New Spin(7) holonomy metrics admitting G_2 holonomy reductions and M-theory/IIA dualities

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Abstract

We construct several Spin(7) holonomy metrics which admit a G_2 holonomy reduction along one isometry. The resulting G_2 holonomy metrics admit a further reduction to 6-dimensional Kahler metrics, therefore realizing the pattern Spin(7) $\rightarrow G_2 \rightarrow$ (Kahler) proposed in [26] and which describe an M-theory/IIA superstring duality. An infinite class of such metrics are found, which are locally R^3-fibrations over the Eguchi-Hanson gravitational instanton and which, to our knowledge, were not considered in the literature before. Some of the supersymmetric cycles inside these metrics are characterized. At the end the constructed G_2 holonomy examples are lifted to a non geometrical IIA supergravity solution.

1. Introduction

Spaces of special holonomy namely, G_2 and Spin(7) holonomy, were the only two cases of the Berger classification of the possible holonomy groups for Riemanian geometry [1] whose existence was in doubt. This situation changed completely with the construction of explicit non compact examples in [2]-[3] and the proof of the existence of compact ones given in [4]-[5]. Since the appearance of these works, further special holonomy metrics were found in [6]-[21]. These spaces are relevant for constructing supersymmetric solutions of supergravity theories or vacuum solutions of superstring theories. For instance, in eleven dimensions in absence of fluxes the backgrounds $R_{1,2} \times M_8$ or $R_{1,3} \times M_7$ are supersymmetric if M_7 or M_8 are of special holonomy. Reduction to 4 or 3 dimensions gives N=1 supersymmetric theories. For heterotic string theory these spaces also provide N=1 supersymmetry in 3 and 2 dimensions [27]. This picture breaks down in presence of branes, and for this reason the study of compactifications with fluxes is also of importance.

For another side, the present understanding of the dynamics of N=1 supersymmetric theories relies partially in the existence of dual realizations of a given theory. For instance [23]-[24] it was found that D6 branes of IIA superstring theory wrapping a supersymmetric cycle inside a CY manifold are dual to M theory compactified on certain G_2 manifold (see also [25]). More recently dualities between D6 branes of IIA superstrings wrapping supersymmetric four cycles (co-associative) of a G_2 manifold and M theory on Spin(7) manifolds were studied in [26].

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This duality refers to N=1 theories in 3 dimensions. In addition it was considered in [26] the analogous relation between IIA superstrings wrapping a special lagrangian sub-manifold of a CY manifold and M-theory on G_2 manifolds, and it was shown that under certain assumptions the 6-metric in the string frame is Kahler. This realize a duality between N=1 four dimensional theories. The interesting thing about both cases is that they relate a background with RR fluxes (the IIA background) to one which is entirely geometrical (the M-theory one).

The pattern of dualities described above is the main motivation of the present work. In principle, not every G_2 manifold is related to 6 dimensional Kahler manifold by a Kaluza-Klein reduction along an isometry. As it was shown in [20] that if a G_2 manifold possess an isometry preserving the G_2 structure and admits a 6-dimensional Kahler reduction, then it has a second isometry commuting with the first one. Thus this G_2 metric is necessarily toric, and there further conditions are to be satisfied. For another side one does not expect *any* G_2 holonomy metric to be toric or that this extra conditions would be identically satisfied. Therefore, this kind of dualities should be realized only for an specific (though, by the results of [21] and [22], infinite) class of G_2 and Kahler metrics. The same considerations must be true for Spin(7) and G_2 pairs. Moreover, it is not evident whether or not the G_2 metrics participating in the (Kahler, G_2) and (G_2, Spin(7)) dualities are related. The interesting fact we prove here is that *any* G_2 metric admitting a Kahler reduction can be uplifted to an Spin(7) metric. We are not in position at the moment to prove or reject the inverse statement. The conclusion is that the set of G_2 metrics participating in the Spin(7) duality is equal or *bigger* than the one for the Kahler duality.

We are also able to construct an infinite class of sequences (Kahler)$\rightarrow G_2 \rightarrow$Spin(7) metrics which arise as fibrations over the Eguchi-Hanson gravitational instanton. Such metrics are parameterized in terms of an holomorphic function defined over a complex sub-manifold of the Eguchi-Hanson manifold. We were not able to repeat the calculation for other hyperkahler metrics, though it does not mean that this can not be done. It is interesting to note that the Eguchi-Hanson instanton is the main ingredient to construct Ricci-flat metrics in K_3 [33] or in the construction of the G_2 or Spin(7) compact manifolds of [4] and it happen also to be the main ingredient in our construction. It is not clear (at least for the authors) if there is a deep reason for this coincidence or not.

The present work is organized as follows. In section 2 a system of equations for uplifting a given G_2 holonomy metric to an Spin(7) one is found. Also a brief characterization of the G_2 holonomy metrics which admit a Kahler reduction is given and it is shown that *any* of these metrics can be uplifted to an Spin(7) one by means of the mentioned system of equations. In section 3 some known examples of these G_2 metrics [20]-[21] are reviewed and the uplifting to Spin(7) metrics is presented explicitly. In section 4 the mentioned infinite family of Spin(7) metrics fibered over the Eguchi-Hanson gravitational instanton is presented. By Kaluza-Klein reduction along one of the isometries we obtain a non geometrical IIA background related to the underlying G_2 holonomy metrics.

2. Spin(7) metrics admitting G_2 reductions

2.1 The defining equations

Our starting point is an 8-dimensional space M_8 with metric

$$g_8 = e^{6f}(dz + A)^2 + e^{-2f} g_7 \tag{2.1}$$

for which the 1-form A, the 7-metric g_7 and the function f are independent on the coordinate z. This condition means that $V = \partial_z$ is a local Killing vector, which induce a local decomposition $M_8 = M_7 \times R_z$ if z is non compact or $M_8 = M_7 \times U(1)_z$ if z is an angular coordinate. In the following we will impose that g_8 is of Spin(7) holonomy and that g_7 is of G_2 holonomy and we will derive the consequences of this statement, with the further assumption that $V = \partial_z$ also preserve the Spin(7) structure. The last assumption is for simplicity.

By defining the one form $e^8 = e^{3f}(dz+A)$ one can decompose the Spin(7) equivariant 4-form corresponding to g_8 as

$$\Omega_8 = e^8 \wedge \widetilde{\Phi} + *\widetilde{\Phi}, \tag{2.2}$$

being $\widetilde{\Phi}$ and $*\widetilde{\Phi}$ a pair of G_2 invariant 3 and 4 forms for the metric $e^{-2f}g_7$. It is clear that the 4-form (2.2) is preserved by $V = \partial_z$ as f is z-independent. Furthermore $\widetilde{\Phi} = e^{-3f}\Phi$ and $*\widetilde{\Phi} = e^{-4f} * \Phi$ being Φ and $*\Phi$ certain G_2 invariant 3 and 4 forms for the metric g_7. The four form (2.2) can be expressed in terms of Φ and $*\Phi$ as

$$\Omega_8 = (dz + A) \wedge \Phi + e^{-4f} * \Phi. \tag{2.3}$$

As is well known g_7 has holonomy in G_2 if and only if $d\Phi = d * \Phi = 0$. By assuming that this is the case the Spin(7) condition $d\Omega_8 = 0$ gives the following system

$$F \wedge \Phi + d(e^{-4f}) \wedge *\Phi = 0, \tag{2.4}$$

being $F = dA$. By construction F is a closed two form.

Equation (2.4) constitutes an apparently simple method to lift a known starting G_2 holonomy metric to an Spin(7) one, but is not obvious that for any G_2 metric it will exist a non trivial solution. Note also that this system do not classify completely all the Spin(7) metrics admitting a G_2 holonomy reduction. Even if $d\Phi = d * \Phi \neq 0$ there could exist a rotation of the tetrad frame of g_7 such that $d\Phi' = d * \Phi' = 0$ for certain new calibration forms. In the present work we will take a modest approach and we will just find several particular solutions of (2.4). The G_2 metrics from which we will start are an special class of G_2 holonomy metrics which are defined by admitting Kahler reductions [26], [21], [20] and [22].

2.2 G_2 holonomy metrics admitting Kahler reductions

As was mentioned above, any Spin(7) metric with an isometry preserving the metric and the calibration form induce a G_2 structure which will not be closed, unless (2.4) is satisfied. Analogous consideration hold for G_2 metrics with an isometry which preserve the calibration forms Φ and $*\Phi$. In this case a 6-dimensional $SU(3)$ structure is induced [31] and if the associated $SU(3)$ structure is Kahler, then it admits another isometry which commute with the former one [20]. Therefore any of such G_2 metrics is toric from the very beginning. Also it is possible to make a further reduction with respect to the second isometry and describe the seven and six metrics as fibrations over certain Kahler 4-dimensional metric which we will specify below. [1]

Let us describe schematically the local form of the G_2 holonomy metrics in question, further details can be found in the original reference [20]. These metrics possess the following local form

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \mu \left(u\, d\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu) \right). \tag{2.5}$$

[1] Note that the uplifting of these metrics to 8 dimensions by (2.4) will give an Spin(7) metric with three commuting isometries, as the initial metric is toric.

All the quantities defining g_7 are independent on the coordinates α and β, therefore (2.5) is toric with Killing vectors ∂_α and ∂_β. The metric $g_4(\mu)$ is Kahler and defined over a four manifold M and it depends on μ as a parameter. It also admits a complex μ-independent symplectic 2-form $\Omega = \omega_2 + i\omega_3$, where being "symplectic" means that it is closed, $d\Omega = 0$. On the other hand, being "complex" implies that

$$\omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3, \qquad \omega_2 \wedge \omega_3 = 0, \tag{2.6}$$

and that the equation

$$\omega_2(J_1 \cdot, \cdot) = \omega_3(\cdot, \cdot). \tag{2.7}$$

define a complex structure J_1. In other words, the Niejenhuis tensor of J_1 vanishes identically or equivalently J_1 is integrable. The corresponding two form $\widetilde{\omega}_1(\mu)$ in general will be μ dependent and is also closed on M. The function u in (2.5) depends on the coordinates of M and on the parameter μ, and is defined through the relation

$$2\mu\, \widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu) = u\, \Omega \wedge \overline{\Omega}. \tag{2.8}$$

This function always exists because the wedge products in (2.8) are proportional to the volume form $V(g_4)$ of $g_4(\mu)$. In fact

$$\widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu) = V(g_4)$$

The forms H_1 and H_2 are defined on $M \times \mathbf{R}_\mu$ and M respectively by the equations

$$dH_1 = (d^c_M u) \wedge d\mu + \frac{\partial \widetilde{\omega}_1}{\partial \mu}, \qquad dH_2 = -\omega_2, \tag{2.9}$$

with $d^c_M = J_1 d_M$. The last equation can always be solved locally as the forms $\widetilde{\omega}_1$ and ω_2 are closed. The construction described in [20] states that if the quantities appearing in (2.5) are related by the evolution equation

$$\frac{\partial^2 \widetilde{\omega}_1}{\partial^2 \mu} = -d_M d^c_M u, \tag{2.10}$$

then the metric (2.5) will have G_2 holonomy. This statement is not difficult to see. The calibration 3-form corresponding to the metrics (2.5) is

$$\begin{aligned} \Phi = \widetilde{\omega}_1(\mu) \wedge (d\alpha + H_2) + d\mu \wedge (d\beta + H_1) \wedge (d\alpha + H_2) \\ +\mu\ \left(\omega_2 \wedge (d\beta + H_1) + u\omega_3 \wedge d\mu \right), \end{aligned} \tag{2.11}$$

and the dual form $*\Phi$ corresponding to (2.11) is given by [21]

$$\begin{aligned} *\Phi = \mu^2 \widetilde{\omega}_1(\mu) \wedge d\mu \wedge (d\beta + H_1) + u\omega_2 \wedge (d\alpha + H_2) \wedge d\mu \\ +\omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu^2 \widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu). \end{aligned} \tag{2.12}$$

By means of (2.9), (2.10) and (2.8) it follows that $d\Phi = d * \Phi = 0$.

Proof. Taking the exterior derivative of (2.11) gives

$$\begin{aligned} d\Phi = d\widetilde{\omega}_1(\mu) \wedge (d\alpha + H_2) + \widetilde{\omega}_1(\mu) \wedge dH_2 + d\mu \wedge dH_1 \wedge (d\alpha + H_2) + d\mu \wedge (d\beta + H_1) \wedge dH_2 \\ +du \wedge\ \left(\omega_2 \wedge (d\beta + H_1) + u\omega_3 \wedge d\mu \right) + \mu\ \left(\omega_2 \wedge dH_1 + du \wedge \omega_3 \wedge d\mu \right). \end{aligned} \tag{2.13}$$

As $\widetilde{\omega}_1$ is closed on M we have that

$$d\widetilde{\omega}_1 = \partial_\mu \widetilde{\omega}_1 \wedge d\mu$$

Also the relation $\widetilde{\omega}_1 \wedge \omega_2 = 0$ and the μ-independence of ω_2 imply that

$$\partial_\mu \widetilde{\omega}_1 \wedge \omega_2 = 0.$$

Inserting the last two equalities into (2.13) and taking into account (2.6) and (2.9) gives

$$d\Phi = \partial_\mu \widetilde{\omega}_1(\mu) \wedge d\mu \wedge (d\alpha + H_2) + d\mu \wedge dH_1 \wedge (d\alpha + H_2) - d\mu \wedge (d\beta + H_1) \wedge \omega_2$$
$$+ d\mu \wedge \ (\ \omega_2 \wedge (d\beta + H_1) + u\omega_3 \wedge d\mu\ ) + \mu\ (\ \omega_2 \wedge d_M^c u \wedge d\mu + d_4 u \wedge \omega_3 \wedge d\mu\ ) . \qquad (2.14)$$

The first two terms cancel by virtue of the first of (2.6). The third and the four terms cancel identically. The fifth is identically zero as it contains $d\mu$ twice. Now, the definition of J_1 given in (2.7) implies that

$$\omega_2 \wedge A = -\omega_3 \wedge J_1 A$$

being A any 1-form. In particular by taking $A = d_4 u$ and remembering that $d_4^c = J_1 d_4$ it is obtained that the last two terms in (2.14) cancel, thus $d\Phi = 0$. An analogous calculation shows that $d * \Phi = 0$. Q. E. D.

The G_2 metrics (2.5) are fibered over the six dimensional metric

$$g_6 = u\ d\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu), \qquad (2.15)$$

which is Kähler with Kähler form

$$K = (d\beta + H_1) \wedge d\mu + \widetilde{\omega}_1. \qquad (2.16)$$

The condition to be Kahler is usually referred as the “strong supersymmetry condition” in the physical literature [26]. The converse of this statement is also true. That is, for given a G_2 holonomy manifold Y with a metric g_7 possessing a Killing vector that preserves the calibration forms Φ and $*\Phi$ and such that the six-dimensional metric g_6 obtained from the orbits of the Killing vector is Kähler, then there exists a coordinate system in which g_7 takes the form (2.5) being $g_4(\mu)$ a one-parameter four-dimensional metric admitting a complex symplectic structure Ω and a complex structure J_1, being the quantities appearing in this expression related by (2.7) and the conditions (2.9), (2.10) and (2.8). This is the most involved part of the proofs and we refer the reader to the original reference [20].

Note that the Killing vector fields preserve the metric and Φ, therefore preserve $*\Phi$ and thus the whole G_2 structure. Another interesting fact is that

$$*\Phi|_M = V(g_4),$$

therefore the Kahler base g_4 is a co-associative sub-manifold . In the same way for fixed value of the coordinates of g_4 one obtains from (2.5) the three dimensional metric

$$g_3 = \frac{d\alpha^2}{\mu^2} + u\ d\mu^2 + \mu \frac{d\beta^2}{u}. \qquad (2.17)$$

defined on certain space M_3, and it follows that

$$*\Phi|_{M_3} = V(g_3),$$

therefore M_3 is an associative sub-manifold. These are calibrated sub-manifolds [35] and are supersymmetric from the physical point of view [36].

2.3 Uplifting to Spin(7) metrics

In this subsection we show that any of the G_2 holonomy metrics (2.5) described above can be uplifted to an Spin(7) holonomy one by means of the uplifting formula (2.4). The two form appearing in the equation $F = dA$ must be closed. As we have a the sympletic form $\Omega = \omega_2 + i\omega_3$ and it is seen from (2.9) that $dH_2 = -\omega_2$ the most natural anzatz is to put

$$F = dA = -\omega_3. \tag{2.18}$$

The system (2.4) reduce in this case to

$$\omega_3 \wedge \Phi = d(e^{-4f}) \wedge *\Phi. \tag{2.19}$$

From (2.12) it follows that the right hand of (2.19) is

$$d(e^{4f}) \wedge *\Phi = d(e^{-4f}) \wedge \Big(\mu^2\widetilde{\omega}_1(\mu) \wedge d\mu \wedge (d\beta + H_1) + u\omega_2 \wedge (d\alpha + H_2) \wedge d\mu$$

$$+\omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu^2\widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu)\Big) \tag{2.20}$$

The left hand side is obtained from (2.11) and is

$$\omega_3 \wedge \Phi = \widetilde{\omega}_1(\mu) \wedge (d\alpha + H_2) \wedge \omega_3 + d\mu \wedge (d\beta + H_1) \wedge (d\alpha + H_2) \wedge \omega_3$$

$$+\mu\ \left(\omega_2 \wedge (d\beta + H_1) + u\omega_3 \wedge d\mu \ \right) \wedge \omega_3 \tag{2.21}$$

But from (2.8) we see that $\omega_2 \wedge \omega_3 = \widetilde{\omega}_1 \wedge \omega_3 = 0$ and also that

$$\mu u\omega_3 \wedge \omega_3 = \mu^2\widetilde{\omega}_1 \wedge \widetilde{\omega}_1.$$

With these relations (2.21) get simplified as

$$\omega_3 \wedge \Phi = d\mu \wedge \Big((d\beta + H_1) \wedge (d\alpha + H_2) \wedge \omega_3 + \mu^2\widetilde{\omega}_1 \wedge \widetilde{\omega}_1\Big) \tag{2.22}$$

Equating (2.22) to (2.20) gives the equation

$$\begin{aligned} &d(e^{-4f}) \wedge d\mu \wedge \Big(\mu^2\widetilde{\omega}_1(\mu) \wedge (d\beta + H_1) - u\omega_2 \wedge (d\alpha + H_2)\Big) \\ &+d(e^{-4f}) \wedge \Big(\omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu^2\widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu)\Big) \\ &= d\mu \wedge \Big(\omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu^2\widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu)\Big) \end{aligned} \tag{2.23}$$

The solution of this system is immediate. If $d(e^{-4f}) = d\mu$ the two first terms of the left hand side vanishes and the two last ones equal to the right hand side. We choose then $e^{-4f} = a\mu + b$ and our Spin(7) metrics become

$$g_8 = \frac{(dz + H_3)^2}{(a\mu + b)^{3/2}} + (a\mu + b)^{1/2} g_7 \tag{2.24}$$

being $dH_3 = -\omega_3$ and g_7 the G_2 holonomy metrics described in the previous section. The metrics (2.24) are an infinite family of Spin(7) metrics admitting G_2 reductions, realizing the pattern described in [26]. The reason for which the family is infinite is because the G_2 family over which are fibered is also infinite [20], [21].

3. Explicit Spin(7) examples

3.1 Two different ramifications

As the Spin(7) metrics (2.24) are completely determined in terms of the G_2 metric g_7, the task to find them has been reduced to solve equations (2.7)-(2.9) defining the G_2 geometry together with the new condition $dH_3 = -\omega_3$. A simple solution is obtained by assuming that the function u does not depends on the coordinates of M but only on the coordinate μ. Then it is obtained from (2.10) that $\widetilde{\omega}_1 = (c\mu + d)\omega_1$ being ω_1 independent on μ. As $\widetilde{\omega}_1$ is closed on M_4 it follows that $d_4\omega_1 = 0$. This means that if one starts with an hyperkahler triplet ω_i of some hyperkahler manifold M all the conditions (2.5)-(2.9) are solved except (2.8), which becomes then an algebraic equation defining u. The solution is $u = \mu(c\mu + d)^2$. Also $g_4(\mu) = (c\mu + d)\overline{g}_4$ being $\overline{g}_4$ the hyperkahler metric corresponding to ω_i. The resulting 7-metrics (2.5) have the following expression

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \frac{(d\beta + H_1)^2}{(c\mu^2 + d)^2} + \mu^2(c\mu + d)^2\, d\mu^2 + \mu(c\mu + d)\overline{g}_4. \tag{3.25}$$

Moreover the equations (2.9) are in this case

$$dH_1 = \omega_1, \qquad dH_2 = -\omega_2. \tag{3.26}$$

These metrics are usually well behaved away from the point $\mu = 0$ or $\mu = -b/a$ if $b/a < 0$.

The second type of solutions corresponds to the case when u depends on μ and also varies on M. This case is more hard but still we will find below several of these solutions. Consider as before an hyperkahler structure ω_i with its Ricci flat metric $\overline{g}_4$ and make an anzatz for $\widetilde{\omega}_1(\mu)$ of the form

$$\widetilde{\omega}_1(\mu) = \omega_1 - d_4 d_4^c G, \tag{3.27}$$

being G a function on $M \times R_\mu$. With this anzatz equations (2.6) and (2.7) are satisfied. Inserting it into the evolution equation (2.10) gives

$$\partial_\mu^2 G = 2u, \tag{3.28}$$

therefore u is determined in terms of G. The equation for G is found from (2.9). The relation

$$\widetilde{\omega}_1(\mu) \wedge \widetilde{\omega}_1(\mu) = (\omega_1 - d_4 d_4^c G) \wedge (\omega_1 - d_4 d_4^c G) = M(G)\omega_1 \wedge \omega_1 \tag{3.29}$$

defines a non linear operator $M(G)$ called Monge-Ampere operator. This operator always exist as all the expressions in (3.29) are proportional to the volume form of $g_4(\mu)$. In terms of $M(G)$ the insertion of (3.28) into (2.9) gives

$$2\mu M(G) = \partial_\mu^2 G, \tag{3.30}$$

which is the equation defining G. Also, from (2.10) it follows that

$$H_1 = -d_4^c \partial_\mu G \tag{3.31}$$

It should be remarked that metric tensor $g_4(\mu)$ in (2.5) is *not* the hyperkahler metric $\overline{g}_4$ in general. If K denote the Kahler potential corresponding to ω_1 then the metric $g_4(\mu)$ is the one which corresponds to the modified Kahler potential $\overline{K} = K - G$. This metric will be Kahler, but not necessarily hyperkahler. Equations (3.28)-(3.31) define a new family of G_2 metrics and all the objects defining the metric are related essentially to a single function G satisfying (3.30).

The difficulty in solving the previous equations reside in the non linearity of the Monge Ampere operator $M(G)$, due to the presence of the term $d_4 d_4^c G \wedge d_4 d_4^c G$. But in the special cases in which

$$d_4 d_4^c G \wedge d_4 d_4^c G = 0, \tag{3.32}$$

it is obtained that [20]

$$M(G) = 1 + \Delta_4 G$$

being Δ_4 the laplacian over the starting hyperkahler metric $\overline{g}_4$. This happens when the function G is defined over a complex sub-manifold on the hyperkahler manifold M [32]. Perhaps is better to express this in other words. The starting hyperkahler structure ω_i is obviously Kahler, thus M is complex and parameterized in terms of certain complex coordinates (z_1, z_2) and their complex conjugates. The equation (3.32) will be satisfied if for the complex coordinate system z_i which diagonalize J_1, the function G is of the form $G = G(w, \overline{w})$ being w a single complex function of the z_i and $\overline{w}$ its complex conjugate. The equation (3.30) will be reduced to

$$2\mu(1 + \Delta_4 G) = \partial_\mu^2 G. \tag{3.33}$$

The advantage of imposing this condition is that one has to solve a Laplace type equation instead a non linear one, though to find solutions of a Laplace equation in a curved space is not easy in general.

3.2 Simple known examples

Both type of solutions described above are constructed starting with an hyperkahler structure. It will be convenient to consider simple examples first. The simplest hyperkahler manifold is R^4 with its flat metric $g_4 = dx^2 + dy^2 + dz^2 + d\varsigma^2$ and with the closed hyperkahler triplet

$$\omega_1 = d\varsigma \wedge dy - dz \wedge dx, \qquad \omega_2 = d\varsigma \wedge dx - dy \wedge dz, \qquad \omega_3 = d\varsigma \wedge dz - dx \wedge dy.$$

This innocent looking case is indeed very interesting. The forms H_i such that $dH_i = \omega_i$ are given by

$$H_1 = -xdz + yd\varsigma, \qquad H_2 = -ydz + xd\varsigma, \qquad H_3 = -ydx + zd\varsigma \tag{3.34}$$

and by selecting $c = 1$ and $d = 0$ in (3.25) the resulting G_2 metric is

$$g_7 = \frac{(d\alpha - xdz + yd\varsigma)^2}{\mu^2} + \frac{(d\beta - ydz - xd\varsigma)^2}{\mu^2} + \mu^4\, d\mu^2 + \mu^2\, (\, dx^2 + dy^2 + dz^2 + d\varsigma^2\,). \tag{3.35}$$

The metrics (3.35) have been already obtained in the physical literature [19]. Even in this simple case and though the base 4-metric has trivial holonomy, it has been shown that (3.35) is irreducible and has holonomy exactly G_2, not a subgroup.

Turning on the attention to the second ramification, a possible choice of complex coordinates for R^4 is $z_1 = x + iy$, $z_2 = z + i\varsigma$ and complex conjugates. If the functional dependence the function G is assumed to $G = G(\mu, z_1, \overline{z}_1)$ then the Monge-Ampere operator $M(G)$ reduce to the laplacian operator in flat space

$$G'' + \mu(\partial_{xx} G + \partial_{yy} G) = 2\mu. \tag{3.36}$$

The separable solutions in the variable μ are of the form

$$G = \frac{1}{3}\mu^3 + V(x,y)K(\mu).$$

By introducing $G = G(\mu, x, y)$ into (3.36) it follows that $K(\mu)$ and $V(x,y)$ are solutions of the equations

$$K''(\mu) = p\ \mu\ K(\mu), \qquad \partial_{xx}V + \partial_{yy}V + p\ V = 0, \tag{3.37}$$

being p a parameter. By defining the $\widetilde{\mu} = \mu/p^{1/3}$ the first of the equations (3.37) reduce to the Airy equation. The second is reduced to find eigenfunctions of the two dimensional Laplace operator, which is a well known problem in electrostatics. For $p > 0$ periodical solutions are obtained and for $p < 0$ there will appear exponential solutions.

A simple example is given by the eigenfunction $V = q\ \sin(p\ x)$, being q a constant. A solution of the Airy equation is given by

$$K = Ai(\widetilde{\mu}) = \frac{1}{3}\widetilde{\mu}^{1/2}(J_{1/3}(\tau) + J_{-1/3}(\tau)), \qquad \tau = i\frac{2\ \mu^{3/2}}{3\ p^{1/2}}.$$

Then the function G is

$$G = \frac{1}{3}\mu^3 + q\ \sin(p\ x)Ai(\frac{\mu}{p^{1/3}}),$$

From (3.31) it is obtained that

$$H_1 = -p\ qAi(\widetilde{\mu})'\cos(p\ x)dy, \qquad u = \mu(1 + p\ q\ Ai(\widetilde{\mu})\ \sin(p\ x)). \tag{3.38}$$

$$g_4(\mu) = \frac{u}{\mu}(dx^2 + dy^2) + dz^2 + d\varsigma^2$$

By defining the new function $H(\mu, x, y) = (1 + p\ q\ Ai(\widetilde{\mu})\ \sin(p\ x))$ it is obtained the following G_2 holonomy metric [20]

$$g_7 = \frac{(d\chi - xdz + yd\varsigma)^2}{\mu^2} + \frac{(dv - p\ qAi(\widetilde{\mu})'\cos(p\ x)dy)^2}{H} + \mu\ (\ Hdx^2 + Hdy^2 + dz^2 + d\varsigma^2\ ) + \mu^2 H\ d\mu^2. \tag{3.39}$$

Again the metric is known to have holonomy *exactly* G_2 [20].

The corresponding $Spin(7)$ metrics are obtained from (2.24) and (3.34), and are given by

$$g_8 = \frac{(dz + ydx - zd\varsigma)^2}{(a\mu + b)^{3/2}} + (a\mu + b)^{1/2}g_7 \tag{3.40}$$

being g_7 any of (3.39) or (3.35). The curvature tensor is irreducible for these metrics and the holonomy is not reduced to a subgroup.

4. Two fibrations over the Eguchi-Hanson gravitational instanton

In this section the solutions of the two ramifications described previously when the starting hyperkahler metric is the Eguchi-Hanson gravitational instanton [28] will be constructed. As is well known this is an hyperkahler metric with a the Killing vector which is tri-holomorphic, namely, one satisfying

$$\mathcal{L}_K\omega_1 = \mathcal{L}_K\omega_2 = \mathcal{L}_K\omega_3 = 0.$$

For any metric admitting a tri-holomorphic Killing vector ∂_t there exists a coordinate system in which it takes generically the Gibbons-Hawking form [37]

$$g = V^{-1}(dt + A)^2 + V dx_i dx_j \delta^{ij}, \tag{4.41}$$

with a 1-form A and a function V satisfying the linear system of equations

$$\nabla V = \nabla \times A. \tag{4.42}$$

These metrics are hyper-Kähler with respect to the hyper-Kähler triplet

$$\begin{aligned}
\omega_1 &= (dt + A) \wedge dx - V dy \wedge dz \\
\omega_2 &= (dt + A) \wedge dy - V dz \wedge dx \\
\omega_3 &= (dt + A) \wedge dz - V dx \wedge dy
\end{aligned} \tag{4.43}$$

which is actually t-independent. The Eguchi-Hanson solution corresponds to take two monopoles on the z axis. Without losing generality, it can be considered that the monopoles are located in the positions $(0, 0, \pm c)$. The potentials for this configurations are

$$V = \frac{1}{r_+} + \frac{1}{r_-}, \qquad A = A_+ + A_- = \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right) d\arctan(y/x), \qquad r_\pm^2 = x^2 + y^2 + (z \pm c)^2.$$

This case corresponds to the Eguchi-Hanson instanton, whose metric, in Cartesian coordinates, reads

$$g = \left(\frac{1}{r_+} + \frac{1}{r_-}\right)^{-1} \left(d\tau + \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right) \, d\arctan(y/x) \right)^2 + \left(\frac{1}{r_+} + \frac{1}{r_-} \right) (\; dx^2 + dy^2 + dz^2 \;), \tag{4.44}$$

where $z_\pm = z \pm c$. In order to recognize the Eguchi-Hanson metric in its standard form it is convenient to introduce a new parameter $a^2 = 8c$, and the elliptic coordinates defined by [30]

$$x = \frac{r^2}{8}\sqrt{1 - (a/r)^4} \sin\varphi \cos\theta, \quad y = \frac{r^2}{8}\sqrt{1 - (a/r)^4} \sin\varphi \sin\theta, \quad z = \frac{r^2}{8} \cos\varphi.$$

In this coordinate system it can be checked that

$$r_\pm = \frac{r^2}{8}\left(1 \pm (a/r)^2 \cos\varphi\right), \qquad z_\pm = \frac{r^2}{8}\left(\cos\varphi \pm (a/r)^2\right), \qquad V = \frac{16}{r^2}\left(1 - (a/r)^4 \cos^2\varphi\right)^{-1},$$

$$A = 2\,\left(1 - (a/r)^4 \cos^2\varphi\right)^{-1} \left(1 - (a/r)^4\right)\, \cos\varphi \, d\theta,$$

and, with the help of these expressions, it is found

$$g = \frac{r^2}{4}\left(\, 1 - (a/r)^4 \,\right)\, (\; d\theta + \cos\varphi d\tau\;)^2 + \left(\, 1 - (a/r)^4 \,\right)^{-1} dr^2 + \frac{r^2}{4}\, (\; d\varphi^2 + \sin^2\varphi d\tau \;) \tag{4.45}$$

This is actually a more familiar expression for the Eguchi-Hanson instanton, indeed. Its isometry group is $U(2) = U(1) \times SU(2)/\mathbf{Z}_2$. The holomorphic Killing vector is ∂_τ. This space is asymptotically locally Euclidean (ALE), which means that it asymptotically approaches the Euclidean metric; and therefore the boundary at infinity is locally S^3. However, the situation is rather different in what regards its global properties. This can be seen by defining the new coordinate

$$u^2 = r^2 \left(1 - (a/r)^4\right)$$

for which the metric is rewritten as

$$g = \frac{u^2}{4} \; (\; d\theta + \cos\varphi d\tau \;)^2 + \left(\; 1 + (a/r)^4 \; \right)^{-2} \; du^2 + \frac{r^2}{4} \; (\; d\varphi^2 + \sin^2\varphi d\tau \;). \tag{4.46}$$

The apparent singularity at $r = a$ has been moved now to $u = 0$. Near the singularity, the metric looks like

$$g \simeq \frac{u^2}{4} \; (\; d\theta + \cos\varphi d\tau \;)^2 + \frac{1}{4} du^2 + \frac{a^2}{4} \; (\; d\varphi^2 + \sin^2\varphi d\tau \;),$$

and, at fixed τ and φ, it becomes

$$g \simeq \frac{u^2}{4} \; d\theta^2 + \frac{1}{4} du^2.$$

This expression "locally" looks like the removable singularity of $\mathbf{R}^2$ that appears in polar coordinates. However, for actual polar coordinates, the range of θ covers from 0 to 2π, while in spherical coordinates in $\mathbf{R}^3$, $0 \leq \theta < \pi$. This means that the opposite points on the geometry turn out to be identified and thus the boundary at infinite is the lens space $S^3/\mathbf{Z}_2$.

The known example

The task to find the G_2 metrics (3.25) corresponding to the Eguchi-Hanson metric was already solved in [21]. The result is

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \frac{(d\beta + H_1)^2}{(c\mu^2 + d)^2} + \mu^2 (c\mu + d)^2 \; d\mu^2 + \mu(c\mu + d)\overline{g}_4 \tag{4.47}$$

being $\overline{g}_4$ the Eguchi-Hanson metric described above and the forms H_i are such that $dH_i = \omega_i$, being ω_i given by (4.43). Their explicit expression is [21]

$$H_1 = -x \; d\tau + (\log(r_+ + z_+) + \log(r_- + z_-)) \, dy - 2a \; x \; d\arctan(y/x), \tag{4.48}$$

$$H_2 = +y \; d\tau \; + (\log(r_+ + z_+) + \log(r_- + z_-)) \; \; dx + 2a \; y \; d\arctan(y/x), \tag{4.49}$$

$$H_3 = -z d\tau - a \; (\; r_+ + r_- \;) \; d\arctan(y/x). \tag{4.50}$$

The corresponding uplift to Spin(7) follows immediately from (4.47) and (2.24).

The new examples

Here we find a solution of the second ramification which, to our knowledge, was not worked out before in the literature. It will be convenient to define a new coordinate $\rho = r^2/4$ for which the Eguchi-Hanson metric (4.45) takes the form

$$g_{EH} = \frac{\rho}{\rho^2 - a^2} d\rho^2 + \rho(\sigma_1^2 + \sigma_2^2) + \frac{\rho^2 - a^2}{\rho} \sigma_3^2 \tag{4.51}$$

being

$$\sigma_1 = \frac{1}{2}(\cos\theta d\varphi + \sin\theta\sin\varphi d\tau),$$
$$\sigma_2 = \frac{1}{2}(-\sin\theta d\varphi + \cos\theta\sin\varphi d\tau),$$
$$\sigma_3 = \frac{1}{2}(d\theta + \cos\varphi d\tau).$$

The problem is now to find a function G satisfying the laplace type equation (3.33) and the condition (3.32) on the curved space (4.51). As this function is necessarily defined on a complex sub-manifold it will be necessary to find a complex coordinate system for (4.51). A well known coordinate system is the one which diagonalize the complex structure J_3 corresponding to the Kahler form

$$\omega_3 = e^0 \wedge e^3 - e^1 \wedge e^2,$$

being e^i the basis for (4.51) defined by

$$e^0 = \sqrt{\frac{\rho}{\rho^2 - a^2}}d\rho, \quad e^{1,2} = \sqrt{\rho}\sigma_{1,2}, \quad e^3 = \sqrt{\frac{\rho^2 - a^2}{\rho}}\sigma_3.$$

These coordinates are [29]

$$z_1 = (\rho^2 - a^2)^{1/4}\cos(\frac{\varphi}{2})\exp(i\frac{\theta+\tau}{2}),$$
$$z_2 = (\rho^2 - a^2)^{1/4}\sin(\frac{\varphi}{2})\exp(i\frac{\theta-\tau}{2}). \qquad (4.52)$$

The Eguchi-Hanson metric (4.51) expressed in this coordinates is

$$g_{1\overline{1}} = \frac{\rho^2|z_2|^2 + \eta^2|z_1|^2}{\rho\eta^2},$$
$$g_{2\overline{2}} = \frac{\rho^2|z_1|^2 + \eta^2|z_2|^2}{\rho\eta^2}, \qquad (4.53)$$
$$g_{1\overline{2}} = \frac{\eta^2 - \rho^2}{\rho\eta^2}z_2\overline{z}_1,$$

which is symmetric under the interchange $z_1 \leftrightarrow z_2$. We have denoted $\eta = |z_1|^2 + |z_2|^2 = \sqrt{\rho^2 - a^2}$. The advantage to consider these coordinates becomes clear when calculating the laplacian

$$\Delta_{EH} = \frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}g^{ij}\partial_j),$$

In this coordinates $\det(g) = 1$ and the inverse metric is simply

$$g^{1\overline{1}} = g_{2\overline{2}}, \qquad g^{2\overline{2}} = g_{1\overline{1}}, \qquad g^{1\overline{2}} = -g_{2\overline{1}}.$$

Moreover, after certain calculation is obtained that

$$\partial_1(g^{1\overline{1}}) = -\partial_2(g^{2\overline{1}}), \qquad \partial_{\overline{1}}(g^{\overline{1}1}) = -\partial_{\overline{2}}(g^{\overline{2}1}).$$

The last equalities are more easily checked by using *Mathematica* than by hand. By virtue of them it follows that the action of the laplacian acting on a function $U(z_1, \overline{z}_1)$ is

$$\Delta_{EH} U = g^{1\overline{1}} \partial_1 \partial_{\overline{1}} U. \tag{4.54}$$

For the linearization of the Monge-Ampere equation (3.30) to work the dependence of G with respect to the complex coordinates should be of the form $G = G(w, \overline{w})$ being $w = w(z_1, z_2)$ an holomorphic function of z_1 and z_2. We assume that $w(z_1, z_2) = z_1$ in order to apply the simple expression (4.54) for the laplacian. Then $G = G(\mu, z_1, \overline{z}_1)$ and the equation (3.33) becomes

$$\mu(1 + g^{1\overline{1}} \partial_1 \partial_{\overline{1}} G) = \partial_\mu^2 G. \tag{4.55}$$

But $g^{1\overline{1}}$ is a function of z_2 and G, by assumption, is not. In consequence

$$\partial_1 \partial_{\overline{1}} G = 0,$$

which together with (4.55) implies that

$$\partial_\mu^2 G = \mu.$$

The most general solution G of (4.55) is simply

$$G = \frac{\mu^3}{3} + \mu\left(F(z_1) + \overline{F}(\overline{z}_1)\right) + H(z_1) + \overline{H}(\overline{z}_1), \tag{4.56}$$

being F and H functions on the complex coordinate z_1 and $\overline{F}$ and $\overline{H}$ their complex conjugated.

Once the solution G is found, it is direct to construct the corresponding G_2 and Spin(7) metrics. From (3.28) and (4.56) it follows that

$$u = \frac{\mu}{2}.$$

Also

$$d_4 = \partial_{z_i} dz^i + \partial_{\overline{z}_i} d\overline{z}^i, \qquad d_4^c = i\partial_{z_i} dz^i - i\partial_{\overline{z}_i} d\overline{z}_i$$

and the action of these operators over (4.56) gives

$$d_4 d_4^c G = 0.$$

From (3.27) and the last equation it is seen that

$$\widetilde{\omega}_1(\mu) = \omega_1,$$

and in this case $g_4 = \overline{g}_4$ will coincide with the Eguchi-Hanson metric g_{EH}. From (3.31) it is obtained that

$$H_1 = i\mu(F' dz^1 - \overline{F}' d\overline{z}_1) + i(H' dz^1 - \overline{H}' d\overline{z}_1) = \Im\left((\mu F' + H')dz_1\right) \tag{4.57}$$

which takes real values. Here ' means the derivative with respect to the argument. With all the quantities described above the corresponding G_2 and Spin(7) metrics are easily read from (2.5) and (2.24). By redefining $d\beta$ by a total differential the resulting G_2 metric is

$$g_7 = \frac{\mu^2}{2} d\mu^2 + (d\beta + \Im(F)d\mu)^2 + \frac{(d\alpha + H_2)^2}{\mu^2} + \mu\, g_{EH}, \tag{4.58}$$

and the Spin(7) one is

$$g_8 = \frac{(dz + H_3)^2}{(a\mu + b)^{3/2}} + (a\mu + b)^{1/2}\Big(\frac{\mu^2}{2}d\mu^2 + 2(d\beta + \Im(F)d\mu)^2 + \frac{(d\alpha + H_2)^2}{\mu^2} + \mu\, g_{EH}\Big), \quad (4.59)$$

where H_1 and H_2 are any of (4.48) or (4.49).

It is worthy to check if the metrics constructed above has a signature change problem or not. By defining the proper coordinate $\tau = \mu^2/2$ the G_2 metric (4.58) becomes

$$g_7 = d\tau^2 + \frac{(\tau d\beta + \Im(F)d\tau)^2}{\tau} + \frac{(d\alpha + H_2)^2}{\tau} + \tau^{1/2} g_{EH}, \quad (4.60)$$

and we see from the square root that τ take positive values and there is no change in the signature. Also, by selecting $a = 1$, $b = 0$ in the Spin(7) metric and defining $\eta = \mu^{9/4}$ it is obtained the following expression

$$g_8 = d\eta^2 + \frac{(\eta^{5/9}d\beta + \Im(F)d\eta)^2}{\eta^{8/9}} + \frac{(dz + H_3)^2}{\eta^{2/3}} + \frac{(d\alpha + H_2)^2}{\eta^{2/3}} + \eta^{2/3} g_{EH} \quad (4.61)$$

for (4.59) and in this case the powers of η are all even and there is not signature change problem as η goes from positive to negative values.

The metrics (4.59) and (4.58) depend on an arbitrary choice of an holomorphic function $F(z_1)$. This is the only freedom to construct them. In fact both metrics arise as an R_β-fibration and by reduction along this isometry the same 6 or 7 dimensional metric is obtained. The function F indicate how the uplift of these 6 or 7 dimensional metrics to a G_2 or Spin(7) holomy one is performed. Therefore (4.59) and (4.58) describe an infinite family of special holonomy metrics.

To conclude, we mention that the metrics (4.61) can be extended to a purely geometrical solution of 11 dimensional supergravity of the form

$$g_{11} = g_{(1,2)} + g_8, \quad (4.62)$$

and of course with all the other fields equal to zero. With respect to the isometry generated by z this metric can be rewritten in the IIA form

$$g_{11} = e^{-\phi} g_{10} + e^{2\phi}(dz + H_3)^2, \quad (4.63)$$

where the dilaton ϕ is defined through the relation $e^{2\phi} = \eta^{-2/3}$. The usual reduction to ten dimensions gives

$$g_{IIA} = \eta^{1/3} g_{(1,2)} + \eta^{-1/9} g_7, \qquad F = \omega_3, \quad (4.64)$$

being g_7 the G_2 holonomy metric. Then the seven dimensional internal part of the background (4.64) is *conformal* to the G_2 metric. This is in agreement with the results of [26]. The same consideration will follow for all the Spin(7) metrics constructed here. We would like also to mention that our metrics are probably non conical, therefore not suitable for studying compactifications giving chiral matter or non trivial gauge groups [38]. Nevertheless there exist several contexts in which the conical property does not have special relevance [39]-[57] and in which the construction of these metrics could be of interest.

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References

[1] M. Berger Bull.Soc.Math.Fr. 83, 279.

[2] R. Bryant Annals Math. 126, 525; R.Bryant and S. Salamon Duke Math.J.58 (1989) 829.

[3] G. Gibbons, D. Page and C. Pope Commun.Math.Phys.127 (1990) 529.

[4] D. Joyce J.Diff.Geom. 43, 2; J.Diff.Geom. 43, 329.

[5] A. Kovalev J. Reine Angew. Math. 565 (2003), 125.

[6] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope Nucl.Phys. B620 (2002) 29; J.Geom.Phys. 49 (2004) 350; Annals Phys. 310 (2004) 265; Phys.Rev. D65 (2002) 106004.

[7] Y. Konishi and M. Naka Class.Quant.Grav. 18 (2001) 5521.

[8] G. Etesi Phys.Lett. B521 (2001) 391.

[9] A.Brandhuber and P.Berglund, Nucl.Phys. B641 (2002) 351.

[10] A. Misra J.Math.Phys. 47 (2006) 033504; A. Franzen, P. Kaura, A. Misra and R. Ray Fortsch.Phys. 54 (2006) 207.

[11] I. Bakas, E. Floratos, A. Kehagias, Phys.Lett. B445 (1998) 69.

[12] K.Behrndt,G. Dall'Agata, D. Lust and S. Mahapatra, JHEP 0208 (2002) 027; K.Behrndt, Nucl.Phys. B635 (2002) 158.

[13] Z.W. Chong, M. Cvetic, G.W. Gibbons, H. Lu, C.N. Pope and P. Wagner Nucl.Phys. B638 (2002) 459.

[14] L. Anguelova and C. Lazaroiu JHEP 0301 (2003) 066; JHEP 0210 (2002) 038.

[15] O.P.Santillan Nucl.Phys. B660 (2003) 169.

[16] A.Bilal, J.Derendinger and K.Sfetsos, Nucl.Phys. B628 (2002) 112.

[17] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope Class.Quant.Grav. 20 (2003) 4239.

[18] A. Brandhuber, J. Gomis, S. Gubser and S. Gukov Nucl.Phys. B611 (2001) 179.

[19] G.Gibbons, H. Lu, C. Pope and K. Stelle Nucl.Phys. B **623** (2002) 3.

[20] V. Apostolov and S. Salamon Comm. Math. Phys **246** (2004) 43.

[21] G. Giribet and O. Santillan Commun.Math.Phys. 275 (2007) 373.

[22] O.P.Santillan Phys.Rev. D73 (2006) 126011.

[23] M. Atiyah, J. Maldacena and C. Vafa J.Math.Phys. 42 (2001) 3209.

[24] B. Acharya "On Realising N=1 Super Yang-Mills in M theory" hep-th/0011089.

[25] M.Aganagic and C.Vafa JHEP 0305 (2003) 061; "G_2 Manifolds, Mirror Symmetry and Geometric Engineering" hep-th/0110171.

[26] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello HEP 0209 (2002) 033.

[27] S. Shatashvili and C. Vafa Selecta Math. 1 (1995) 347; G.Papadopoulos and K. Townsend Phys.Lett. B357 (1995) 300.

[28] T. Eguchi and A. Hanson, Phys. Lett. **B74** (1978). 249; Annals Phys. **120** (1979) 82; Gen.Rel.Grav. **11** (1979) 315.

[29] G. Gibbons and C. Pope Comm.Math.Phys **66** (1979) 267.

[30] M. Prasad Phys. Lett B 83 (1979) 310.

[31] S. Chiossi and S. Salamon Proc. conf. Differential Geometry Valencia 2001.

[32] E. Bedford and M. Kalka Commun. Pure. Appl. Math **30** (1977) 543.

[33] C. LeBrun, M. Singer Math. Ann. 300 (1994) 165.

[34] R. Kobayashi Adv. Stud. Pure. Math 18 II (1990) 137.

[35] R. Harvey and H. B. Lawson Acta Math.148 (1982) 47.

[36] K.Becker, M. Becker and A. Strominger Nucl.Phys. B456 (1995) 130.

[37] G. Gibbons and S. Hawking Phys. Lett B 78 (1978) 430.

[38] M. Atiyah and E. Witten JHEP 0606 (2006) 051.

[39] K. Becker JHEP 0105 (2001) 003.

[40] Melanie Becker, Dragos Constantin, S. James Gates, Jr., William D. Linch III, Willie Merrell and J. Phillips Nucl.Phys. B683 (2004) 67; D. Constantin Nucl.Phys. B706 (2005) 221.

[41] K. Becker and M. Becker Nucl.Phys. B477 (1996) 155.

[42] B. Acharya and S. Gukov JHEP 0209 (2002) 047.

[43] A. Belhaj J.Phys. A36 (2003) 4191.

[44] A. Belhaj, M.P. Garcia del Moral, A. Restuccia, A. Segui and J.P. Veiro "The Supermembrane with Central Charges on a G2 Manifold" arXiv:0803.1827; A. Belhaj, L. Boya and A. Segui "Relation Between Holonomy Groups in Superstrings, M and F-theories" arXiv:0806.4265.

[45] U. Gursoy, C. Nunez and M. Schvellinger JHEP 0206 (2002) 015.

[46] K.Becker, M.Becker, D.R.Morrison, H.Ooguri, Y.Oz and Z.Yin Nucl.Phys. B480 (1996) 225.

[47] G.W. Gibbons and G. Papadopoulos Commun.Math.Phys. 202 (1999) 593.

[48] A. Fayazzuddin and T. Hussein Phys.Rev. D75 (2007) 065017.

[49] Sergei Gukov, Shing-Tung Yau, Eric Zaslow "Duality and Fibrations on G_2 Manifolds" hep-th/0203217.

[50] S. Frolov and A. Tseytlin Nucl.Phys. B632 (2002) 69; H. Lu, C.N. Pope, K.S. Stelle, P.K. Townsend JHEP 0410 (2004) 019; JHEP 0507 (2005) 075.

[51] S. Hartnoll and C. Nunez JHEP 0302 (2003) 049.

[52] J.D. Edelstein, A. Paredes and A.V. Ramallo JHEP 0301 (2003) 011.

[53] J. de Boer, P. Medeiros, S. El-Showk and A. Sinkovics JHEP 0802 (2008) 012; Class.Quant.Grav.25 (2008) 075006.

[54] G. Bonelli and M. Zabzine JHEP 0509 (2005) 015; G. Bonelli, A. Tanzini and M. Zabzine Adv.Theor.Math.Phys. 10 (2006) 239; JHEP 0703 (2007) 023.

[55] J. Gutowski and G. Papadopoulos Nucl.Phys. B615 (2001) 237.

[56] B. Acharya "Confining Strings from G_2 holonomy spacetimes" hep-th/0101206.

[57] P. Svrcek and E. Witten JHEP 0606 (2006) 051.