A Parameter–Uniform Finite Difference Method for a Singularly Perturbed Initial Value Problem: a Special Case

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Summary. A system of singularly perturbed ordinary differential equations of first order with given initial conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These parameters are not necessarily equal. The components of the solution exhibit overlapping layers. A Shishkin piecewise–uniform mesh is constructed, which is used, in conjunction with a classical finite difference discretisation, to form a new numerical method for solving this problem. It is proved, in a special case, that the numerical approximations obtained from this method are essentially first order convergent uniformly in all of the parameters. Numerical results are presented in support of the theory.

1 Introduction

We consider the initial value problem for the singularly perturbed system of linear first order differential equations

$$E\mathbf{u}'(t) + A(t)\mathbf{u}(t) = \mathbf{f}(t), \quad t \in (0,T], \quad \mathbf{u}(0) \ \text{given} \tag{1}$$

Here $\mathbf{u}$ is a column n-vector, E and $A(t)$ are $n\times n$ matrices, $E = diag(\varepsilon_1, \ \ldots, \ \varepsilon_n)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ \ldots, \ \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1 \ldots n$. We assume that $T \geq 2\max_i(\varepsilon_i)/\alpha$, which ensures that we are solving over a domain that includes all of the layers. For this it suffices to take $T \geq 2/\alpha$. For convenience we assume the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n$$

all of the inequalities being strict. We write the problem in the operator form

$$\mathbf{L}\mathbf{u} = \mathbf{f}, \quad \mathbf{u}(0) \text{ given}$$

where the operator $\mathbf{L}$ is defined by

$$\mathbf{L} = ED + A(t) \text{ and } D = \frac{d}{dt}$$

We assume that, for all $t \in [0, 1]$, the functions a_{ij} satisfy the inequalities

$$a_{ii}(t) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(t)| \text{ for } i = 1, \ldots, n, \text{ and } a_{ij}(t) \leq 0 \text{ when } i \neq j.$$

We take α to be any number such that

$$0 < \alpha < \min_{\substack{t \in (0,1] \\ i=1, \ldots, n}} (\sum_{j=1}^{n} a_{ij}(t))$$

The analogous problem to (1), with all the ε_i equal, was treated in [2].

2 Analytical results

The operator $\mathbf{L}$ satisfies the following maximum principle

Lemma 1. *Let the above assumptions on the matrix $A(t)$ hold. Let $\boldsymbol{\psi}(t)$ be any function in the domain of $\mathbf{L}$ such that $\boldsymbol{\psi}(0) \geq 0$. Then $\mathbf{L}\boldsymbol{\psi}(t) \geq 0$ for all $t \in (0, T]$ imples that $\boldsymbol{\psi}(t) \geq 0$ for all $t \in [0, T]$.*

We remark that the maximum principle is not necessary for the results that follow, but it is a convenient sufficient condition. An immediate consequence of it is the following stability result.

Lemma 2. *Let the above assumptions on the matrix $A(t)$ hold. If $\boldsymbol{\psi}(t)$ is any function in the domain of $\mathbf{L}$ such that $\boldsymbol{\psi}(0) \geq 0$, then*

$$\| \boldsymbol{\psi}(t) \| \leq C \max \left\{ \| \boldsymbol{\psi}(0) \|, \frac{1}{\alpha} \| \mathbf{L}\boldsymbol{\psi}(t) \| \right\}, \qquad t \in [0, T]$$

where C is a constant independent of t and $\boldsymbol{\varepsilon}$.

The Shishkin decomposition of the solution $\mathbf{u}$ of (1) is given by $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is the solution of $\mathbf{L}\mathbf{v} = \mathbf{f}$ on $(0, T]$ with $\mathbf{v}(0) = A^{-1}(0)\mathbf{f}(0)$ and $\mathbf{w}$ is the solution of $\mathbf{L}\mathbf{w} = \mathbf{0}$ on $(0, T]$ with $\mathbf{w}(0) = \mathbf{u}(0) - \mathbf{v}(0)$. Here $\mathbf{v}$, $\mathbf{w}$ are, respectively, the smooth and singular components of $\mathbf{u}$.
Bounds on the smooth component and its derivatives are contained in

Lemma 3. *There exists a constant C, independent of $\boldsymbol{\varepsilon}$, such that for each $i = 1, \ldots, n$, $\| v_i^{(k)} \| \leq C$ for $k = 0, 1$ and $\| \varepsilon_i v_i'' \| \leq C$.*

We define the layer functions $B_i, i = 1, \ldots, n$, associated with the solution $\mathbf{u}$ by

$$B_i(t) = e^{-\alpha t/\varepsilon_i}, \; t \in [0, \infty).$$

Some properties of the layer functions are given in

Lemma 4. *Let $1 \leq i < j \leq n$ and $0 \leq s < t < \infty$. Then*

$$B_i(t) < B_j(t), \; \textit{for all} \quad t > 0$$

$$B_i(s) > B_i(t), \quad \textit{for all} \quad 0 \leq s < t \leq 1$$

$$B_i(0) = 1 \quad \textit{and} \quad 0 < B_i(t) \leq 1 \quad \textit{for all} \quad t$$

$$B_i(t) \; \textit{is a monotonically decreasing function of} \quad t \in [0, \infty).$$

Bounds on the singular component and its derivatives are contained in

Lemma 5. *There exists a constant C, independent of $\boldsymbol{\varepsilon}$, such that, for each $t \in [0, T]$ and $i = 1, \ldots, n$,*

$$\begin{aligned} |w_i(t)| &\leq CB_n(t) \\ |w_i'(t)| &\leq C\big[\varepsilon_i^{-1}B_i(t) + \cdots + \varepsilon_n^{-1}B_n(t)\big] \\ |\varepsilon_i w_i''(t)| &\leq C\left[\varepsilon_1^{-1}B_1(t) + \cdots + \varepsilon_n^{-1}B_n(t)\right] \end{aligned}$$

For each $i \neq j$ we now define the point $t_{i,j}$ by

$$\frac{B_i(t_{i,j})}{\varepsilon_i} = \frac{B_j(t_{i,j})}{\varepsilon_j}$$

It is easy to see that this point exists and is unique for each i and j, since for $i < j$ we have $\varepsilon_i < \varepsilon_j$ and the ratio of the two sides of this equation, namely

$$\frac{B_i(t)}{\varepsilon_i}\frac{\varepsilon_j}{B_j(t)} = \frac{\varepsilon_j}{\varepsilon_i}\exp\left(-\alpha t\left(\frac{1}{\varepsilon_i} - \frac{1}{\varepsilon_j}\right)\right)$$

is monotonically decreasing from $\frac{\varepsilon_j}{\varepsilon_i} > 1$ to 0 as t increases from 0 to ∞. Also, the following inequalities hold, for all i, j with $1 \leq i < j \leq n$

$$\varepsilon_i^{-1}B_i(t) > \varepsilon_j^{-1}B_j(t) \qquad \text{on } [0, t_{ij}) \tag{2}$$

$$\varepsilon_i^{-1}B_i(t) < \varepsilon_j^{-1}B_j(t) \qquad \text{on } (t_{ij}, \infty) \tag{3}$$

and if $\varepsilon_i \leq \varepsilon_j/2$ then $t_{ij} \in (0, T]$.

Lemma 6. *The points $t_{i,j}$ satisfy the following inequalities*

$$t_{i,j} < t_{i+1,j}, \text{ if } i + 1 < j$$

and

$$t_{i,j} < t_{i,j+1}, \text{ if } i < j$$

3 The discrete problem

We define n transition points between uniform meshes of different mesh size by

$$\sigma_i = \min\{\frac{\sigma_{i+1}}{2}, \frac{\varepsilon_i}{\alpha} \ln N\}$$

for $i = 1, \ldots, n-1$ and

$$\sigma_n = \min\{\frac{1}{2}, \frac{\varepsilon_n}{\alpha} \ln N\}$$

Clearly

$$0 < \sigma_1 < \cdots < \sigma_n < 1$$

We divide the interval $[0, T]$ into $n+1$ subintervals using these n transition points

$$[0, 1] = [0, \sigma_1] \cup [\sigma_1, \sigma_2] \cup \ldots [\sigma_{n-1}, \sigma_n] \cup [\sigma_n, 1]$$

We construct a piecewise uniform mesh on $[0, T]$, consisting of N subintervals and $N+1$ mesh points t_j, by placing a uniform mesh with $\frac{N}{2}$ subintervals on $[\sigma_n, 1]$ and uniform meshes with $\frac{N}{2n}$ subintervals on each of $[0, \sigma_1]$ and $[\sigma_i, \sigma_{i+1}], i = 1, \ldots, n-1$. Using these we obtain 2^n Shishkin piecewise uniform meshes $M_{\mathbf{b}}$, where $\mathbf{b}$ denotes an n–vector with b_i equal to either 0 or 1 depending on whether the left–hand or right–hand value of σ_i is used. We remark that, on any such mesh at any point t_j, we have

$$|t_j - t_{j-1}| \leq CN^{-1}$$

and for $i = 1, \ldots, n$

$$\sigma_i \leq C\varepsilon_i \ln N$$

On these meshes we now consider the discrete solutions defined by the backward Euler finite difference scheme

$$ED^- \mathbf{U} + A(t)\mathbf{U} = \mathbf{f}, \qquad \mathbf{U}(0) = \mathbf{u}(0)$$

or in operator form

$$\mathbf{L}^N \mathbf{U} = \mathbf{f}, \qquad \mathbf{U}(0) = \mathbf{u}(0)$$

where

$$\mathbf{L}^N = ED^- + A(t)$$

and D^- is the backward difference operator

$$D^- \mathbf{U}(t_j) = \frac{\mathbf{U}(t_j) - \mathbf{U}(t_{j-1})}{t_j - t_{j-1}}$$

We have the following discrete maximum principle analogous to the continuous case

Lemma 7. *Let the above assumptions on the matrix $A(t)$ hold. Then, for any mesh function $\mathbf{\Psi}$, the inequalities $\mathbf{\Psi}(0) \geq \mathbf{0}$ and $\mathbf{L}^N\mathbf{\Psi}(t_j) \geq \mathbf{0}$ for $1 \leq j \leq N$, imply that $\mathbf{\Psi}(t_j) \geq \mathbf{0}$ for $0 \leq j \leq N$.*

An immediate consequence of this is the following discrete stability result.

Lemma 8. *Let the above assumptions on the matrix $A(t)$ hold. Then, for any mesh function $\mathbf{\Psi}$,*

$$\| \mathbf{\Psi}(t_j) \| \leq C \max\left\{ \| \mathbf{\Psi}(0) \|, \frac{1}{\alpha} \| L^N\mathbf{\Psi}(t_j) \| \right\}, 0 \leq j \leq N,$$

where C is a constant independent of j and $\boldsymbol{\varepsilon}$

4 The local truncation error

From the Lemma 8, we see that in order to bound the error $\| \mathbf{U} - \mathbf{u} \|$ it suffices to bound $L^N(\mathbf{U} - \mathbf{u})$. But this expression satisfies

$$L^N(\mathbf{U} - \mathbf{u}) = L^N(\mathbf{U}) - L^N(\mathbf{u}) = \mathbf{f} - L^N(\mathbf{u}) = L(\mathbf{u}) - L^N(\mathbf{u}) =$$

$$(L - L^N)\mathbf{u} = -E(D^- - D)\mathbf{u}$$

which is the local truncation of the first derivative. We have

$$E(D^- - D)\mathbf{u} = E(D^- - D)\mathbf{v} + E(D^- - D)\mathbf{w}$$

and so, by the triangle inequality,

$$\| L^N(\mathbf{U} - \mathbf{u}) \| \leq \| E(D^- - D)\mathbf{v} \| + \| E(D^- - D)\mathbf{w} \|$$

Thus, we can treat the smooth and singular components of the local truncation error separately We note first that, for any smooth function ψ, we have the following two distinct estimates of the local truncation error of the first derivative

$$|(D^- - D)\psi(t_j)| \leq \max_{s\in[t_{j-1},t_j]} |\psi''(s)| \frac{(t_j - t_{j-1})}{2} \tag{4}$$

$$|(D^- - D)\psi(t_j)| \leq 2 \max_{s\in[t_{j-1},t_j]} |\psi'(s)| \tag{5}$$

Note that the first involves just the first derivative, while the second involves the second derivative.

5 Error estimate: the special case n=3

We estimate the smooth component of the local truncation error in the following lemma.

Lemma 9. *For each $i = 1, \dots, n$ and $j = 1, \dots, N$ we have*

$$|\varepsilon_i(D^- - D)v_i(t_j)| \le CN^{-1}$$

For the singular component we obtain a similar estimate, but we must distinguish between the different types of mesh. We need the following preliminary lemmas.

Lemma 10. *On each mesh of the form $M_{\mathbf{b}}$, for $i = 1, 2, 3$ we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C\frac{t_j - t_{j-i}}{\varepsilon_1}$$

Lemma 11. *On each mesh of the form $M_{1b_2b_3}$, for $i = 1, 2, 3$ there exists a decomposition*

$$w_i = w_{i,1} + w_{i,2}$$

for which we have the estimates

$$|\varepsilon_i w'_{i,1}(t)| \le CB_1(t)$$

$$|\varepsilon_i w''_{i,1}(t)| \le C\frac{B_1(t)}{\varepsilon_1}, \qquad |\varepsilon_i w''_{i,2}(t)| \le C(\frac{B_2(t)}{\varepsilon_2} + \frac{B_3(t)}{\varepsilon_3})$$

Furthermore

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C(B_1(t_{j-1}) + \frac{t_j - t_{j-i}}{\varepsilon_2})$$

Lemma 12. *On each mesh of the form $M_{b_1 1 b_3}$, for $i = 1, 2, 3$ there exists a decomposition*

$$w_i = w_{i,1} + w_{i,2} + w_{i,3}$$

for which we have the estimates

$$|\varepsilon_i w'_{i,j}(t)| \le CB_j(t) \ \text{ for } \ j = 1, 2$$

$$|\varepsilon_i w''_{i,j}(t)| \le C\frac{B_j(t)}{\varepsilon_j} \ \text{ for } \ j = 1, 2, 3$$

Furthermore

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C(B_2(t_{j-1}) + \frac{t_j - t_{j-i}}{\varepsilon_3})$$

Lemma 13. *On each mesh of the form $M_{\mathbf{b}}$, for $i = 1, 2, 3$ we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le CB_3(t_{j-1})$$

Using the above preliminary lemmas on appropriate subintervals we obtain the desired estimate of the singular component of the local truncation error in the following.

Lemma 14. *For $i = 1, 2, 3$ and $j = 1, \ldots, N$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \leq CN^{-1} \ln N$$

Let **u** denote the exact solution of 1 and **U** the discrete solution. Then, using Lemmas 9, 14, we have the following ε-uniform error estimate

Theorem 1. *There exists a constant C, independent of N and ε, such that*

$$\| \mathbf{U} - \mathbf{u} \| \leq CN^{-1} \ln N$$

for all $N > 1$

6 Numerical results

The above numerical method is applied to the following singularly perturbed initial value problem

$$\varepsilon_1 {u_1}'(t) + 4u_1(t) - u_2(t) - u_3(t) = t \quad (6)$$

$$\varepsilon_2 {u_2}'(t) - u_1(t) + 4u_2(t) - u_3(t) = 1 \quad (7)$$

$$\varepsilon_3 {u_3}'(t) - u_1(t) - u_2(t) + 4u_3(t) = 1 + t^2 \quad (8)$$

for $t \in (0, 1]$ and $\mathbf{u}(0) = 0$. For various values of ε_1, fixed values $\varepsilon_2 = 2^{-10}$, $\varepsilon_3 = 2^{-7}$ and $N = 128$, the computed order of $\boldsymbol{\varepsilon}$–uniform convergence and the computed $\boldsymbol{\varepsilon}$–uniform error constant are found using the general methodology from [1],[3]. The results, presented in Table 1 below, exhibit the behaviour expected from an $\boldsymbol{\varepsilon}$–uniform method.

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Table 1. *Values of* D^N_ε, D^N, p^N, p^*, *and* $C^N_{p^*}$ *for various* ε_1 *and fixed* $\varepsilon_2 = 2^{-6}$, $\varepsilon_3 = 2^{-4}$

	Number of mesh points N								
ε_1	128	256	512	1024	2048	4096	8192	16384	32768
2^{-7}	0.135-2	0.832-3	0.485-3	0.276-3	0.154-3	0.853-4	0.466-4	0.253-4	0.136-4
2^{-11}	0.195-2	0.118-2	0.688-3	0.391-3	0.215-3	0.117-3	0.625-4	0.332-4	0.175-4
2^{-15}	0.230-2	0.136-2	0.808-3	0.469-3	0.262-3	0.145-3	0.792-4	0.430-4	0.232-4
2^{-19}	0.232-2	0.138-2	0.810-3	0.476-3	0.266-3	0.147-3	0.805-4	0.436-4	0.235-4
2^{-23}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-27}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-31}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-35}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-39}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
2^{-43}	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
D^N	0.232-2	0.138-2	0.810-3	0.477-3	0.266-3	0.147-3	0.806-4	0.437-4	0.236-4
p^N	0.753+0	0.767+0	0.765+0	0.842+0	0.855+0	0.867+0	0.882+0	0.891+0	
$C^N_{0.753}$	0.221+0	0.221+0	0.219+0	0.217+0	0.204+0	0.190+0	0.176+0	0.161+0	0.146+0
Computed order of $\boldsymbol{\varepsilon}$–uniform convergence = 0.753									
Computed $\boldsymbol{\varepsilon}$–uniform error constant = 0.221									

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