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The 't Hooft-Polyakov Monopole in the Presence of an 't Hooft Operator

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Abstract

We present explicit BPS field configurations representing one non-abelian monopole with one minimal weight 't Hooft operator insertion. We explore the $SO(3)$ and $SU(2)$ gauge groups.

In the case of $SU(2)$ gauge group the minimal 't Hooft operator can be completely screened by the monopole. If the gauge group is $SO(3)$, however, such screening is impossible. In the latter case we observe a different effect of the gauge symmetry enhancement in the vicinity of the 't Hooft operator.

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1 Introduction

't Hooft operators [1] play a central role in recent studies of the Montonen-Olive duality [2] as electric-magetic duals of the Polyakov-Wilson operators [3]. Their significance as Hecke operators in the geometric Langlands program is elucidated in [4].

By a 't Hooft operator we understand an insertion of a Dirac monopole imbedded into the gauge group in question. In other words, in the vicinity of an insertion point in the three-dimensional space, we impose the following boundary conditions [5] on the gauge fields

$$F = \frac{B}{2} d\Omega_2, \tag{1}$$

where $d\Omega_2$ is the volume form of a unit sphere surrounding the point and B is the Lie algebra element satisfying $\exp(2\pi i B) = \mathbb{I}$. The 't Hooft charge of such an operator takes values in $H^2(S^2, \pi_1(G))$. It vanishes if the gauge group G is $SU(2)$ and is $\mathbb{Z}/2\mathbb{Z}$ valued if G is $SO(3)$. Strictly speaking only the operators with nonzero 't Hooft charge are significant in [1], but here we forgo this restriction and adopt the more general definition of [3] and [4]. A minimal 't Hooft operator is an insertion of a Dirac monopole of the lowest possible charge. Here we focus on such minimal operator insertions.

BPS monopoles [7, 6] with 't Hooft operator insertions are solutions of the Bogomolny equation [7]

$$F_{ij} = -\epsilon_{ijk}[D_k, \Phi], \tag{2}$$

with prescribed Dirac type singularities. Explicitly, condition (1) implies that the Higgs field near the insertion point at $\vec{z} = \vec{0}$ is gauge equivalent to

$$SO(3): \quad \Phi_{jk} = -i\epsilon_{1jk}\frac{1}{2z} + \mathrm{O}(z^0), \tag{3}$$

$$SU(2): \quad \Phi_{\alpha\beta} = \sigma^3_{\alpha\beta}\frac{1}{2z} + \mathrm{O}(z^0). \tag{4}$$

Here ϵ_{ijk} is a completely antisymmetric tensor and σ^1, σ^2, and σ^3 are the Pauli sigma matrices. We used the technique of the Nahm transform [8] to obtain the explicit solutions presented here. The detailed derivation will appear in [9]. The solutions below have been explicitly verified numerically and analytically.

When the separation $\vec{d}$ between the 't Hooft operator insertion and the nonabelian monopole is large, i.e. $d = |\vec{d}| \gg 1, 1/\lambda$, we expect the fields $\Phi = (\Phi_{ij})$ and $A = (A_{ij})$ with

$$SO(3): \quad \Phi_{ij} = -2i\epsilon_{ijk}\phi^k, A_{ij} = -2i\epsilon_{ijk}A^k, \tag{5}$$

$$SU(2): \quad \Phi = \vec{\sigma}\cdot\vec{\phi}, \quad A = \vec{\sigma}\cdot\vec{A}, \tag{6}$$

to approach those of the 't Hooft-Polyakov BPS monopole solution [10, 11, 6]

$$\vec{\phi} = \Big(\lambda\coth(2\lambda r) - \frac{1}{2r}\Big)\frac{\vec{r}}{r}, \tag{7}$$

$$\vec{A} = \Big(\frac{\lambda}{\sinh(2\lambda r)} - \frac{1}{2r}\Big)\frac{\vec{r}\times d\vec{x}}{r}. \tag{8}$$

For the 't Hooft operator insertion at $\vec{p}$ and the monopole position parameter $\vec{m}$ we denote the separation parameter by $\vec{d} = \vec{m} - \vec{p}$, the position with respect to the monopole by $\vec{r} = \vec{x} - \vec{m}$, and the position with respect to the singularity by $\vec{z} = \vec{x} - \vec{p}$. We let $r = |\vec{r}|$ and $z = |\vec{z}|$.

2 Solutions

It is convenient to introduce $\mathcal{D} = 2zd + 2\vec{z}\cdot\vec{d} = (z+d)^2 - r^2$, and to use the vector-valued functions $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ and $\vec{A} = (A^1, A^2, A^3)$. Then the monopole solutions of the Bogomolny Eq. (2) satisfying the boundary conditions (3) and (4) are provided by Eq.(5) and Eq.(6) above with $\vec{\phi}$ and $\vec{A}$ given respectively as follows:

$\mathbf{SO(3)}$ Case:

$$\begin{aligned}
\vec{\phi} &= \left(\Big(\lambda + \frac{1}{4z}\Big)\frac{k}{l} - \frac{1}{2r}\right)\frac{\vec{r}}{r} - \frac{r}{2zl\sqrt{\mathcal{D}}}\left(\vec{d} - \frac{\vec{r}\cdot\vec{d}}{r^2}\vec{r}\right), \\
\vec{A} &= \left(\Big(\lambda + \frac{z+d}{2\mathcal{D}}\Big)\frac{\sqrt{\mathcal{D}}}{l} - \frac{1}{2r}\right)\frac{\vec{r}\times d\vec{x}}{r} \\
&\quad - \frac{r}{2l\sqrt{\mathcal{D}}}\left(\frac{\vec{z}\times d\vec{x}}{z} + \Big(\frac{k}{\sqrt{\mathcal{D}}} - 1\Big)\frac{(\vec{r}\cdot(\vec{z}\times d\vec{x}))}{rz}\frac{\vec{r}}{r}\right),
\end{aligned} \tag{9}$$

where

$$l = (z+d)\sinh(2\lambda r) + r\cosh(2\lambda r), \tag{10}$$

$$k = (z+d)\cosh(2\lambda r) + r\sinh(2\lambda r). \tag{11}$$

SU(2) Case:

$$\begin{aligned}
\vec{\phi} &= \left(\left(\lambda + \frac{1}{2z}\right)\frac{\mathcal{K}}{\mathcal{L}} - \frac{1}{2r}\right)\frac{\vec{r}}{r} - \frac{r}{z\mathcal{L}}\left(\vec{d} - \frac{\vec{r}\cdot\vec{d}}{r^2}\vec{r}\right), \\
\vec{A} &= \left(\left(\lambda + \frac{z+d}{\mathcal{D}}\right)\frac{\mathcal{D}}{\mathcal{L}} - \frac{1}{2r}\right)\frac{\vec{r}\times d\vec{x}}{r} \\
&\quad -\frac{r}{\mathcal{L}}\left(\frac{\vec{z}\times d\vec{x}}{z} + \left(\frac{\mathcal{K}}{\mathcal{D}} - 1\right)\frac{(\vec{r}\cdot(\vec{z}\times d\vec{x}))}{rz}\frac{\vec{r}}{r}\right),
\end{aligned} \tag{12}$$

where

$$\mathcal{L} = ((z+d)^2 + r^2)\sinh(2\lambda r) + 2r(z+d)\cosh(2\lambda r), \tag{13}$$

$$\mathcal{K} = ((z+d)^2 + r^2)\cosh(2\lambda r) + 2r(z+d)\sinh(2\lambda r). \tag{14}$$

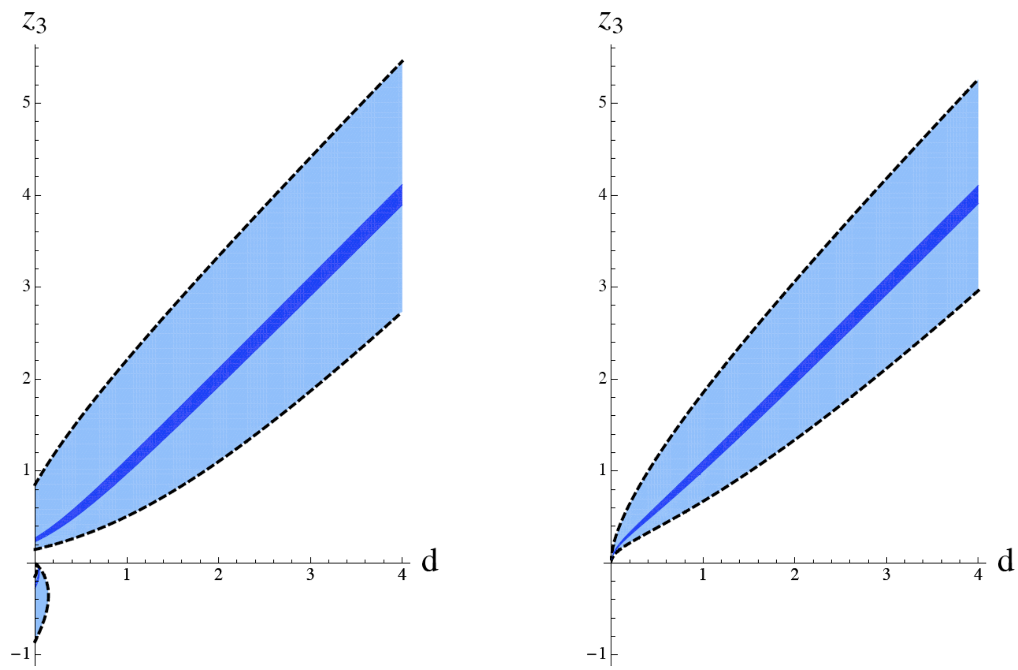
3 Analysis

The form of the expressions (9) and (12) makes the large separation limit transparent. Indeed, in this limit the fields near the 't Hooft operator insertion ($d \to \infty, z$ finite) reproduce those of Eqs. (3,4), while near the monopole core ($d \to \infty, r$ finite) they approach the 't Hooft-Polyakov solution (7, 8).

There is a substantial difference in the behavior of these solutions as we decrease d and the nonabelian monopole and the 't Hooft operator collide. The $SO(3)$ solution at $d = 0$ becomes another 't Hooft operator with the Higgs field

$$\vec{\phi} = \left(\lambda - \frac{1}{4r}\right)\frac{\vec{r}}{r}, \tag{15}$$

while the $SU(2)$ solution in this limit becomes trivial with $F = 0$, and Φ constant. The latter illustrates the screening effect, in which a nonabelian monopole completely screens the point-like singularity of the 't Hooft operator. A priori one might think such screening impossible since the 't Hooft-Polyakov monopole has a finite size of order $1/\lambda$ and finite energy density in the core, while the Dirac singularity of the 't Hooft operator is pointlike with the energy density divergent at one point. Electric-magnetic duality, however, suggests such screening as a possible dual explanation of screening of Wilson line operators by quarks. Our solution (12) resolves this seeming contradiction as we now explain.



(a) Gauge group $SO(3)$.

(b) Gauge group $SU(2)$.

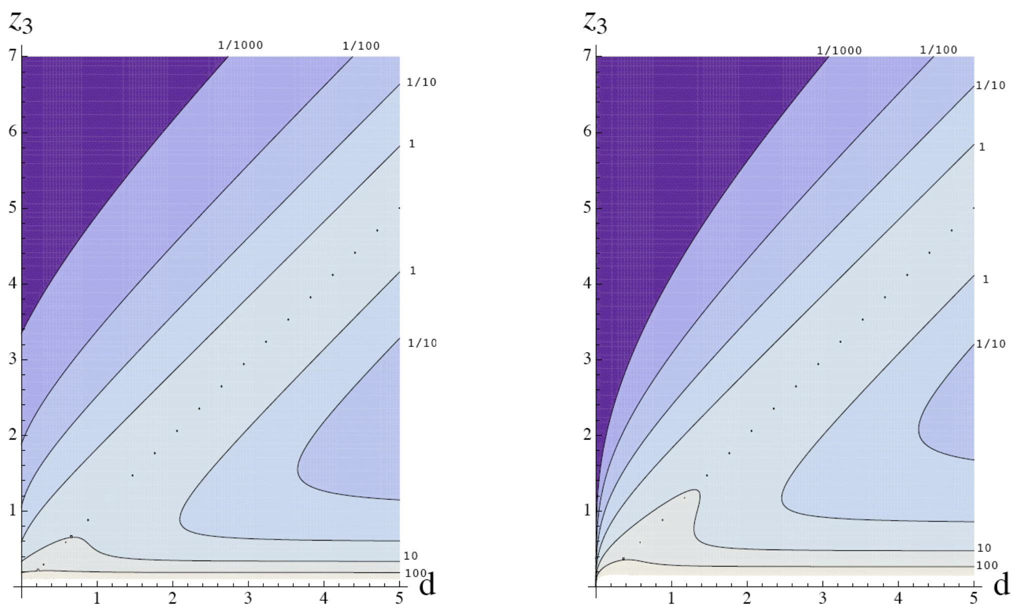
Figure 1: Higgs field profiles for $\lambda = 1$. Dashed lines correspond to $|\Phi|^2 = \frac{1}{2}$, the shaded area $|\Phi|^2 \leq \frac{1}{2}$, and the dark region indicates the position of the monopole core.

There is a number of ways to explore the size of the monopole. One is via the energy density distribution $\mathcal{E} \sim \frac{1}{2}\mathrm{Tr}(F^2 + (D\Phi)^2) = (\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3})\mathrm{Tr}\Phi^2$ and another is by how much the gauge symmetry is broken by the Higgs field. In particular, we might think of the position of the monopole as the point where the Higgs field vanishes and the gauge symmetry is fully restored. A word of caution is due here. Even though the parameter $\vec{d}$ is a good indication of the monopole relative position when d is large, it is not the point where the Higgs field vanishes, rather, at $\vec{d}$ (i.e. at $\vec{r} = \vec{0}$) we have

$$SO(3) : |\vec{\phi}| = \frac{1}{4d(1 + 4\lambda d)}, \quad SU(2) : |\vec{\phi}| = \frac{1}{4d(1 + 2\lambda d)}. \tag{16}$$

For the two gauge groups the profiles of $|\Phi|^2 = \frac{1}{2}\mathrm{Tr}\Phi^2$ at large separation parameter d look remarkably similar to each other. They do differ drastically, however, for small values of d. One can infer from Figs. 1(a) and 1(b) how the position and size of the monopole vary with d. The shaded areas in these graphs corresponds to the values of $|\Phi|^2 \leq 1/2$ and we choose the asymptotic condition to be $|\Phi(\infty)| = \lambda = 1$. z_3 is the coordinate along the line originating at the 't Hooft operator and passing through the monopole. The dark area in the middle corresponds to the values of $|\Phi|^2 < 0.007$, giving a good indication of the position of the monopole center.

In the case of the $SU(2)$ gauge group, Fig. 1(b), the monopole shrinks to zero size as it approaches the singularity and screens the 't Hooft operator completely. In the case of the $SO(3)$ gauge group, as the parameter $d \to 0$, the longitudinal size of the monopole decreases somewhat, approaching a constant. Instead of the monopole shrinking to zero size, we observe it spreading transversely and eventually encircling the singularity. This is also indicated by the expression (15) representing the limiting configuration to be an 't Hooft operator surrounded by a sphere of vanishing Higgs field at $z = \frac{1}{2\lambda}$.



(a) Gauge group $SO(3)$.

(b) Gauge group $SU(2)$.

Figure 2: Energy density contour plots for $\lambda = 1$.

Energy density plots of Figs. 2(a) and 2(b) support this picture. They present the contour levels of $\Delta\vec{\phi}^{\,2}$, which is proportional to the energy density

$\mathcal{E}$, due to Bogomolny Eq. (2) and the Bianchi identity. As $d \to 0$ the energy density for the $SO(3)$ gauge group case approaches a steady distribution diverging at the origin, while for the $SU(2)$ case the energy density decreases uniformly and, at $d = 0$, vanishes everywhere.

4 Comments

One can view the two solutions discussed here as part of the same picture since every $SU(2)$ solution can be viewed as one with the gauge group $SO(3)$ by factoring out the center of the group. Thus, we can reinterpret Eq.(12) combined with Eq.(5) as an $SO(3)$ monopole with the charge two 't Hooft operator insertion. Such an interpretation provides a topological reason for our observations in the previous section. For the gauge group $SO(3)$ the 't Hooft charge takes value in $\mathbb{Z}/2\mathbb{Z}$. The solution (5, 9) has 't Hooft charge one operator insertion and thus it is topologically protected. The solution (5, 12), however, has Dirac charge two and thus vanishing 't Hooft charge operator insertion which, as a result, can be screened completely. One can view the latter charge two configuration as a limit of two minimal 't Hooft operators approaching each other. We explore such a limit in more detail in the near future [9].

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