

A PARAMETER–UNIFORM FINITE DIFFERENCE METHOD FOR MULTISCALE SINGULARLY PERTURBED LINEAR DYNAMICAL SYSTEMS

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Abstract. A system of singularly perturbed ordinary differential equations of first order with given initial conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These parameters are assumed to be distinct and they determine the different scales in the solution to this problem. A Shishkin piecewise–uniform mesh is constructed, which is used, in conjunction with a classical finite difference discretization, to form a new numerical method for solving this problem. It is proved that the numerical approximations obtained from this method are essentially first order convergent uniformly in all of the parameters. Numerical results are presented in support of the theory.

Key words. linear dynamical system, multiscale, initial value problem, singularly perturbed, finite difference method, parameter–uniform convergence

AMS subject classifications. 65L05, 65L12, 65L20, 65L70

1. Introduction. We consider the initial value problem for the singularly perturbed system of linear first order differential equations

$$(1.1) \quad E\vec{u}'(t) + A(t)\vec{u}(t) = \vec{f}(t), \quad t \in (0, T], \quad \vec{u}(0) \text{ given}$$

Here \vec{u} is a column n -vector, E and $A(t)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1 \dots n$. For convenience we assume the ordering

$$\varepsilon_1 < \dots < \varepsilon_n$$

These n distinct parameters determine the n distinct scales in this multiscale problem. Cases with some of the parameters coincident are not considered here. We write the problem in the operator form

$$\vec{L}\vec{u} = \vec{f}, \quad \vec{u}(0) \text{ given}$$

where the operator \vec{L} is defined by

$$\vec{L} = ED + A(t) \quad \text{and} \quad D = \frac{d}{dt}$$

We assume that, for all $t \in [0, T]$, the components $a_{ij}(t)$ of $A(t)$ satisfy the inequalities

$$(1.2) \quad a_{ii}(t) > \sum_{\substack{j \neq i \\ j=1}}^n |a_{ij}(t)| \quad \text{for } i = 1, \dots, n, \quad \text{and } a_{ij}(t) \leq 0 \quad \text{when } i \neq j.$$

We take α to be any number such that

$$(1.3) \quad 0 < \alpha < \min_{\substack{t \in (0,1] \\ 1 \leq i \leq n}} \left(\sum_{j=1}^n a_{ij}(t) \right)$$

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We also assume that $T \geq 2 \max_i(\varepsilon_i)/\alpha$, which ensures that the solution domain contains all of the layers. This condition is fulfilled if, for example, $T \geq 2/\alpha$. We introduce the norms $\|\vec{V}\| = \max_{1 \leq k \leq n} |V_k|$ for any n-vector \vec{V} , $\|y\| = \sup_{0 \leq t \leq T} |y(t)|$ for any scalar-valued function y and $\|\vec{y}\| = \max_{1 \leq k \leq n} \|y_k\|$ for any vector-valued function \vec{y} . Throughout the paper C denotes a generic positive constant, which is independent of t and of all singular perturbation and discretization parameters.

The initial value problems considered here arise in many areas of applied mathematics; see for example [1]. Parameter uniform numerical methods for simpler problems of this kind, when all the singular perturbation parameters are equal, were considered in [4]. A special case of the present problem with $n = 3$ was considered in [3]. A general introduction to parameter uniform numerical methods is given in [2] and [7].

2. Analytical results. The operator \vec{L} satisfies the following maximum principle

LEMMA 2.1. *Let $A(t)$ satisfy (1.2) and (1.3). Let $\vec{\psi}(t)$ be any function in the domain of \vec{L} such that $\vec{\psi}(0) \geq 0$. Then $\vec{L}\vec{\psi}(t) \geq 0$ for all $t \in (0, T]$ implies that $\vec{\psi}(t) \geq 0$ for all $t \in [0, T]$.*

Proof: Let i^*, t^* be such that $v_{i^*}(t^*) = \min_{i,t} v_i(t)$ and assume that the lemma is false. Then $v_{i^*}(t^*) < 0$. From the hypotheses we have $t^* \neq 0$ and $v'_{i^*}(t^*) \leq 0$. Thus

$$\begin{aligned} (\vec{L}\vec{v}(t^*))_{i^*} &= \varepsilon_{i^*} v'_{i^*}(t^*) + a_{i^*,i^*}(t^*)v_{i^*}(t^*) + \sum_{j=1, j \neq i^*}^n a_{i^*,j}(t^*)v_j(t^*) \\ &< v_{i^*}(t^*) \sum_{j=1, j \neq i^*}^n a_{i^*,j} < 0 \end{aligned}$$

which contradicts the assumption and proves the result for \vec{L} .

Let $A'(t)$ be any principal sub-matrix of $A(t)$ and \vec{L}' the corresponding operator. To see that any \vec{L}' satisfies the same maximum principle as \vec{L} , it suffices to observe that the elements of $A'(t)$ satisfy *a priori* the same inequalities as those of $A(t)$.

We remark that the maximum principle is not necessary for the results that follow, but it is a convenient tool in the analysis that follows.

LEMMA 2.2. *Let $A(t)$ satisfy (1.2) and (1.3). If $\vec{\psi}(t)$ is any function in the domain of \vec{L} such that $\vec{\psi}(0) \geq 0$, then*

$$\|\vec{\psi}(t)\| \leq \max \left\{ \|\vec{\psi}(0)\|, \frac{1}{\alpha} \|\vec{L}\vec{\psi}\| \right\}, \quad t \in [0, T]$$

Proof: Define the two functions

$$\vec{\psi}^\pm(t) = \max\{\|\vec{\psi}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|\} \vec{e} \pm \vec{v}(t)$$

where $\vec{e} = (1, \dots, 1)'$ is the unit column vector. Using the properties of A it is not hard to verify that $\vec{\psi}^\pm(0) \geq 0$ and $\vec{L}\vec{\psi}^\pm(t) \geq 0$. It follows then from Lemma 2.1 that $\vec{\psi}^\pm(t) \geq 0$ for all $t \in [0, T]$. This completes the proof.

The Shishkin decomposition of the solution \vec{u} of (1) is given by $\vec{u} = \vec{v} + \vec{w}$ where \vec{v} is the solution of $\vec{L}\vec{v} = \vec{f}$ on $(0, T]$ with $\vec{v}(0) = A^{-1}(0)\vec{f}(0)$ and \vec{w} is the solution of $\vec{L}\vec{w} = \vec{0}$ on $(0, T]$ with $\vec{w}(0) = \vec{u}(0) - \vec{v}(0)$. Here \vec{v}, \vec{w} are, respectively, the smooth

and singular components of \vec{u} .

Bounds on the smooth component \vec{v} of \vec{u} and its derivatives are contained in

LEMMA 2.3. *Let $A(t)$ satisfy (1.2) and (1.3). Then there exists a constant C , independent of $\varepsilon, \vec{u}(0)$ and \vec{f} , such that*

$$\|\vec{v}\| \leq C \|\vec{f}\|, \quad \|\vec{v}'\| \leq C(\|\vec{f}\| + \|\vec{f}'\|)$$

and, for all $1 \leq i \leq n$,

$$\|\varepsilon_i v_i''\| \leq C(\|\vec{f}\| + \|\vec{f}'\|)$$

Proof: We introduce the two functions $\vec{\psi}^\pm(t) = C\|\vec{f}\|\vec{e} \pm \vec{v}(t)$ where \vec{e} is the unit column vector. Noting that $\vec{v}(0) = A^{-1}(0)\vec{f}(0)$, it is not hard to see that $\vec{\psi}^\pm(0) \geq 0$ and $\vec{L}\vec{\psi}^\pm(t) \geq 0$. It follows from Lemma 2.1 that $\vec{\psi}^\pm(t) \geq 0$ for all $t \in [0, T]$ and so $\|\vec{v}\| \leq C \|\vec{f}\|$. To estimate the derivative we now define the two functions $\vec{\phi}^\pm(t) = C(\|\vec{f}\| + \|\vec{f}'\|)\vec{e} \pm \vec{v}'(t)$. Since $\vec{v}'(0) = 0$ and $\vec{L}\vec{v}' = \vec{f}' - A'\vec{v}$, it may be verified that $\vec{\phi}^\pm(0) \geq 0$ and $\vec{L}\vec{\phi}^\pm(t) \geq 0$. Again by Lemma 2.1 we have $\vec{\phi}^\pm(t) \geq 0$, which proves the result. Finally, differentiating the equation $\varepsilon_i v_i' + (A\vec{v})_i = f_i$ and using the estimates of \vec{v} and \vec{v}' , we obtain the required bound on $\varepsilon_i v_i''$.

We define the layer functions $B_i, i = 1, \dots, n$, associated with the solution \vec{u} by

$$B_i(t) = e^{-\alpha t/\varepsilon_i}, \quad t \in [0, \infty).$$

The following elementary properties of the layer functions, for all $1 \leq i < j \leq n$, should be noted:

- (i) $B_i(t) < B_j(t)$, for all $t > 0$.
- (ii) $B_i(s) > B_i(t)$, for all $0 \leq s < t < \infty$.
- (iii) $B_i(0) = 1$ and $0 < B_i(t) < 1$ for all $t > 0$.

Bounds on the singular component \vec{w} of \vec{u} and its derivatives are contained in

LEMMA 2.4. *Let $A(t)$ satisfy (1.2) and (1.3). Then there exists a constant C , such that, for each $t \in [0, T]$ and $i = 1, \dots, n$,*

$$\begin{aligned} |w_i(t)| &\leq C B_n(t) \\ |w_i'(t)| &\leq C[\varepsilon_i^{-1} B_i(t) + \dots + \varepsilon_n^{-1} B_n(t)] \\ |\varepsilon_i w_i''(t)| &\leq C[\varepsilon_1^{-1} B_1(t) + \dots + \varepsilon_n^{-1} B_n(t)] \end{aligned}$$

Proof: First we obtain the bound on \vec{w} . We define the two functions $\vec{\psi}^\pm = C B_n \vec{e} \pm \vec{w}$. Then clearly $\vec{\psi}^\pm \geq 0$ and $L(\vec{\psi}^\pm) = CL(B_n \vec{e})$. Then, for $i = 1, \dots, n$, $(L\vec{\psi}^\pm)_i = C(\sum_{j=1}^n \alpha_{i,j} - \alpha_{\varepsilon_i}) B_n > 0$. By the Lemma 2.1 $\vec{\psi}^\pm \geq 0$, which leads to the required bound on \vec{w} .

To establish the bound on \vec{w}' we begin with the n^{th} equation in $\vec{L}\vec{w} = 0$, namely

$$\varepsilon_n w_n' + a_{n,1} w_1 + \dots + a_{n,n} w_n = 0$$

from which the bound for $i = n$ follows. We now bound w_i' for $i = 1, \dots, n-1$. We define $\vec{p} = (w_1, \dots, w_{n-1})$ and, taking the first $n-1$ equations satisfied by \vec{w} , we get $\vec{A}\vec{p} = \vec{g}$ where \vec{A} is the principal sub-matrix obtained from A by deleting its last row and last column and \vec{g} is given by $g_j = -a_{j,n} w_n$, $1 \leq j \leq n$. Using the bounds

already obtained for \vec{w} we see that \vec{g} is bounded by $CB_n(t)$ and its derivative by $C\frac{B_n(t)}{\varepsilon_n}$. The initial condition for \vec{p} is $\vec{p}(0) = \vec{u}(0) - \vec{w}^0(0)$ where \vec{u}^0 is the solution of the reduced problem $\vec{u}^0 = A^{-1}\vec{f}$. The initial condition for \vec{p} is therefore bounded by $C(\|\vec{u}(0)\| + \|\vec{f}(0)\|)$. Decomposing \vec{p} into smooth and singular components we get

$$\vec{p} = \vec{q} + \vec{r}, \quad \vec{p}' = \vec{q}' + \vec{r}'$$

Applying Lemma 2.3 to \vec{q} , from the bounds on the inhomogeneous term \vec{g} and its derivative \vec{g}' , we conclude that $\|\vec{q}'(t)\| \leq C\frac{B_n(t)}{\varepsilon_n}$. We now use mathematical induction. We assume that Lemma 2.4 holds for all systems with $n-1$ equations. Then Lemma 2.4 applies to \vec{r} and so for $i = 1, \dots, n-1$

$$|r'_i(t)| \leq C\left(\frac{B_i(t)}{\varepsilon_i} + \dots + \frac{B_{n-1}(t)}{\varepsilon_{n-1}}\right)$$

Combining the bounds for q_i and r_i we obtain

$$|p'_i(t)| \leq C\left(\frac{B_i(t)}{\varepsilon_i} + \dots + \frac{B_n(t)}{\varepsilon_n}\right)$$

Recalling the definition of \vec{p} this is the same as

$$|w'_i(t)| \leq C\left(\frac{B_i(t)}{\varepsilon_i} + \dots + \frac{B_n(t)}{\varepsilon_n}\right)$$

We have thus proved that Lemma 2.4 holds for our system with n equations. Since Lemma 2.4 is true for a system with one equation, we conclude by mathematical induction that it is true for any system of $n > 1$ equations.

Finally, to estimate the second derivative, we differentiate the i^{th} equation of the system $\vec{L}\vec{w} = 0$ to get

$$\varepsilon_k w''_i = -(A\vec{w}'_i + A'\vec{w})_i$$

and we see that the bound on w''_i follows easily from the bounds on \vec{w} and \vec{w}' .

DEFINITION 2.5. For each $1 \leq i \neq j \leq n$ we define the point $t_{i,j}$ by

$$(2.1) \quad \frac{B_i(t_{i,j})}{\varepsilon_i} = \frac{B_j(t_{i,j})}{\varepsilon_j}$$

In the next lemma it is shown that these points exist, are uniquely defined and have an interesting ordering. Sufficient conditions for them to lie in the domain $[0, T]$ are also provided.

LEMMA 2.6. For all i, j with $1 \leq i < j \leq n$ the points $t_{i,j}$ exist, are uniquely defined and satisfy the following inequalities

$$(2.2) \quad \varepsilon_i^{-1}B_i(t) > \varepsilon_j^{-1}B_j(t) \quad t \in [0, t_{i,j})$$

$$(2.3) \quad \varepsilon_i^{-1}B_i(t) < \varepsilon_j^{-1}B_j(t) \quad t \in (t_{i,j}, \infty)$$

In addition the following ordering holds

$$(2.4) \quad t_{i,j} < t_{i+1,j}, \text{ if } i+1 < j \text{ and } t_{i,j} < t_{i,j+1}, \text{ if } i < j$$

and

$$(2.5) \quad \varepsilon_i \leq \varepsilon_j/2 \text{ implies that } t_{ij} \in (0, T] \text{ for all } i < j$$

Proof: Existence, uniqueness, (2.2) and (2.3) all follow from the observation that for $i < j$ we have $\varepsilon_i < \varepsilon_j$ and the ratio of the two sides of (2.1), namely

$$\frac{B_i(t)}{\varepsilon_i} \frac{\varepsilon_j}{B_j(t)} = \frac{\varepsilon_j}{\varepsilon_i} \exp\left(-\alpha t \left(\frac{1}{\varepsilon_i} - \frac{1}{\varepsilon_j}\right)\right)$$

is monotonically decreasing from $\frac{\varepsilon_j}{\varepsilon_i} > 1$ to 0 as t increases from 0 to ∞ .

Rearranging (2.1) gives

$$t_{i,j} = \frac{\ln\left(\frac{1}{\varepsilon_i}\right) - \ln\left(\frac{1}{\varepsilon_j}\right)}{\alpha\left(\frac{1}{\varepsilon_i} - \frac{1}{\varepsilon_j}\right)}$$

Writing $\varepsilon_k = \exp(-p_k)$ for some $p_k > 0$ and all k gives

$$t_{i,j} = \frac{p_i - p_j}{\alpha(\exp p_i - \exp p_j)}$$

The inequality $t_{i,j} < t_{i+1,j}$ is equivalent to

$$\frac{p_i - p_j}{\exp p_i - \exp p_j} < \frac{p_{i+1} - p_j}{\exp p_{i+1} - \exp p_j}$$

which can be written in the form

$$(p_{i+1} - p_j) \exp(p_i - p_j) + (p_i - p_{i+1}) - (p_i - p_j) \exp(p_{i+1} - p_j) > 0$$

With $a = p_i - p_j$ and $b = p_{i+1} - p_j$ it is not hard to see that $a > b > 0$ and $a - b = p_i - p_{i+1}$. Moreover, the previous inequality is then equivalent to

$$\frac{\exp a - 1}{a} > \frac{\exp b - 1}{b}$$

which is true because $a > b$ and proves the first part of (2.4). The second part is proved by a similar argument.

Finally, to prove (2.5) it suffices to rearrange (2.1) in the form

$$t_{i,j} = \frac{\ln\left(\frac{\varepsilon_j}{\varepsilon_i}\right)}{\alpha\left(\frac{1}{\varepsilon_i} - \frac{1}{\varepsilon_j}\right)}$$

Since $T > \frac{2}{\alpha}$ and $\varepsilon_i \leq \frac{\varepsilon_j}{2}$ it follows that $\ln\left(\frac{\varepsilon_j}{\varepsilon_i}\right) \leq \frac{\varepsilon_j}{\varepsilon_i}$ and $t_{i,j} \in (0, T]$. //

3. The discrete problem. We construct a piecewise uniform mesh with N mesh-intervals and mesh-points $\{t_i\}_{i=0}^N$ by dividing the interval $[0, T]$ into $n + 1$ sub-intervals as follows

$$[0, T] = [0, \sigma_1] \cup (\sigma_1, \sigma_2] \cup \dots \cup (\sigma_{n-1}, \sigma_n] \cup (\sigma_n, T]$$

Then, on the sub-interval $[0, \sigma_1]$, a uniform mesh with $\frac{N}{2^n}$ mesh-intervals is placed, and similarly on $(\sigma_i, \sigma_{i+1}]$, $1 \leq i \leq n - 1$, a uniform mesh of $\frac{N}{2^{n-i+1}}$ mesh-intervals and on $(\sigma_n, T]$ a uniform mesh of $\frac{N}{2}$ mesh-intervals. In practice it is convenient to take $N = 2^n k$ where k is some positive power of 2. The n transition points between the uniform meshes are defined by

$$\sigma_i = \min\left\{\frac{\sigma_{i+1}}{2}, \frac{\varepsilon_i}{\alpha} \ln N\right\}$$

for $i = 1, \dots, n - 1$ and

$$\sigma_n = \min\left\{\frac{T}{2}, \frac{\varepsilon_n}{\alpha} \ln N\right\}$$

Clearly

$$0 < \sigma_1 < \dots < \sigma_n \leq \frac{T}{2}$$

This construction leads to a class of 2^n possible Shishkin piecewise uniform meshes $M_{\vec{b}}$, where \vec{b} denotes an n -vector with $b_i = 0$ if $\sigma_i = \frac{\sigma_{i+1}}{2}$ and $b_i = 1$ otherwise. Writing $\delta_j = t_j - t_{j-1}$ we remark that, on any $M_{\vec{b}}$, we have

$$\delta_j \leq CN^{-1}, \quad 1 \leq j \leq N$$

and

$$\sigma_i \leq C\varepsilon_i \ln N, \quad 1 \leq i \leq n$$

On any $M_{\vec{b}}$ we now consider the discrete solutions defined by the backward Euler finite difference scheme

$$ED^- \vec{U} + A(t) \vec{U} = \vec{f}, \quad \vec{U}(0) = \vec{u}(0)$$

or in operator form

$$\vec{L}^N \vec{U} = \vec{f}, \quad \vec{U}(0) = \vec{u}(0)$$

where

$$\vec{L}^N = ED^- + A(t)$$

and D^- is the backward difference operator

$$D^- \vec{U}(t_j) = \frac{\vec{U}(t_j) - \vec{U}(t_{j-1})}{\delta_j}$$

We have the following discrete maximum principle analogous to the continuous case.

LEMMA 3.1. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for any mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi}(0) \geq \vec{0}$ and $\vec{L}^N \vec{\Psi}(t_j) \geq \vec{0}$ for $1 \leq j \leq N$, imply that $\vec{\Psi}(t_j) \geq \vec{0}$ for $0 \leq j \leq N$.*

Proof: Let i^*, j^* be such that $V_{i^*}(t_{j^*}) = \min_{i,j} V_i(t_j)$ and assume that the lemma is false. Then $V_{i^*}(t_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0$ and $V_{i^*}(t_{j^*}) - V_{i^*}(t_{j^*-1}) \leq 0$. Thus

$$\begin{aligned} (L^N \vec{V}(t_{j^*}))_{i^*} &= \varepsilon_{i^*} \frac{V_{i^*}(t_{j^*}) - V_{i^*}(t_{j^*-1})}{\delta_{j^*}} + a_{i^*,i^*}(t_{j^*})V_{i^*}(t_{j^*}) + \sum_{k=1, k \neq i^*}^n a_{i^*,k}(t_{j^*})V_k(t_{j^*}) \\ &< V_{i^*}(t_{j^*}) \sum_{k=1, k \neq i^*}^n a_{i^*,k} < 0 \end{aligned}$$

which contradicts the assumption and completes the proof.

An immediate consequence of this is the following discrete stability result.

LEMMA 3.2. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for any mesh function $\vec{\Psi}$,*

$$\|\vec{\Psi}(t_j)\| \leq \max \left\{ \|\vec{\Psi}(0)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \right\}, 0 \leq j \leq N$$

Proof: Define the two functions

$$\vec{\Psi}^\pm(t) = \max\{\|\vec{V}(0)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{V}\|\} \vec{e} \pm \vec{V}(t)$$

where $\vec{e} = (1, \dots, 1)$ is the unit vector. Using the properties of A it is not hard to verify that $\vec{\Psi}^\pm(0) \geq 0$ and $\vec{L}^N \vec{\Psi}^\pm(t_j) \geq 0$. It follows then from Lemma 3.1 that $\vec{\Psi}^\pm(t_j) \geq 0$ for all $0 \leq j \leq N$. This completes the proof.

4. The local truncation error. From Lemma 3.2, we see that in order to bound the error $\|\vec{U} - \vec{u}\|$ it suffices to bound $\vec{L}^N(\vec{U} - \vec{u})$. But this expression satisfies

$$\begin{aligned} \vec{L}^N(\vec{U} - \vec{u}) &= \vec{L}^N(\vec{U}) - \vec{L}^N(\vec{u}) = \vec{f} - \vec{L}^N(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^N(\vec{u}) = \\ &(\vec{L} - \vec{L}^N)\vec{u} = -E(D^- - D)\vec{u} \end{aligned}$$

which is the local truncation of the first derivative. We have

$$E(D^- - D)\vec{u} = E(D^- - D)\vec{v} + E(D^- - D)\vec{w}$$

and so, by the triangle inequality,

$$(4.1) \quad \|\vec{L}^N(\vec{U} - \vec{u})\| \leq \|E(D^- - D)\vec{v}\| + \|E(D^- - D)\vec{w}\|$$

Thus, we can treat the smooth and singular components of the local truncation error separately. In view of this we note that, for any smooth function ψ , we have the following two distinct estimates of the local truncation error of its first derivative

$$(4.2) \quad |(D^- - D)\psi(t_j)| \leq \max_{s \in [t_{j-1}, t_j]} |\psi''(s)| \frac{\delta_j}{2}$$

$$(4.3) \quad |(D^- - D)\psi(t_j)| \leq 2 \max_{s \in [t_{j-1}, t_j]} |\psi'(s)|$$

5. Error estimate. Here we establish the error estimate by generalizing the approach based on Shishkin decompositions used in [3]. For a reaction-diffusion boundary value problem in the special case $n = 2$ a parameter uniform numerical method was analyzed in [6] by a similar technique and in the general case in [5] using discrete Green's functions.

We estimate the smooth component of the local truncation error in the following lemma.

LEMMA 5.1. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, we have*

$$|\varepsilon_i(D^- - D)v_i(t_j)| \leq CN^{-1}$$

Proof: Using (4.3), Lemma 2.3 and the geometry of the mesh we obtain

$$|\varepsilon_i(D^- - D)v_i(t_j)| \leq C\delta_j \max_{s \in I_j} |\varepsilon_i v_i''(s)| \leq C\delta_j \leq CN^{-1}$$

For the singular component we obtain a similar estimate, but we must distinguish between the different types of mesh. We need the following preliminary lemmas.

LEMMA 5.2. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$ on each mesh $M_{\bar{b}}$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \leq C \frac{\delta_j}{\varepsilon_1}$$

Proof: From (4.3) and Lemma 2.4, we have

$$\begin{aligned} |\varepsilon_i(D^- - D)w_i(t_j)| &\leq C\delta_j \max_{s \in I_j} |\varepsilon_i w_i''(s)| \\ &\leq C\delta_j [\varepsilon_1^{-1} B_1(t) + \dots + \varepsilon_n^{-1} B_n(t)] \\ &\leq C \frac{\delta_j}{\varepsilon_1} \end{aligned}$$

In what follows we make use of second degree polynomials of the form

$$p_{i;\theta} = \sum_{k=0}^2 \frac{(t - t_\theta)^k}{k!} w_i^{(k)}(t_\theta)$$

where θ denotes a pair of integers, for example $\theta = 2, 3$.

LEMMA 5.3. *Let $A(t)$ satisfy (1.2) and (1.3) and consider, for $k = 1, \dots, n-1$, any mesh $M_{\bar{b}}$ with $b_k = 1$. Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, there exists a decomposition*

$$w_i = \sum_{p=1}^{k+1} w_{i,p}$$

for which we have the following estimates for $q = 1, \dots, k$

$$|\varepsilon_i w'_{i,q}(t)| \leq CB_q(t), \quad |\varepsilon_i w''_{i,q}(t)| \leq C \frac{B_q(t)}{\varepsilon_q}$$

and

$$|\varepsilon_i w''_{i,k+1}(t)| \leq C \sum_{p=k+1}^n \frac{B_p(t)}{\varepsilon_p}$$

Furthermore

$$|\varepsilon_i (D^- - D)w_i(t_j)| \leq C(B_k(t_{j-1}) + \frac{\delta_j}{\varepsilon_{k+1}})$$

Proof: Since $b_k = 1$ we have $\varepsilon_k \leq \varepsilon_{k+1}/2$, so $t_{k,k+1} \in (0, T]$ and we can define the decomposition

$$w_i = \sum_{p=1}^{k+1} w_{i,p}$$

where the components of the decomposition are defined by

$$w_{i,k+1} = p_{i,k,k+1} \text{ on } [0, t_{k,k+1})$$

$$w_{i,k+1} = w_i \text{ otherwise}$$

and, for $k \geq m \geq 2$,

$$w_{i,m} = p_{i,m-1,m} \text{ on } [0, t_{m-1,m})$$

$$w_{i,m} = w_i - \sum_{p=m+1}^{k+1} w_{i,p} \text{ otherwise}$$

and

$$w_{i,1} = w_i - \sum_{p=2}^{k+1} w_{i,p} \text{ on } [0, T]$$

We note that, for $1 \leq m \leq k$, $w_{i,m} = 0$ on $[t_{m,m+1}, T]$.

For the bounds on the second derivatives we observe that:

in $[t_{k,k+1}, T]$, using Lemma 2.4 and $t \geq t_{k,k+1}$, we obtain

$$|\varepsilon_i w''_{i,k+1}(t)| = |\varepsilon_i w''_i(t)| \leq C \sum_{p=1}^n \frac{B_p(t)}{\varepsilon_p} \leq C \sum_{p=k+1}^n \frac{B_p(t)}{\varepsilon_p}$$

in $[0, t_{k,k+1}]$, using Lemma 2.4 and $t \leq t_{k,k+1}$, we obtain

$$|\varepsilon_i w''_{i,k+1}(t)| = |\varepsilon_i w''_i(t_{k,k+1})| \leq \sum_{p=1}^n \frac{B_p(t_{k,k+1})}{\varepsilon_p} \leq \sum_{p=k+1}^n \frac{B_p(t_{k,k+1})}{\varepsilon_p} \leq \sum_{p=k+1}^n \frac{B_p(t)}{\varepsilon_p}$$

and for $2 \leq m \leq k$ taken in the reverse order $m = k; \dots 2$ we see that

$$\text{in } [t_{m,m+1}, T], w''_{i,m} = 0$$

in $[t_{m-1,m}, t_{m,m+1}]$, using Lemma 2.4, we obtain

$$|\varepsilon_i w''_{i,m}(t)| \leq |\varepsilon_i w''_i(t)| + \sum_{p=m+1}^{k+1} |\varepsilon_i w''_{i,p}(t)| \leq C \sum_{p=1}^n \frac{B_p(t)}{\varepsilon_p} \leq C \frac{B_m(t)}{\varepsilon_m}$$

in $[0, t_{m-1,m}]$, using Lemma 2.4 and $t \leq t_{m-1,m}$, we obtain

$$|\varepsilon_i w''_{i,m}(t)| = |\varepsilon_i w''_i(t_{m-1,m})| \leq C \sum_{p=1}^n \frac{B_p(t_{m-1,m})}{\varepsilon_p} \leq C \frac{B_m(t_{m-1,m})}{\varepsilon_m} \leq C \frac{B_m(t)}{\varepsilon_m}$$

in $[t_{1,2}, T]$, $w''_{i,1} = 0$

in $[0, t_{1,2}]$, using Lemma 2.4,

$$|\varepsilon_i w''_{i,1}(t)| \leq |\varepsilon_i w''_i(t)| + \sum_{p=2}^{k+1} |\varepsilon_i w''_{i,p}(t)| \leq C \sum_{p=1}^n \frac{B_p(t)}{\varepsilon_p} \leq C \frac{B_1(t)}{\varepsilon_1}$$

For the bounds on the first derivatives we observe that for $1 \leq j \leq k$:

in $[t_{j,j+1}, T]$, $w'_{i,j} = 0$

in $[0, t_{j,j+1}]$ $\int_t^{t_{j,j+1}} \varepsilon_i w''_{i,j}(s) ds = \varepsilon_i w'_{i,j}(t_{j,j+1}) - \varepsilon_i w'_{i,j}(t) = -\varepsilon_i w'_{i,j}(t)$ so

$$|\varepsilon_i w'_{i,j}(t)| \leq \int_t^{t_{j,j+1}} |\varepsilon_i w''_{i,j}(s)| ds \leq \frac{C}{\varepsilon_j} \int_t^{t_{j,j+1}} B_j(s) ds \leq C B_j(t)$$

Finally, since

$$|\varepsilon_i (D^- - D)w_i(t_j)| \leq \sum_{m=1}^{n-1} |\varepsilon_i (D^- - D)w_{i,m}(t_j)| + |\varepsilon_i (D^- - D)w_{i,n}(t_j)|$$

using (4.3) on the last term and (4.2) on all other terms on the right hand side, we obtain

$$|\varepsilon_i (D^- - D)w_i(t_j)| \leq C \left(\sum_{m=1}^{n-1} \max_{s \in I_j} |\varepsilon_i w'_{i,m}(s)| + \delta_j \max_{s \in I_j} |\varepsilon_i w''_{i,n}(s)| \right)$$

The desired result follows by applying the bounds on the derivatives in the first part of this lemma.

LEMMA 5.4. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, on each mesh $M_{\vec{v}}$, we have the estimate*

$$|\varepsilon_i (D^- - D)w_i(t_j)| \leq C B_n(t_{j-1})$$

Proof: From (4.2) and Lemma 2.4, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, we have

$$\begin{aligned} |\varepsilon_i (D^- - D)w_i(t_j)| &\leq C \max_{s \in I_j} |\varepsilon_i w'_i(s)| \\ &\leq C \varepsilon_i [\varepsilon_i^{-1} B_i(t) + \dots + \varepsilon_n^{-1} B_n(t)] \\ &\leq C B_n(t_{j-1}) \end{aligned}$$

Using the above preliminary lemmas on appropriate subintervals we obtain the desired estimate of the singular component of the local truncation error in the following.

LEMMA 5.5. *Let $A(t)$ satisfy (1.2) and (1.3). Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} \ln N$$

Proof: We consider each subinterval separately.

In the subinterval $(0, \sigma_1]$ we have $\delta_j \leq CN^{-1}\sigma_1$.

On any mesh $M_{\vec{b}}$, using Lemma 5.2, we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} \frac{\sigma_1}{\varepsilon_1} \leq CN^{-1} \ln N$.

In the subinterval $(\sigma_1, \sigma_2]$ we have $\delta_j \leq CN^{-1}\sigma_2$.

On any mesh $M_{\vec{b}}$ with $b_1 = 0$, we have $\sigma_2 = 2\sigma_1$. Using Lemma 5.2 we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} \frac{\sigma_2}{\varepsilon_1} \leq CN^{-1} \frac{\sigma_1}{\varepsilon_1} \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_1 = 1$, we have $\sigma_1 = \frac{\varepsilon_1}{\alpha} \ln N$. Using Lemma 5.3 with $k = 1$ we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq C(B_1(\sigma_1) + N^{-1} \frac{\sigma_2}{\varepsilon_2}) \leq CN^{-1} \ln N$.

In a general subinterval $(\sigma_m, \sigma_{m+1}]$ $2 \leq m \leq n - 1$, we have $\delta_j \leq CN^{-1}\sigma_{m+1}$.

On any mesh $M_{\vec{b}}$ with $b_q = 0$, $q = 1, \dots, m$, we have $\sigma_{m+1} = C\sigma_1$. Using Lemma 5.2 we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} \frac{\sigma_{m+1}}{\varepsilon_1} \leq CN^{-1} \frac{\sigma_1}{\varepsilon_1} \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_1 = 1, b_q = 0, q = 2, \dots, m$, we have $\sigma_1 = \frac{\varepsilon_1}{\alpha} \ln N$, $\sigma_{m+1} = C\sigma_2$. Using Lemma 5.3 with $k = 1$ we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq C(B_1(\sigma_m) + N^{-1} \frac{\sigma_{m+1}}{\varepsilon_2}) \leq C(B_1(\sigma_1) + N^{-1} \frac{\sigma_2}{\varepsilon_2}) \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_k = 1, b_q = 0, q = k + 1, \dots, m$, we have $\sigma_k = \frac{\varepsilon_k}{\alpha} \ln N$, $\sigma_{m+1} = C\sigma_{k+1}$. Using Lemma 5.3 with general k we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq C(B_k(\sigma_m) + N^{-1} \frac{\sigma_{m+1}}{\varepsilon_{k+1}}) \leq C(B_k(\sigma_k) + N^{-1} \frac{\sigma_{k+1}}{\varepsilon_{k+1}}) \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_m = 1$, we have $\sigma_m = \frac{\varepsilon_m}{\alpha} \ln N$. Using Lemma 5.4 we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} B_m(\sigma_m) \leq CN^{-1} \ln N$.

In the subinterval $(\sigma_n, T]$ we have $\delta_j \leq CN^{-1}$.

On any mesh $M_{\vec{b}}$ with $b_q = 0, q = 1, \dots, n$, we have $1/\varepsilon_1 \leq C \ln N$. Using Lemma 5.2 we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1}/\varepsilon_1 \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_1 = 1, b_q = 0, q = 2, \dots, n$, we have $\sigma_1 = \frac{\varepsilon_1}{\alpha} \ln N$, $1/\varepsilon_2 \leq C \ln N$. Using Lemma 5.3 with $k = 1$ we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq C(B_1(\sigma_n) + N^{-1}/\varepsilon_2) \leq C(B_1(\sigma_1) + N^{-1}/\varepsilon_2) \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_k = 1, b_q = 0, q = k + 1, \dots, n, 2 \leq k \leq n - 1$, we have $\sigma_k = \frac{\varepsilon_k}{\alpha} \ln N$, $1/\varepsilon_{k+1} \leq C \ln N$. Using Lemma 5.3 with general k we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq C(B_k(\sigma_n) + N^{-1}/\varepsilon_{k+1}) \leq C(B_k(\sigma_k) + N^{-1}/\varepsilon_{k+1}) \leq CN^{-1} \ln N$.

On any mesh $M_{\vec{b}}$ with $b_n = 1$, we have $\sigma_n = \frac{\varepsilon_n}{\alpha} \ln N$. Using Lemma 5.4 we get $|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} B_n(\sigma_n) \leq CN^{-1} \ln N$.

It is not hard to verify that on each of the $n + 1$ subintervals we have obtained the required estimate for all of the 2^n possible meshes. This completes the proof of the lemma.

Let \vec{u} denote the exact solution of (1.1) and \vec{U} the discrete solution. Then, the main result of this paper is the following ε -uniform error estimate

THEOREM 5.6. *Let $A(t)$ satisfy (1.2) and (1.3). Then there exists a constant C such that*

$$\|\vec{U} - \vec{u}\| \leq CN^{-1} \ln N$$

for all $N > 1$

Proof: This follows immediately by applying Lemmas 5.1 and 5.5 to (4.1) and using Lemma 3.2.

TABLE 6.1
 Values of D_ε^N , D^N , p^N , p^* , and $C_{p^*}^N$ for various ε_1 and N with fixed $\varepsilon_2 = 2^{-6}$, $\varepsilon_3 = 2^{-4}$

Number of mesh points N									
ε_1	128	256	512	1024	2048	4096	8192	16384	32768
2^{-7}	0.135-2	0.832-3	0.485-3	0.276-3	0.154-3	0.853-4	0.466-4	0.253-4	0.136-4
2^{-11}	0.195-2	0.118-2	0.688-3	0.391-3	0.215-3	0.117-3	0.625-4	0.332-4	0.175-4
2^{-15}	0.230-2	0.136-2	0.808-3	0.469-3	0.262-3	0.145-3	0.792-4	0.430-4	0.232-4
2^{-19}	0.232-2	0.138-2	0.810-3	0.476-3	0.266-3	0.147-3	0.805-4	0.436-4	0.235-4
2^{-23}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-27}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-31}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-35}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-39}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-43}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
D^N	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
p^N	0.753+0	0.767+0	0.765+0	0.842+0	0.855+0	0.867+0	0.882+0	0.891+0	
$C_{0.753}^N$	0.221+0	0.221+0	0.219+0	0.217+0	0.204+0	0.190+0	0.176+0	0.161+0	0.146+0
Computed order of ε_1 -uniform convergence = 0.753									
Computed ε_1 -uniform error constant = 0.221									

6. Numerical results. The above numerical method is applied to the following singularly perturbed initial value problem

$$(6.1) \quad \varepsilon_1 u_1'(t) + 4u_1(t) - u_2(t) - u_3(t) = t$$

$$(6.2) \quad \varepsilon_2 u_2'(t) - u_1(t) + 4u_2(t) - u_3(t) = 1$$

$$(6.3) \quad \varepsilon_3 u_3'(t) - u_1(t) - u_2(t) + 4u_3(t) = 1 + t^2$$

for $t \in (0, 1]$ and $\vec{u}(0) = 0$. For various values of ε_1 , fixed values $\varepsilon_2 = 2^{-6}$, $\varepsilon_3 = 2^{-4}$ and $N = 2^r$, $r = 7, \dots, 15$, the computed order of $\vec{\varepsilon}_1$ -uniform convergence and the computed $\vec{\varepsilon}_1$ -uniform error constant are found using the general methodology from [2],[7]. The results, presented in Table 1 below, exhibit the behaviour expected from an $\vec{\varepsilon}_1$ -uniform method. Similar numerical experiments illustrate separate $\vec{\varepsilon}_2$ - and $\vec{\varepsilon}_3$ uniform behaviour.

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