not{T} + \not{\nu}_j. \tag{35}$$

5 The Monopole

Given any solution $(\vec{T}, B_j)$ of the moment map conditions (34), we construct a singular monopole solution with monopole position parameter $-\vec{T}$. In order to find the value of A and Φ at a point $\vec{t}$ let us introduce the relative position $\vec{z} = \vec{t} + \vec{T}$ with respect to the monopole. We also introduced $\vec{t}_j = \vec{t} - \vec{\nu}_j$ which are the relative positions with respect to the singularities. Now consider the twisted Weyl or Dirac operator

$$\mathcal{D}^\dagger = \left(\frac{d}{ds} - \not{z}\right) \oplus \sum_j \delta(s - t(j))(b_j^-, B_j^-) \oplus \sum_j \delta(s - h(j))(B_j^+, b_j^+). \tag{36}$$

This operator acts on $\Psi = (\psi(s), v_j)$ with $\psi(s)$ a section of $E_j \otimes e_j \otimes S$, where $E_j \to I_j$ is the line bundle of the large representation over the interval I_j, e_j is the line bundle of the small representation over the interval I_j, and S is the two-dimensional chiral spin bundle, while $v_j = \begin{pmatrix} v_j^+ \\ v_j^- \end{pmatrix}$ with $v_j^+ \in e_{h(j)} \otimes E_{t(j)}$, $v_j^- \in E_{h(j)} \otimes e_{t(j)}$. Thus the equation $\mathcal{D}^\dagger \mathbf{\Psi} = 0$ amounts to

$$\left(\frac{d}{ds} - \not{z}\right) \psi(s) = 0 \qquad \text{within each interval,} \tag{37}$$

$$\psi(t(j)) = (b_j^-, B_j^-) v_j, \qquad \psi(h(j)) = -(B_j^+, b_j^+) v_j. \tag{38}$$

If the columns of $\mathbf{\Psi}$ form an orthonormal basis of solutions of $\mathcal{D}^\dagger \mathbf{\Psi} = 0$, then the resulting self-dual connection on the multi-Taub-NUT is

$$\hat{A} = \left(\mathbf{\Psi}, \left(i dt_a \frac{d}{dt_a} + a_s\right) \mathbf{\Psi}\right). \tag{39}$$

Here we use the most natural norm

$$(\mathbf{\Psi}, \mathbf{\Psi}) \equiv \int \psi^\dagger(s) \psi(s) ds + \sum_{j=1}^{k} v_j^\dagger v_j. \tag{40}$$

Together with Kronheimer's reduction (29) and the expression for the abelian instanton a_s of Eq. (28) this leads to the monopole expression

$$\Phi = \left(\mathbf{\Psi}, \Big(s + \sum_{j=1}^{\mathrm{int}(s)} \frac{1}{2t_j}\Big)\mathbf{\Psi}\right), \tag{41}$$

$$A = \left(\mathbf{\Psi}, \Big(idt_a \frac{d}{dt_a} + \sum_{j=1}^{\mathrm{int}(s)} \omega_j\Big)\mathbf{\Psi}\right). \tag{42}$$

Before we proceed solving for $\mathbf{\Psi}$ we introduce $\mathcal{P}_j = \sqrt{2(t_jT_j - \vec{t}_j\cdot\vec{T}_j)} = \sqrt{(t_j+T_j)^2 - z^2}$ and observe the following useful relations

$$\mathcal{P}_j = B_j^{\pm\dagger} b_j^{\mp} = b_j^{\pm\dagger} B_j^{\mp} = B_j^{+} b_j^{-\dagger} + b_j^{+} B_j^{-\dagger} = B_j^{-} b_j^{+\dagger} + b_j^{-} B_j^{+\dagger}, \tag{43}$$

$$(b_j^-, B_j^-)(b_j^-, B_j^-)^\dagger = T_j + t_j - \not{z}, \qquad (B_j^+, b_j^+)(B_j^+, b_j^+)^\dagger = T_j + t_j + \not{z}, \tag{44}$$

and

$$\mathcal{P}_j = (b_j^-, B_j^-)(B_j^+, b_j^+)^\dagger = (B_j^+, b_j^+)^\dagger(b_j^-, B_j^-) \tag{45}$$

$$\mathcal{P}_j = (B_j^+, b_j^+)(b_j^-, B_j^-)^\dagger = (b_j^-, B_j^-)^\dagger(B_j^+, b_j^+). \tag{46}$$

In a way Eqs. (43) hold up to a phase factor $e^{i\phi}$. We put this factor to identity which amounts to choosing a gauge in which our solution will be written.

Due to Eq. (37) within each interval $\psi(s) = e^{s\not{z}}\Pi_j$ for some s-independent Π_j, while the matching conditions (38) give

$$v_j = \frac{(B_j^+, b_j^+)^\dagger}{\mathcal{P}_j}\psi(t(j)) \qquad \text{and} \qquad \psi(h(j)) = -\frac{T_j + t_j + \not{z}}{\mathcal{P}_j}\psi(t(j)). \tag{47}$$

Therefore the factors Π_j on consecutive intervals are related by

$$\Pi_j = -\frac{T_j + t_j + \not{z}}{\mathcal{P}_j}\Pi_{j-1}, \tag{48}$$

so that the choice of Π_0 completely determines the solution $\mathbf{\Psi}$. As we shall need an orthonormal basis of solutions we shall fix Π_0 accordingly after we compute the normalization.

In our computation the factor $T_j + t_j + \not{z}$ plays a special role, with this in mind we observe that

$$T_j + t_j \pm \not{z} = \mathcal{P}_j e^{\pm 2\alpha_j \not{z}}, \tag{49}$$

where

$$\alpha_j = \frac{1}{4z} \ln \frac{T_j + t_j + z}{T_j + t_j - z}. \tag{50}$$

Also, let us introduce $\alpha = \sum_{j=1}^{k} \alpha_j$.

It is most convenient to choose the value of Π_0 so that the normalisation factor

$$N^2 = (\mathbf{\Psi}, \mathbf{\Psi}) = \sum_{j=0}^{k} \int_{p_j^L}^{p_j^R} ds\, \Pi_j^\dagger e^{2s\bar{x}} \Pi_j + \sum_{j=1}^{k} v_j^\dagger v_j, \tag{51}$$

is indeed just a scalar factor (times the identity matrix $\mathbb{I}_{2\times 2}$). This dictates our choice below.

The normalised solution in this case can be written as $\mathbf{\Psi}_N = \frac{1}{N}\mathbf{\Psi}$. Differentiating $(\mathbf{\Psi}_N, \mathbf{\Psi}_N) = 1$, one verifies that

$$\begin{aligned} \left(\mathbf{\Psi}_N, \frac{d}{dt_a}\mathbf{\Psi}_N\right) &= \frac{1}{2}\left(\left(\mathbf{\Psi}_N, \frac{d}{dt_a}\mathbf{\Psi}_N\right) - \left(\frac{d}{dt_a}\mathbf{\Psi}_N, \mathbf{\Psi}_N\right)\right) \\ &= \frac{1}{N^2}\left(\left(\mathbf{\Psi}, \frac{d}{dt_a}\mathbf{\Psi}\right) - \left(\frac{d}{dt_a}\mathbf{\Psi}, \mathbf{\Psi}\right)\right). \end{aligned} \tag{52}$$

This allows us to work with the solution $\mathbf{\Psi}$ satisfying $(\mathbf{\Psi}, \mathbf{\Psi}) = \frac{\sinh 2(\lambda+\alpha)z}{z}$ when we compute the Higgs field and the connection below.

From our expressions for the monopole fields in Eq. (41) the Higgs field satisfies

$$\begin{aligned} N^2\Phi =& \sum_{j=0}^{k} \int_{p_j^L}^{p_j^R} ds\, \Pi_j^\dagger s e^{2s\bar{x}} \Pi_j + \sum_{j=1}^{k} v_j^\dagger \begin{pmatrix} p_j^L & 0 \\ 0 & p_{j-1}^R \end{pmatrix} v_j \\ &+ \sum_{j=1}^{k} \frac{1}{2t_j} \left(\sum_{i=j+1}^{k} v_i^\dagger v_i + (v_j^+)^\dagger v_j^+ \right) + \sum_{j=1}^{k} \sum_{i=1}^{j} \frac{1}{2t_i} \int_{p_j^L}^{p_j^R} ds\, \Pi_j^\dagger e^{2s\bar{x}} \Pi_j \end{aligned} \tag{53}$$

and the connection satisfies

$$\begin{aligned} N^2 A =& \frac{i}{2} \sum_{j=0}^{k} \int_{p_j^L}^{p_j^R} ds \left(\psi_j^\dagger(s) d\psi_j(s) - d\psi_j^\dagger(s) \psi_j(s) \right) + \frac{i}{2} \sum_{j=1}^{k} \left(v_j^\dagger dv_j - dv_j^\dagger v_j \right) \\ &+ \sum_{j=1}^{k} \omega_j \left(\sum_{i=j+1}^{k} v_i^\dagger v_i + (v_j^+)^\dagger v_j^+ \right) + \sum_{j=1}^{k} \sum_{i=1}^{j} \omega_i \int_{p_j^L}^{p_j^R} ds\, \Pi_j^\dagger e^{2s\bar{x}} \Pi_j \end{aligned} \tag{54}$$

where $\psi_j(s) = e^{s\not{z}}\Pi_j$, with $\Pi_j = (-1)^j e^{2(\alpha_1+\cdots+\alpha_j)\not{z}}\Pi_0$ and

$$v_j = \begin{pmatrix} v_j^+ \\ v_j^- \end{pmatrix} = (-1)^j \frac{(b_j^-, B_j^-)^\dagger}{\mathcal{P}_j} e^{p_{j-1}^R \not{z}} e^{2(\alpha_1+\cdots+\alpha_j)\not{z}}\Pi_0. \tag{55}$$

Except for the total length of all the intervals in the bow, the sizes of the individual intervals did not play any role in our discussion so far. Nor will they. From this point on we put all of the intervals in the bow to zero size with the exception of the one interval that contains the two λ-points. This interval is of length l. Every other interval that is now shrunk to a point is at $s = 0$. The resulting Cheshire bow representation is in Figure 7. This puts $p_0^L = -\lambda, p_k^R = \lambda$, and all other $p_j^L = p_j^R = 0$. This substantially simplifies our computation.

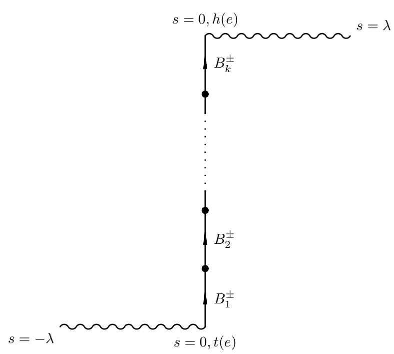

Figure 7: The Cheshire representation of Figure 6 with all but one intervals shrunk to zero size. It is important to keep in mind the relation of this diagram with the TN_k bow.

5.1 Normalization

The normalisation factor (51) is now given by

$$N^2 = (\mathbf{\Psi}, \mathbf{\Psi}) = \int_{-\lambda}^{0} ds\, \Pi_0^\dagger e^{2s\not{z}}\Pi_0 + \int_0^\lambda ds\, \Pi_k^\dagger e^{2s\not{z}}\Pi_k + \sum_{j=1}^{k} v_j^\dagger v_j \tag{56}$$

The integrals over s are straightforward and one can show that the contribution from the $s = 0$ endpoints cancels with the sum of v_j terms; this latter calculation in fact implies that

$$\sum_{i=j+1}^{k} v_i^\dagger v_i = \frac{1}{2z^2}\left(\Pi_k^\dagger \not{z}\Pi_k - \Pi_j^\dagger \not{z}\Pi_j\right) \tag{57}$$

Using the fact that $\Pi_k = (-1)^k e^{2\alpha\not{z}}\Pi_0$ one ends up with an expression for N^2 which is proportional to $\Pi_0^\dagger e^{2\alpha\not{z}}\Pi_0$. This suggests a natural choice of orthogonal basis of solutions given by $\Pi_0 = e^{-\alpha\not{z}}$. In this basis the normalization factor is indeed a scalar

$$N = \sqrt{\frac{\sinh 2(\lambda+\alpha)z}{z}}, \tag{58}$$

and all basis elements have the same norm and are orthogonal to each other.

5.2 Higgs Field

With our labelling of the interval endpoints, the Higgs field Φ of Eq. (53) becomes

$$\begin{aligned} N^2\Phi = & \int_{-\lambda}^{0} ds\, \Pi_0^\dagger\, s e^{2s\not{z}}\Pi_0 + \int_0^\lambda ds\, \Pi_k^\dagger\, s e^{2s\not{z}}\Pi_k \\ & + \left(\sum_{j=1}^{k}\frac{1}{2t_j}\right)\int_0^\lambda ds\, \Pi_k^\dagger\, e^{2s\not{z}}\Pi_k \\ & + \sum_{j=1}^{k}\frac{1}{2t_j}\left((v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i\right), \end{aligned} \tag{59}$$

Our choice for Π_0 from the previous section simplifies the computation of the integrals greatly, leaving us with simple integrals involving the exponentials $e^{2(s\pm\alpha)\not{z}}$. The result of the integration is

$$\begin{aligned} \Phi = \sum_{j=1}^{k}\frac{1}{4t_j} + & \left(\left[\lambda + \sum_{j=1}^{k}\frac{1}{4t_j}\right]\coth 2(\lambda+\alpha)z - \frac{1}{2z}\right)\frac{\not{z}}{z} \\ & + \frac{z}{\sinh 2(\lambda+\alpha)z}\left[\frac{\not{z}}{2z^3}\sinh 2\alpha z - \frac{\not{z}}{z^2}e^{2\alpha\not{z}}\sum_{j=1}^{k}\frac{1}{4t_j}\right. \\ & \qquad \left. + \sum_{j=1}^{k}\frac{1}{2t_j}\left((v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i\right)\right]. \end{aligned} \tag{60}$$

The $\sum_i v_i^\dagger v_i$ term can be replaced with a much simpler expression using Eq. (57). After substituting in $v_j^+ = (-1)^j (b_j^-)^\dagger e^{-\alpha \slashed{z}} e^{2(\alpha_1+\cdots+\alpha_j)\slashed{z}}$ and bringing the remaining pieces together one finds, after some manipulation of sums of exponentials of slashed terms that the final expression is

$$\Phi = \sum_{j=1}^{k} \frac{1}{4t_j} + \left(\left(\lambda + \sum_{j=1}^{k} \frac{1}{4t_j} \right) \coth 2(\lambda+\alpha)z - \frac{1}{2z} \right) \frac{\slashed{z}}{z} \\ + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{1}{2t_j \mathcal{P}_j^2} \slashed{T}_{j\perp}. \quad (61)$$

The second term in the first line of this expression is reminiscent of the Higgs field of the 't Hooft-Polyakov monopole:

$$\Phi(\vec{z}) = \left(\lambda \coth 2\lambda z - \frac{1}{2z} \right) \frac{\slashed{z}}{z}. \quad (62)$$

One can see for example that the size of the nonabelian monopole is modulated by the presence of the singularities with $\lambda + \sum_{j=1}^{k} \alpha_j$ playing the role of the size controlling λ in the 't Hooft-Polyakov case.

5.3 Vector Potential

From (54) the vector potential is given by

$$N^2 A = \frac{i}{2} \int_{-\lambda}^{0} ds \left(\psi_0^\dagger(s) d\psi_0(s) - h.c. \right) + \frac{i}{2} \int_{0}^{\lambda} ds \left(\psi_k^\dagger(s) d\psi_k(s) - h.c. \right) \\ + \sum_{j=1}^{k} \omega_j \int_0^\lambda ds\, \Pi_k^\dagger e^{2s\slashed{z}} \Pi_k + \sum_{j=1}^{k} \omega_j \left((v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i + \right) \\ + \frac{i}{2} \sum_{j=1}^{k} (v_j^\dagger d v_j - h.c.). \quad (63)$$

The integrals in the first line are straightforward to compute after writing $\psi_0(s) = e^{(s-\alpha)\slashed{z}}$ and $\psi_k(s) = (-1)^k e^{(s+\alpha)\slashed{z}}$, while the integral and the summation in the second line are the same as those that occur in the calculation of Φ. One also needs

$$
\begin{aligned}
v_j^\dagger dv_j - dv_j^\dagger v_j = {}& \frac{1}{\mathcal{P}_j^2} e^{-(\alpha-2[\alpha_1+\cdots+\alpha_j])\not{z}} \left(b_j^- db_j^{-\dagger} - db_j^- b_j^{-\dagger} \right) e^{-(\alpha-2[\alpha_1+\cdots+\alpha_j])\not{z}} \\
&+ \frac{1}{\mathcal{P}_j} \frac{[\not{z}, d\not{z}]}{z^2} \sinh(\alpha - 2[\alpha_1 + \cdots + \alpha_{j-1}])z \, \sinh(\alpha - 2[\alpha_1 + \cdots + \alpha_j])z.
\end{aligned}
\tag{64}
$$

and

$$
b_j^- db_j^{-\dagger} - db_j^- b_j^{-\dagger} = 2i\omega_j (t_j - \not{t}_j) + \frac{1}{2t_j} [\not{t}_j, d\not{t}_j] \tag{65}
$$

Using these and the explicit expression $\omega_j = -\frac{1}{\mathcal{P}_j^2 t_j} \vec{z} \cdot (\vec{t}_j \times d\vec{t}_j)$ it is straightforward to simplify the remaining terms obtaining

$$
\begin{aligned}
A = {}& \frac{i}{2z} [\not{z}, d\not{z}] \left(-\frac{1}{\sinh 2(\lambda+\alpha)z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2\mathcal{P}_j^2} \right] + \frac{1}{2z} \right) \\
&+ \sum_{j=1}^{k} \frac{\omega_j}{2} + \sum_{j=1}^{k} \frac{\omega_j}{2} \frac{\not{z}}{z} \coth 2(\lambda+\alpha)z + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{i[\not{t}_j, d\not{t}_j]_\perp}{4\mathcal{P}_j^2 t_j}
\end{aligned}
\tag{66}
$$

Our results, Eqs. (61) and (66), deliver a one monopole with k minimal Dirac singularities at $\vec{\nu}_j$ points. The monopole position is parameterized by $-\vec{T}$, and we used $\vec{T}_j = \vec{T} + \vec{\nu}_j, \vec{t}_j = \vec{t} - \vec{\nu}_j$, and $\mathcal{P}_j^2 = (T_j + t_j)^2 - z^2$.

6 Conclusions

We formulate an alternative Nahm transform for monopoles. This new version of the Nahm transform that we apply here amounts to finding a solution (T, B) of the moment maps of a large Cheshire bow representation and forming a family of Dirac operators $D^\dagger$ determined by the solution (T, B). An orthonormal basis of solutions $\mathbf{\Psi}$ of the Dirac equation $D^\dagger \mathbf{\Psi} = 0$ gives a singular monopole with

$$
\Phi = \left(\mathbf{\Psi}, \left(s + \sum_{j \leq \mathrm{int}(s)} \frac{1}{2t_j} \right) \mathbf{\Psi} \right), \qquad A = i \left(\mathbf{\Psi}, \nabla_a \mathbf{\Psi} \right) dt_a, \tag{67}
$$

with the covariant derivative $\nabla_a = \frac{\partial}{\partial t_a} - i a_a$. One can think of these expressions as an induced Higgs field and connection on the kernel of $D^\dagger$ from the simple abelian monopole family

$$\phi = s + \sum_{j=1}^{\mathrm{int}(s)} \frac{1}{2t_j}, \qquad\qquad a = \sum_{j=1}^{\mathrm{int}(s)} \omega_j. \tag{68}$$

This construction in principle delivers all singular monopoles of any charge, singularity number, and unitary gauge group. As an illustration, we work out the example of one $U(2)$ monopole with k singularities is complete detail. The resulting Higgs fields and connection are given in Eqs. (61) and (66).

In [30] we use this solution to obtain an $SU(2)$ monopole with k minimal singularities and analyze its properties.

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8 Appendix

We describe here in detail the calculations which lead to our expressions for Φ and A. We have set $p_0^L = -\lambda$, $p_k^R = \lambda$ and all other points $p_j^L = p_j^R = 0$. Solving the Dirac equation (37) and (38) we write our data $\mathbf{\Psi} = (\psi(s), v_j)$ in the form

$$\psi(s) = \begin{cases} e^{s\not{z}}\Pi_0 & -\lambda < s < 0 \\ e^{s\not{z}}\Pi_k & 0 < s < \lambda \end{cases}, \tag{69}$$

$$v_j = \begin{pmatrix} v_j^+ \\ v_j^- \end{pmatrix} = \frac{(-1)^j}{\mathcal{P}_j}(b_j^-, B_j^-)^\dagger e^{2[\alpha_1+\cdots+\alpha_j]\not{z}}\Pi_0, \tag{70}$$

with α_j such that $\exp(2\alpha_j z) = \sqrt{\frac{T_j+t_j+z}{T_j+t_j-z}}$, so that

$$\cosh 2\alpha_j z = \frac{T_j + t_j}{\mathcal{P}_j}, \quad \sinh 2\alpha_j z = \frac{z}{\mathcal{P}_j}, \tag{71}$$

$$\cosh 4\alpha_j z = \frac{(T_j + t_j)^2 + z^2}{\mathcal{P}_j^2}, \quad \sinh 4\alpha_j z = \frac{2z(T_j + t_j)}{\mathcal{P}_j^2}. \tag{72}$$

Note as well from (48) that $\Pi_k = (-1)^k e^{2\alpha\not{z}}\Pi_0$.

Normalisation

The first step in our construction is to compute the normalisation factor $N^2 = (\boldsymbol{\Psi}, \boldsymbol{\Psi}) = \int ds\, \psi(s)^\dagger \psi(s) + \sum_j v_j^\dagger v_j$. From (56) this is given by

$$
\begin{aligned}
N^2 =& \int_{-\lambda}^{0} ds\, \Pi_0^\dagger e^{2s\not{z}} \Pi_0 + \int_0^\lambda ds\, \Pi_k^\dagger e^{2s\not{z}} \Pi_k + \sum_{j=1}^{k} v_j^\dagger v_j \\
=& \frac{1}{2z}\left(\sinh 2\lambda z \left[\Pi_0^\dagger \Pi_0 + \Pi_k^\dagger \Pi_k\right] + \frac{1}{z}\cosh 2\lambda z \left[\Pi_k^\dagger \not{z} \Pi_k - \Pi_0^\dagger \not{z} \Pi_0\right]\right) \\
&+ \frac{1}{2z^2}\Pi_0^\dagger \not{z}\Pi_0 - \frac{1}{2z^2}\Pi_k^\dagger \not{z} \Pi_k + \sum_{j=1}^{k} v_j^\dagger v_j .
\end{aligned} \tag{73}
$$

We can write the last three terms as $\frac{1}{2z^2}\Pi_0^\dagger C(k) \Pi_0$ with

$$
C(k) = \not{z}(1 - e^{4(\alpha_1+\cdots+\alpha_k)\not{z}}) + 2z^2 \sum_{j=1}^{k} \frac{1}{\mathcal{P}_j} e^{4(\alpha_1+\cdots+\alpha_j)\not{z}} e^{-2\alpha_j \not{z}}. \tag{74}
$$

Then the difference $C(k) - C(k-1)$ can be written as

$$
C(k) - C(k-1) = e^{4(\alpha_1+\cdots+\alpha_{k-1})\not{z}}\left(-\not{z} e^{4\alpha_k \not{z}} + \not{z} + \frac{2z^2}{\mathcal{P}_k} e^{2\alpha_k \not{z}}\right), \tag{75}
$$

which vanishes, as can be checked by expanding the exponentials and using the relations (71) and (72). Thus $C(k) = C(k-1) = \ldots = C(1) = \left(-\not{z} e^{4\alpha_1\not{z}} + \not{z} + \frac{2z^2}{\mathcal{P}_1} e^{2\alpha_1 \not{z}}\right) = 0$, so we have shown that the last line in Eq. (73) vanishes. Hence, using $\Pi_k = (-1)^k e^{2\alpha \not{z}} \Pi_0$, Eq. (73) becomes

$$
\begin{aligned}
N^2 &= \frac{1}{2z}\Pi_0^\dagger \left(\sinh 2\lambda z \left(e^{4\alpha\not{z}} + 1\right) + \frac{\not{z}}{z}\cosh 2\lambda z \left(e^{4\alpha \not{z}} - 1\right)\right)\Pi_0 \\
&= \frac{1}{z}\left(\sinh 2\lambda z \cosh 2\alpha z + \cosh 2\lambda z \sinh 2\alpha z\right) \Pi_0^\dagger e^{2\alpha\not{z}} \Pi_0
\end{aligned} \tag{76}
$$

This expression suggests a natural choice of orthogonal basis of solutions delivered by $\Pi_0 = e^{-\alpha \not{z}}$. In this basis the normalization factor satisfies

$$
N^2 = \frac{1}{z}\sinh 2(\lambda + \alpha) z. \tag{77}
$$

Higgs Field

Our Higgs field was given by (59):

$$N^2\Phi = \int_{-\lambda}^{0} ds\,\Pi_0^\dagger\, s e^{2s\not{z}}\Pi_0 + \int_0^\lambda ds\,\Pi_k^\dagger\, s e^{2s\not{z}}\Pi_k + \left(\sum_{j=1}^{k}\frac{1}{2t_j}\right)\int_0^\lambda ds\,\Pi_k^\dagger\, e^{2s\not{z}}\Pi_k + \sum_{j=1}^{k}\frac{1}{2t_j}\left((v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i\right). \quad (78)$$

The integrals are straightforward to compute upon substituting $\Pi_0 = e^{-\alpha\not{z}}$, $\Pi_k = e^{\alpha\not{z}}$. One finds

$$\Phi = \sum_{j=1}^{k}\frac{1}{4t_j} + \left(\left[\lambda + \sum_{j=1}^{k}\frac{1}{4t_j}\right]\coth 2(\lambda+\alpha)z - \frac{1}{2z}\right)\frac{\not{z}}{z} + \frac{z}{\sinh 2(\lambda+\alpha)z}\left(\frac{1}{2z^3}\not{z}\sinh 2\alpha z - \frac{\not{z}}{z^2}e^{2\alpha\not{z}}\sum_{j=1}^{k}\frac{1}{4t_j} + \sum_{j=1}^{k}\frac{1}{2t_j}\left[(v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i\right]\right). \quad (79)$$

Now vanishing of the last three terms in Eq. (73) implies

$$\sum_{i=j+1}^{k} v_i^\dagger v_i = \frac{1}{2z^2}\left(\Pi_k^\dagger \not{z}\Pi_k - \Pi_j^\dagger \not{z}\Pi_j\right) = \frac{1}{2z^2}\not{z}e^{2\alpha\not{z}} - \frac{\not{z}}{2z^2}e^{-2\alpha\not{z}}e^{4(\alpha_1+\cdots+\alpha_j)\not{z}}. \quad (80)$$

The Dirac equation (47) gives us v_j and its first component

$$v_j^+ = \frac{(-1)^j}{\mathcal{P}_j}(b_j^-)^\dagger e^{-\alpha\not{z}}e^{2(\alpha_1+\cdots+\alpha_j)\not{z}}, \quad (81)$$

and a short calculation shows that

$$(v_j^+)^\dagger v_j^+ = \frac{1}{\mathcal{P}_j^2}\not{T}_{j\perp} + \frac{1}{\mathcal{P}_j^2}\left(t_j - \frac{\vec{z}\cdot\vec{t}_j}{z}\frac{\not{z}}{z}\right)e^{-2\alpha\not{z}}e^{4(\alpha_1+\cdots+\alpha_j)\not{z}}. \quad (82)$$

Combining these two observations

$$\Phi = \sum_{j=1}^{k} \frac{1}{4t_j} + \left(\left(\lambda + \sum_{j=1}^{k} \frac{1}{4t_j} \right) \coth 2(\lambda+\alpha)z - \frac{1}{2z} \right) \frac{\not{z}}{z}$$
$$+ \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{1}{2t_j \mathcal{P}_j^2} \not{T}_{j\perp} + \frac{z}{\sinh 2(\lambda+\alpha)z} \left\{ \frac{\not{z}}{2z^3} \sinh 2\alpha z \right.$$
$$\left. + \sum_{j=1}^{k} \frac{1}{2t_j} e^{-2\alpha \not{z}} e^{4(\alpha_1+\cdots+\alpha_j)\not{z}} \left(\frac{1}{\mathcal{P}_j^2} \left(t_j - \frac{\vec{z}\cdot\vec{t}_j}{z} \frac{\not{z}}{z} \right) - \frac{\not{z}}{2z^2} \right) \right\}. \quad (83)$$

The last line is simplified using

$$\frac{1}{2t_j} \left(\frac{1}{\mathcal{P}_j^2} \left(t_j - \frac{\vec{z}\cdot\vec{t}_j}{z} \frac{\not{z}}{z} \right) - \frac{\not{z}}{2z^2} \right) = -\frac{1}{2\mathcal{P}_j} \frac{\not{z}}{z^2} e^{-2\alpha_j \not{z}}, \quad (84)$$

and in fact the sum of the terms in the curly brackets in Eq. (83) vanishes if

$$\sinh 2\alpha z = \sum_{j=1}^{k} \sinh 2\alpha_j z e^{2(\alpha_1+\cdots+\alpha_{j-1})\not{z} - 2(\alpha_{j+1}+\cdots+\alpha_k)\not{z}}. \quad (85)$$

This is indeed the case since

$$\sum_{j=1}^{k} \left(e^{2\alpha_j \not{z}} - e^{-2\alpha_j \not{z}} \right) e^{2(\alpha_1+\ldots+\alpha_{j-1}-\alpha_{j+1}-\ldots-\alpha_k)\not{z}} =$$
$$\sum_{j=1}^{k} \left(e^{4(\alpha_1+\ldots+\alpha_{j-1}+\alpha_j)\not{z}} - e^{4(\alpha_1+\ldots+\alpha_{j-1})\not{z}} \right) e^{-2\alpha\not{z}} =$$
$$\left(e^{4(\alpha_1+\ldots+\alpha_k)\not{z}} - 1 \right) e^{-2\alpha\not{z}} = e^{2\alpha\not{z}} - e^{-2\alpha\not{z}}. \quad (86)$$

Vector Potential

The connection A is given by Eq. (63) so that

$$N^2 A = \frac{i}{2} \int_{-\lambda}^{0} ds \, \left(\psi_0^\dagger(s) d\psi_0(s) - \, h.c. \right) + \frac{i}{2} \int_0^{\lambda} ds \, \left(\psi_k^\dagger(s) d\psi_k(s) - \, h.c. \right)$$
$$+ \frac{i}{2} \sum_{j=1}^{k} (v_j^\dagger dv_j - h.c.) + \sum_{j=1}^{k} \omega_j \int_0^{\lambda} ds \, \Pi_k^\dagger e^{2s\not{z}} \Pi_k + \sum_{j=1}^{k} \omega_j \left((v_j^+)^\dagger v_j^+ + \sum_{i=j+1}^{k} v_i^\dagger v_i \right). \quad (87)$$

We can now insert $v_j^\dagger dv_j - dv_j^\dagger v_j$ from Eq. (64) to find

$$
\begin{aligned}
A = \frac{i}{2z}[\not{z}, d\not{z}]\Bigg(&\frac{1}{2z} + \frac{1}{\sinh 2(\lambda+\alpha)z}\Big[-\lambda - \frac{\sinh 2\alpha z}{2z} \\
&+ \sum_{j=1}^{k} \frac{1}{\mathcal{P}_j} \sinh(\alpha - 2[\alpha_1 + \cdots + \alpha_{j-1}])z \sinh(\alpha - 2[\alpha_1 + \cdots + \alpha_j])z\Big]\Bigg) \\
&+ \sum_{j=1}^{k} \frac{1}{2}\omega_j + \sum_{j=1}^{k} \frac{1}{2}\omega_j \frac{\not{z}}{z} \coth 2(\lambda+\alpha)z + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{1}{\mathcal{P}_j^2}\omega_j \not{T}_{j\perp} \\
&+ \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \Bigg\{ \omega_j \left(\frac{1}{\mathcal{P}_j^2} \left(t_j - \frac{\vec{z}\cdot\vec{t}_j}{z}\frac{\not{z}}{z} \right) - \frac{1}{2z^2}\not{z} \right) e^{-2\alpha\not{z}} e^{4(\alpha_1+\cdots+\alpha_j)\not{z}} \\
&+ \frac{i}{2\mathcal{P}_j^2} e^{-(\alpha - 2[\alpha_1+\cdots+\alpha_j])\not{z}} \left(b_j^- db_j^{-\dagger} - db_j^- b_j^{-\dagger} \right) e^{-(\alpha-2[\alpha_1+\cdots+\alpha_j])\not{z}} \Bigg\}. \qquad (88)
\end{aligned}
$$

Simple trigonometric identities and Eq. (71) give

$$
\begin{aligned}
&\sum_{j=1}^{k} \frac{1}{\mathcal{P}_j} \sinh(\alpha - 2[\alpha_1+\cdots+\alpha_{j-1}])z \sinh(\alpha - 2[\alpha_1+\cdots+\alpha_j])z - \frac{\sinh 2\alpha z}{2z} = \\
&-\frac{1}{2}\sum_{j=1}^{k} \frac{T_j + t_j}{\mathcal{P}_j^2} + \frac{1}{2}\sum_{j=1}^{k} \frac{1}{\mathcal{P}_j} \cosh(2\alpha - 4[\alpha_1+\cdots+\alpha_{j-1}] - 2\alpha_j)z - \frac{\sinh 2\alpha z}{2z}.
\end{aligned} \qquad (89)
$$

Now, $\cosh(2\alpha - 4[\alpha_1+\cdots+\alpha_{j-1}] - 2\alpha_j)z = \cosh[-2(\alpha_1+\cdots+\alpha_{j-1})z + 2(\alpha_{j+1}+\cdots+\alpha_k)z]$, and the sum of hyperbolic cosines in (89) cancels against the $\sinh 2\alpha z$ factor due to the trace part of Eq. (85).

This simplifies the $[\not{z}, d\not{z}]$ terms of Eq. (88). Using Eq. (65) for the b_j^- terms and then applying $e^{\beta\not{z}}\not{a}e^{\beta\not{z}} = \not{a}_\perp + \frac{\vec{a}\cdot\vec{z}}{z}\frac{\not{z}}{z}e^{2\beta\not{z}}$, $\not{t}_{j\perp} = -\not{T}_{j\perp}$ and $\omega_j = -\frac{1}{\mathcal{P}_j^2 t_j}\vec{z}\cdot(\vec{t}_j \times d\vec{t}_j)$ we obtain

$$
\begin{aligned}
A = \frac{i}{2z}[\not{z}, d\not{z}] &\left(-\frac{1}{\sinh 2(\lambda+\alpha)z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2\mathcal{P}_j^2} \right] + \frac{1}{2z} \right) \\
&+ \sum_{j=1}^{k} \frac{1}{2}\omega_j + \sum_{j=1}^{k} \frac{1}{2}\omega_j \frac{\not{z}}{z} \coth 2(\lambda+\alpha)z + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{i}{4\mathcal{P}_j^2 t_j} [\not{t}_j, d\not{t}_j]_\perp.
\end{aligned} \qquad (90)
$$

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