From planar graph layout to flat embeddings of 2-complexes

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Abstract

A question about 2-complexes, analogous to graph planarity, is whether they can be embedded in 3 dimensions. This is a difficult question. Also, whereas planar graphs always admit straight-edge embeddings, it is known that 2-complexes may admit embeddings without admitting flat-face embeddings.

We consider three methods for straight-edge planar graph layout: barycentric maps (and convex-combination maps) which have already been studied in 3 dimensions, Read's vertex-deletion method, and the De Fraysseix-Pach-Pollack method. We give examples showing that each method can fail. The last method is based on so-called monotone subdivisions, and not only are some subdivisions intrinsically non-monotone, but some monotone subdivisions will not lead to a flat embedding.

1 Introduction

This paper is concerned with deciding which 2-complexes can be flat-face embedded in three dimensions. The general problem has received a great deal of attention, especially in the case of triangulated manifolds [2]. In principle it is solved, being a decidable question in Tarski geometry, but such a solution is seldom of practical use.

(1) Here we confine our attention to the analogue of triangulated planar graphs, namely, simplicial 2-complexes which are presented as the 2-skeletons of a simplicial subdivision of the closed 2-ball.

In the context of the De Fraysseix-Pach-Pollack method, we also require that the external boundary of the subdivision has four faces, i.e., is a (curvilinear) tetrahedron.

In this introduction we review three straight-edge layout methods whose analogue for 2-complexes will be discussed in the later sections.

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1.1 Barycentric and convex combination maps

Barycentric maps can be a very simple way of constructing straight-edge embeddings of planar graphs. A map f of a plane embedded graph, with prescribed outer face, is *barycentric* if it maps the external boundary C bijectively onto a convex polygon P, mapping the vertices on C to the corners of P (so the outer boundary is required to be a simple cycle), and for any internal vertex v,

$$f(v) = \frac{1}{k} \sum_j f(v_j),$$

where $v_1, \ldots, v_k$ are the neighbours of v in G. More generally, it is a convex combination map if every internal vertex is a proper weighted average of its neighbours.

Tutte [17] showed that if G is connected, then there is a unique solution f to the above equations, and that if G nodally 3-connected with at least 3 nodes, every barycentric map is an embedding. This striking result has been extended in various ways [9,10,14,18].

(2) Proposition *Let G be a connected planar graph with prescribed outer face, and at least three vertices. Then G admits a convex embedding if and only if (a) every face boundary is a simple cycle, (b) the intersection of any two bounded faces is connected, and (c) if a bounded face meets both ends of a boundary edge, then it meets the entire edge (see* [10,14,16].*) In this case, every convex combination map is an embedding.*

This holds in particular when G is triangulated (see [18]). ∎

1.2 Read's method

Another simple method of constructing a straight-edge embedding of a planar graph is using Read's linear-time re-triangulation method, as follows.

We take a schematic plane embedding of the graph G. This is equivalent to a cyclic ordering of the edges incident to every vertex, together with a choice of outer boundary. This also amounts to a topological description of the embedding, without specifying the placement of vertices or edges. The graph is, if necessary, triangulated.

Recursively, it takes a low-degree internal vertex x of G and deletes it, so the faces incident to x are merged into a single face F. F contains at least one vertex y from which new edges can be extended to subdivide F into triangles; call this a fan-triangulation of F. This yields a triangulated graph G' with fewer vertices.

Recursively, a straight-edge embedding of G' is computed. Recursion stops when G' is reduced to a 3-cycle, which can be embedded as a triangle. Having embedded G', one deletes the edges introduced to subdivide F: then all of F is visible from y, and one can embed x close to y so all of F is visible from x, so the edges joining x to the corners of F can be restored.

There is a method due to the present author, based on edge contraction [13]. It is similar to Read's method but requires less space.

Existence of a fan-triangulation can be proved by an interlacing argument, or by the following observation: any *external* triangulation of a polygon includes a triangle incident to two adjacent sides of the polygon. The vertex common to these sides can be made the apex of an internal fan-triangulation.

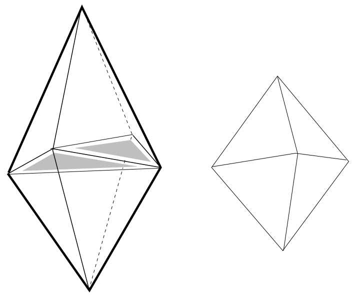

Figure 1: barycentric map not an embedding.

1.3 The De Fraysseix-Pach-Pollack method

De Fraysseix, Pach, and Pollack [4] gave a linear-time algorithm for planar graph layout on a rectangular grid. We are not concerned with a grid layout, but it is based on a simple method which can sometimes be applied to 2-complexes.

Let G be a fully triangulated planar graph (every face, including the outer face, is bounded by a 3-cycle).

Let u, v, w be the external vertices of G.

A *canonical representation* for G is an ordering $v_1, \ldots, v_n$ of its vertices such that $v_1 = u$, $v_2 = v$, $v_n = w$, and for $4 \leq k \leq n$ G_{k-1}, the subgraph generated by $v_1, \ldots, v_{k-1}$, is biconnected, v_k is in its outer face, and the neighbours in G_{k-1} form a nontrivial subinterval of the external boundary of G_{k-1}.

Given a canonical representation for G, straight-edge embeddings for the subgraphs $G_2, \ldots, G_{n-1}$, $G_n = G$ can be constructed iteratively with u and v on the x-axis and the others placed with monotonically increasing heights and such that the external vertices are monotonic in the x-direction and all visible from sufficiently high points. Let N_k be the interval of neighbours of v_k in G_{k-1}, and I_k the interval of x-coordinates spanned by the vertices in N_k. Place v_k so its x-coordinate is within the interval I_k and sufficiently high so that all of N_k is visible from it. This gives a straight-edge embedding of G_k and ultimately of G.

2 Barycentric maps

An obvious question is whether barycentric maps, or more generally convex combination maps, described in the introduction, are embeddings for simplicial 2-complexes, or at least those simplicial 2-complexes which are subdivisions of a closed ball admitting flat embeddings.

A counterexample to baycentric embedding in 3 dimensions was given in [5]. More generally, in [10] a 2-complex was described which does not admit convex combination embeddings in 3 dimensions.

Here are two counterexamples which appear to differ from the ones mentioned. Figure 1 illustrates the first. A 3-dimensional polytope is subdivided into 5 tetrahedral polytopes, and the outer boundary is an octohedron. All vertices are external, so if f is a flat-face mapping taking these vertices to a regular octohedron, then the central tetrahedron collapses to a square. Since there are no internal vertices, f is automatically barycentric but not an embedding.

Of course, the cell is flattened in this example because its four corners are coplanar. It is not enough just to require the corners to be in general position, because the middle cell could be everted and covered three-to-one.

In fact, since there are no internal vertices, it shows that no method can work which begins by mapping the external vertices to the corners of a convex polyhedron.

There is another counterexample, viz., any n-vertex subdivision K of the closed ball whose 1-skeleton is the complete graph K_n. Such subdivisions exist, as mentioned in the next section.

(3) Lemma *Let K be a simplicial 2-complex with $n \geq 4$ vertices and suppose its 1-skeleton is the complete graph K_n. Let four of its vertices be designated as outer vertices and mapped to the corners of a tetrahedron.*

Then in the barycentric extension of this map, all inner vertices map to the barycentre of these corners.

Proof. Let A, B, C, D be the coordinates of the corners, and let $E = (A+B+C+D)/4$. Define f as the map taking the outer vertices to the same corners, and if u is an inner vertex let $f(u) = E$. Then

$$\frac{1}{n-1}\sum_v f(v) = \frac{1}{n-1}(A+B+C+D+(n-5)E) = \frac{1}{n-1}(n-1)E = E = f(u),$$

where v varies over the set of all $(n-1)$ neighbours of u, so f is the unique barycentric extension. **Q.E.D.**

That is, under any barycentric map all internal vertices map to the same point, no matter what points are chosen for the external vertices.

Note that in all the examples given, the complexes admit flat embeddings.

3 Read's method

Read's method has been described in the Introduction. It is obvious how we should attempt to embed 2-complexes (satisfying §1) by this method: delete an internal vertex x, leaving a merged cell F, fan-triangulate F, and recurse.

In attempting a counterexample to a 3-dimensional variant of Read's method, we want a simplicial 3-complex K with the following property. For every internal vertex x, let L_x be the union of faces opposite x, (a 2-complex homeomorphic to the 2-sphere). Then for every vertex u on L_x, there exists another vertex v on L_x, such that $\{u, v\}$ is an edge of K but not of L_x. One might expect the complex described below to have this property.

(4) Proposition *For every $n \geq 4$ there exists a subdivision of a 3-simplex, with n vertices, whose 1-skeleton is the complete graph K_n.*

Proof. Take a neighbourly 4-polytope with n vertices, that is, a simplicial polytope P in $\mathbb{R}^4$ with n vertices whose 1-skeleton is K_n [3]. Choose any face, and a point p outside P close to the centre of the face, and form a perspective projective of ∂P from p into a suitable 3-dimensional hyperplane. This projected complex is a subdivision of a tetrahedron whose 1-skeleton is K_n. ∎

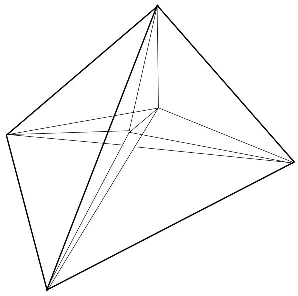

Figure 2: an 8-cell 6-vertex subdivision of a tetrahedral polytope whose 1-skeleton is K_6.

Figure 2 illustrates the proposition with $n = 6$.

However, study of examples with $n \leq 10$ showed that for every vertex x in the complex,

the subcomplex L_x contains a vertex u such that $\{u, v\} \in L_x$ for every $v \neq u, x$. (*)

In other words, these examples don't work.

In fact, this is not surprising since the polytopes P considered were cyclic polytopes, the convex hull of n distinct points on the moment curve in $\mathbb{R}^4$. The edge contraction process can be related to a homotopy which moves two points on the curve together, which causes a polytope with n vertices to be deformed into one with $n-1$ vertices, imitating Read's process in reverse.

There exist non-cyclic neighbourly polytopes, but the 8-vertex non-cyclic polytope given in [11, §7.2], has property (*) and does not furnish a counterexample.

On the other hand, we can exhibit a complex which has *one* internal vertex x, and the inside of L_x cannot be fan-triangulated. This uses a complex whose 1-skeleton contains some complete bipartite graphs.

Let X be a line-segment and let Y be a copy of X obtained by translating it in a direction perpendicular to X. The convex hull $H(X \cup Y)$ is a rectangle. If we displace one end of Y slightly, $H(X \cup Y)$ becomes a shallow tetrahedron.

Given X, form two copies Y and Z by perpendicular translation. $H(X \cup Y \cup Z)$ is a prism. For definiteness, suppose X is parallel to the y-axis directly above the origin, Y and Z are in the xy-plane, and the triangular faces of the prism are equilateral.

Displace the front ends of Y and Z slightly to the left (keeping them parallel and in the xy-plane). The convex hulls of $X \cup Y$ and $X \cup Z$ are now shallow tetrahedra. The inside diagonal of the first meets the outside diagonal of the second and vice-versa. Then displace the front end of Z slightly downwards. This makes $H(Y \cup Z)$ a tetrahedron and the three tetrahedra are as depicted in Figure 3.

Adding the front and back faces completes the figure into a modified prism.

Fix $m \geq 2$, and take $m+1$ points $x_0, x_1, \ldots, x_m$ on X, from front to back, with x_0 the front endpoint of X and x_m the back. Similarly take points y_i, z_j on Y and Z.

$H(X \cup Y)$ can be subdivided into $m \times m$ tetrahedra $H(x_i, x_{i+1}, y_j, y_{j+1})$, $0 \leq i < m, 0 \leq j < m$. Similarly for $H(Y \cup Z)$ and $H(X \cup Z)$. This subdivision is illustrated in the Figure 3.

A point x is placed in the centre and the modified prism is triangulated by including tetrahedra $H(\{x\} \cup T)$ for every triangle T subdividing those surfaces of $H(X \cup Y)$, $H(Y \cup Z)$, and $H(X \cup Z)$, visible from x. The figure shows only the front tetrahedron containing x, as to show more detail would increase the confusion. Let K be the triangulated complex.

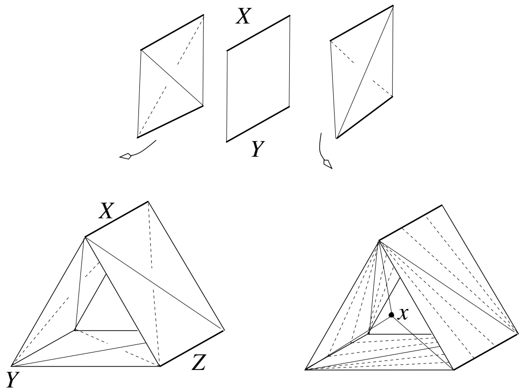

Figure 3: Perturbing a rectangle gives a tetrahedron, whose new sides depend on the direction of perturbation.

For every vertex u of K, except x, $\{u, x\}$ is an edge of K, so the subcomplex L_x contains all vertices except x. For every $u \neq x$, there exists an edge $\{u, v\}$ of K which is not an edge of L_x, for the following reasons.

If $u = x_i$, $1 \leq i < m$, let $v = y_i$. The edge $\{x_i, y_i\}$ is an edge of K interior to $H(X \cup Y)$ and hence not in L_x. Similarly, if $u = y_i$ or z_i, $1 \leq i < m$, there exists an edge $\{u, v\}$ of K not in L_x. The edges $\{x_0, z_m\}$, $\{y_0, x_m\}$, and $\{z_0, y_m\}$, are in K but not in L_x.

Therefore L_x does *not* have property (*), and K cannot be flat-embedded using Read's method as discussed.

Note that K is already flat-embedded.

4 A complex containing a trefoil knot

In general the presence of knots inside 3-complexes makes them less tractable [8], and is an obstruction to flat embeddings.

(5) Definition *A* subdivision *of the closed ball is a simplicial subdivision of the ball embedded in* $\mathbb{R}^3$.

It is a monotone subdivision *if* (a) *it has four external faces,* (b) *one of its external faces is in the* xy*-plane (the base),* (c) *all vertices, except the three base vertices, have different positive* z*-coordinates, and* (d) *except for the base, each 3-simplex in the subdivision is embedded monotonically with respect to the* z*-direction. This means that each 3-simplex except the base has no critical points on its boundary: its intersection with every horizontal plane is simply connected.*

Every flat-face subdivision is a monotone subdivision, or can be made so by changing the coordinate system and perturbing some of the vertices.

We introduce monotone subdivisions because they are more tractable for the purposes of flat embedding. In a later section we shall relate them to the De Fraysseix-Pach-Pollack graph layout method. Here we show that a monotone subdivision cannot contain a knotted triangle, and hence exhibit a subdivision which is not isotopic to a monotone subdivision.

(6) Lemma *Let* L *be a closed curve embedded in three dimensions, with a unique minimum and a unique maximum with respect to the* z*-direction. Then* L *is unknotted.*

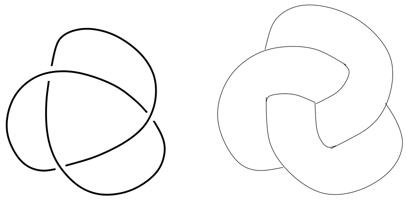

Figure 4: trefoil knot and inflated trefoil knot

Sketch proof. Every horizontal plane, between the minimum and maximum height of L, intersects L in exactly two points. Let T be a triangle in three dimensions whose vertices are at different heights, the lowest and highest being at the same height as the lowest and highest points in L. Thus every horizontal plane intersects T and L in the same number, $0, 1$, or 2, of points. One can construct a homeomorphism of $\mathbb{R}^3$ to itself, taking each horizontal plane Π_z to itself, and taking $\Pi_z \cap L$ bijectively onto $\Pi_z \cap T$. Therefore as knots, L and T are equivalent, and T is trivial. ∎

(7) Corollary *Let* L *be a 3-cycle occurring in the* 1*-skeleton of a monotone embedding. Then* L *is unknotted.*

Sketch proof. Each edge in L is part of a monotone 3-simplex, so it is embedded monotonically with respect to the z-direction. By the above lemma L is unknotted. ∎

More generally, the *stick number* of a knot is the smallest number of sides in an equivalent polygonal knot, if they exist [1]. One can show that any knot L composed of k continuous arcs monotonic in the z-direction has stick number $\leq k$.

Figure 4 illustrates a trefoil knot. It is a true knot because it has projections with an odd number of crossings [1].

If one imagines it formed out of a bicycle tube, one can imagine the tube being inflated to produce an inflated trefoil knot as illustrated in the same figure. It can be arranged that the outer boundary is homeomorphic to a sphere, and by further subdividing the tube interior one can produce a subdivision containing a 3-cycle which is a trefoil knot.

The idea is that several layers can be introduced inside the inflated trefoil knot, and triangulated, so that the innermost layer is a knotted torus containing some knotted 3-cycles.

Further layers (homeomorphic to S^2) need to be added on the outside to coarsen the subdivision of the external boundary and produce a simplicial subdivision with four external faces.

The inflated knot as illustrated is difficult to describe in terms of parametrised surfaces, but the construction can be presented in terms of cubes: details are omitted. Thus there exist subdivisions (with four external faces) which cannot be deformed into monotone subdivisions.

Similar constructions involving knots are described elsewhere (for example, [19]).

5 The De Fraysseix – Pach – Pollack method

Recall from the Introduction that the De Fraysseix-Pach-Pollack method begins with a canonical embedding of a fully triangulated planar graph. An important feature of canonical embeddings is repeated in monotone subdivisions, as follows.

(8) Lemma *Let x be a vertex in a monotone subdivision* (5), *not in the xy-plane. Let L be the union of faces opposite x in the collection of 3-simplexes meeting x from below. Then L is simply connected.*

Sketch proof. If x is the lowest vertex above the base then L is the base itself. Otherwise let U be the union of all those cells whose highest vertex is x.

Consider a horizontal plane Π_t with z-coordinate t moving upwards from the xy-plane. It first intersects U at a single vertex y. For some time its $\Pi_t \cap U$ is restricted to those 3-simplexes incident to the arc xy. Since the embedding is monotonic, $\Pi_t \cap U$ is simply connected. By definition of U, whenever Π_t intersects a new 2-simplex in U, that intersection remains until Π_t meets x. Furthermore, the new simplex contributes a simply-connected region to $\Pi_t \cap U$, beginning on the boundary of $\Pi_t \cap U$. Thus, $\Pi_t \cap U$ is a simply-connected closed disc contracting to a point only when Π_t passes through y and through x. One can deduce that U is homeomorphic to a closed ball and its boundary to a sphere.

The union of faces incident to x on U is also simply-connected, so its complement, whose closure is L, is simply connected. **Q.E.D.**

This leads to a strategy for flat-face embedding:

- Place the bottom three vertices non-collinearly on the xy-plane.
- Recursively, choose the next vertex x in ascending order with respect to z. In the currently embedded subcomplex, the *upper cover* — those external faces differing from the base triangle — is a simply-connected union of faces whose projection onto the xy-plane is injective. Let L be the lower link of x, a simply-connected union of faces on the upper cover. Suppose x is placed so the faces joining it to ∂L project injectively onto the same region as L itself. Then the projection of L is starshaped with respect to the projection of x.

  Therefore: adjust the existing embedding if necessary to make L starshaped, then x can be placed as desired.

6 Simultaneously embeddable families

The notion of simultaneously embeddable graphs arises from the previous discussion of the De Fraisseix-Pach-Pollack method.

(9) Definition *Let $G_1, G_2, \ldots$ be a list of graphs, all sharing the same set V of nodes. A* simultaneous embedding *of these graphs is an injective map $f : V \to \mathbb{R}^2$ such that for each graph G_i, f is a straight-edge embedding of G_i.*

In our 3-dimensional version of the De Fraysseix-Pach-Pollack method, the subgraphs G_i arise from the 1-skeletons of the lower links L: suppose that L be the lower link from x. in order to guarantee that the projection of L_x is starshaped, the appropriate subgraph G_i should consist of ∂L and the edges joining x to the vertices on ∂L.

The topic of simultaneous embeddings has received some attention [6]. It is related to the geometric thickness of graphs, which is probably NP-complete [7].

Figure 5 illustrates a pair of graphs which are not simultaneously embeddable.

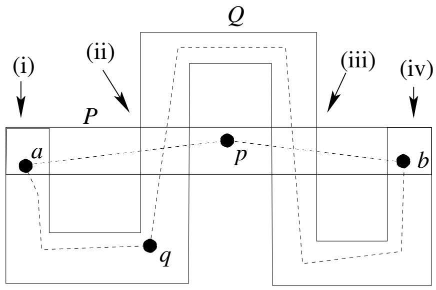

Figure 5: two regions not simultaneously starshaped, crossing twice and overlapping twice.

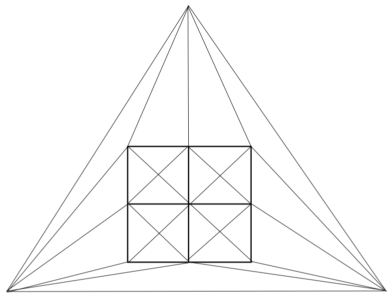

Figure 6:

There exists a triangulated graph containing two simple cycles P and Q with the property that if we place vertices p and q inside P and Q and connect them to all vertices in P (respectively, Q), then in any simultaneously embeddable family, all of P is visible from p and all of Q from q. That is, P and Q are starshaped relative to p and q. We therefore should arrange P and Q to cross in such a way that they cannot simultaneously be made starshaped.

(10) Definition *Let P be a simple closed curve, P^o the bounded open region it encloses, and $\overline{P}$ its closure $P \cup P^o$; similarly for another curve Q. Let the region R be a connected component of $\overline{P} \cap \overline{Q}$.*

Then P and Q cross *at R if $\overline{P} \backslash R$ and $\overline{Q} \backslash R$ are both disconnected, and if they are both connected then P and Q* overlap *at R.*

Figure 5 shows two regions P and Q which intersect four times, twice at crossings. In the figure, two polygons P and Q intersect in four connected regions (i)–(iv). They are supposed starshaped relative to p and q respectively. Paths are taken connecting the points a and b to p and q respectively. These paths are supposed to be straight lines (though that is impossible), so they consist of line-segments ap, pb, aq, qb.

Claim that there can be only one more transverse intersection of these segments besides a and b, w.l.o.g. ap with qb. The only other possibility is aq with bp. For if ap intersects qb transversally and interior to both line-segments, p and a are on opposite sides of the line qb, so the line-segments aq and bp do not intersect, as claimed.

The regions (ii) and (iii) are crossing regions which separate a from b in both P and Q, so the paths must go through these regions and intersect transversally in these regions. However, the paths can intersect in at most one of these regions, so they do not exist.

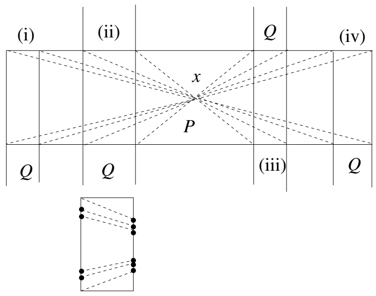

Figure 7:

(11) Corollary *There exists a monotone subdivision which cannot be flat-embedded using the proposed analogue of the De Fraysseix-Pach-Pollack method. Moreover, such a subdivision can itself be a flat embedding.*

Proof. Begin by constructing a partial monotone subdivision whose upper boundary contains a rectangular grid. This is easily accomplished, leading to a subsivision whose upper boundary, viewed from above, resembles figure 6: the rectangular grid can be made arbitrarily fine. One should first place the vertices as depicted in a planar arrangement then perturb them slightly to be in convex position so that the upper boundary of the convex hull has edges as illustrated.

Actually, the vertices are perturbed slightly to put them in convex position, the convex hull of these points and the subdivision is both flat-face and (partially) monotone.

Simple cycles P and Q can be chosen to realise the situation illustrated in Figure 5. P is a cycle of grid edges forming a near-rectangular figure, and Q is another cycle of grid edges which intersects P in the pattern shown in Figure 5. Let x be a new point above all existing vertices, lying vertically above the interior of P. Join x to all triangles inside P. So far we have a flat-face embedding which is, so far, monotone. Projecting this subdivision vertically downwards, part of Q is covered by faces from x, as illustrated in Figure 7, and Q is not on the external boundary. However, new vertices can be added, illustrated in region (ii) in Figure 7, and put in convex position relative to the others, so that a cycle similar to Q can be constructed on the external boundary.

Finally a vertex y can be introduced above Q and connected to all faces inside Q. This is possible while retaining flat faces, but not so that all these faces are visible from above. We have a flat-face embedding which can not be embedded by the proposed analogue of the De Fraysseix-Pach-Pollack method.

Call this embedding G.

One last vertex z can be added and cells formed by connecting z to all vertices, edges, and faces, of the upper boundary of G. This cannot be done with flat faces, but there is no difficulty with a topological subdivision which is monotonic. To expand upon this, the inside of Q can be thought of as a rectangle with three rectangular areas removed. Figure 8 shows the pyramid between y and the middle one of the rectangles removed. This is a actually outside the subdivision: the left face of the pyramid is outside the subdivision and the other four faces are external faces of the subdivision G.

These four faces can be projected onto the left face as illustrated, producing a subdivided triangle (not all faces of the subdivision are illustrated: the rectangle and its projection, a trapezium, are subdivided further). Let R_1 indicate the subdivision of the triangular face arising from this projection. R_1 is outside G.

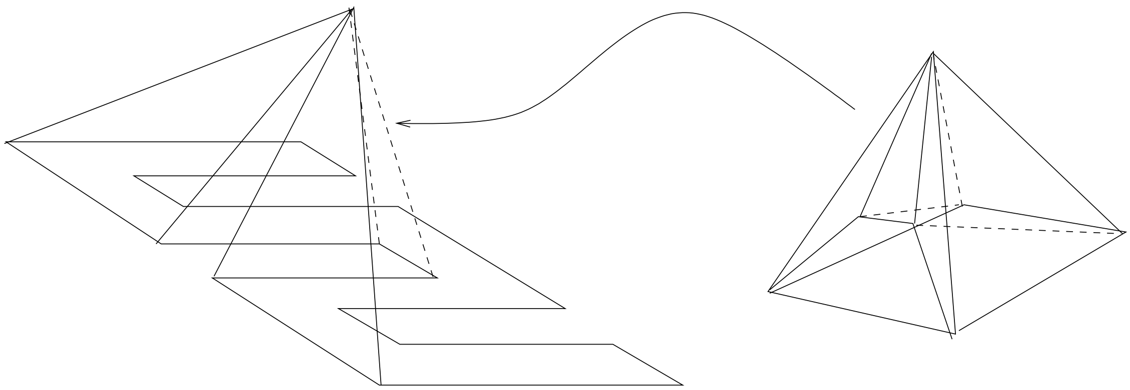

Figure 8:

All of R_1 is visible from above. There are another two pyramids which can be treated similarly, with subdivided faces R_2 and R_3. Then from a sufficiently high point z, all of the outer boundary of G except these three pyramids is visible, and the three subdivided triangles R_i are visible. One can thus produce a non-flat monotonic embedding, with highest vertex z, in which external faces visible from z extend to tetrahedral cells , and external faces not visible from z are piecewise-linear embeddings of tetrahedra formed by joining z to the projected subdivisions R_i and continuing these linearly into the interior of the respective pyramids.

Now we have a monotonic subdivision which cannot be flat-embedded by our proposed analogue of the De Fraysseix-Pach-Pollack method. **Q.E.D.**

7 Acknowledgements

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