

The SF running coupling with four flavours of staggered quarks

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In order to study the running coupling in four-flavour QCD, we review the set-up of the Schrödinger functional (SF) with staggered quarks. Staggered quarks require lattices which, in the usual counting, have even spatial lattice extent L/a while the time extent T/a must be odd. Setting $T = L$ is therefore only possible up to $O(a)$, which introduces different cutoff effects already in the pure gauge theory. We re-define the SF such as to cope with this situation and determine the corresponding classical background field. A perturbative calculation yields the coefficient of the pure gauge $O(a)$ boundary counterterm to one-loop order.

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1. Introduction

The renormalised coupling in the Schrödinger functional (SF) scheme has been defined in [1, 2] and its scale evolution has been studied in QCD with zero and two quark flavours [3, 4]. We here discuss the set-up for studying the running coupling with four quark flavours, in the lattice regularisation with staggered quarks [5, 6]. This writeup is organized as follows. We start with a reminder of the basic definition of the renormalised coupling in a formal continuum notation. Its lattice formulation with staggered quarks will require a modification at $O(a)$ of the standard framework, even in the pure gauge theory. The corresponding tree-level and one-loop computations are then described and we end with an outlook to future work.

2. The renormalised Schrödinger functional coupling

The Schrödinger functional is a useful tool to study the scaling properties of QCD. It is defined as the Euclidean time evolution kernel for a state at time $x_0 = 0$ to another state at Euclidean time $x_0 = T$. Using the transfer matrix formalism, it can be written as a path integral over fields which satisfy Dirichlet boundary conditions at Euclidean times $x_0 = 0$ and T and L -periodic boundary conditions in space. More precisely, one imposes homogeneous boundary conditions on the quark fields,

$$\begin{aligned} (1 + \gamma_0)\psi|_{x_0=0} = 0 &= (1 - \gamma_0)\psi|_{x_0=T} \\ \bar{\psi}(1 - \gamma_0)|_{x_0=0} = 0 &= \bar{\psi}(1 + \gamma_0)|_{x_0=T}, \end{aligned} \quad (2.1)$$

while the spatial components of the gluon fields satisfy

$$A_k(x)|_{x_0=0} = C_k \quad A_k(x)|_{x_0=T} = C'_k. \quad (2.2)$$

The SF is considered a functional of these boundary gauge fields,

$$\mathcal{Z}[C', C] = \int \mathcal{D}[A, \psi, \bar{\psi}] e^{-S[A, \psi, \bar{\psi}]}. \quad (2.3)$$

To define the SF coupling we follow [1] and choose Abelian and spatially constant gauge boundary fields C and C' ,

$$C_k = \frac{i}{L} \text{diag}(\phi_1, \phi_2, \phi_3), \quad C'_k = \frac{i}{L} \text{diag}(\phi'_1, \phi'_2, \phi'_3). \quad (2.4)$$

Under mild conditions on these angular variables, the absolute minimum of the action is uniquely determined up to gauge equivalence. Denoting this minimal action configuration by $B_\mu(x)$, one may thus unambiguously define the effective action

$$\Gamma[B] = -\ln \mathcal{Z}[C, C']. \quad (2.5)$$

In the weak coupling domain, the SF can be computed by performing a saddle point expansion of the functional integral about the induced background field B , leading to an asymptotic series of the form,

$$\Gamma[B] \xrightarrow{g_0 \rightarrow 0} \frac{1}{g_0^2} \Gamma_0[B] + \Gamma_1[B] + g_0^2 \Gamma_2[B] + \dots \quad (2.6)$$

The leading term of the series is given by the classical action, while the higher order contributions are sums of vacuum bubble Feynman diagrams with an increasing number of loops.

The finite size scaling technique is based on the idea of a renormalized coupling that runs with the box size L . In order for L to be the only external scale in the system all dimensionful parameters are taken proportional to L . In particular, one sets $T = L$ and the quark masses are set to zero. Then one lets the background field B depend on a dimensionless parameter η and defines a renormalized coupling,

$$\frac{1}{\bar{g}^2(L)} = \frac{\Gamma'_0}{\Gamma'}, \quad (2.7)$$

where the prime indicates differentiation with respect to η at $\eta = 0$,

$$\Gamma' = \frac{\partial}{\partial \eta} \Gamma[B(\eta)] \Big|_{\eta=0}. \quad (2.8)$$

Note that the numerator Γ'_0 merely serves as a normalization constant to ensure $\bar{g} = g_0$ at tree level. When initially defined with the lattice regularisation in place the coupling is well-defined beyond perturbation theory. In particular, its non-perturbative running has been previously computed for a pure Yang Mills theory in [3] and for QCD with two flavours in [4]. Here, we prepare the set-up for similar studies in QCD with 4 flavours. Staggered fermions seem to be a natural choice, as they come in multiples of 4 species, due to the incomplete elimination of the doubler modes.

3. Subtleties with staggered fermions

As observed previously in [5, 6], the SF with staggered fermions requires lattices where the time extent T/a is odd while the spatial extent L/a has to be even. This is illustrated in Fig. 1. As the Dirac spinors are reconstructed from the one-component fields on the corners of a hypercube (indicated by different colours) the constraint arises from having the degrees of freedom for a multiple of 4 Dirac spinors fit on the lattice, taking into account that half the Dirac spinors satisfy Dirichlet conditions at the time boundaries.

The continuum limit for the SF coupling is usually taken setting $T = L$ already at finite values of the lattice spacing. Obviously, with staggered fermions this can only be done up to terms of $O(a)$. In order to define the continuum limit, it is convenient to imagine a dual lattice, as indicated in Fig.1. For the spatial directions, the dual lattice has the same number of points as the original lattice. However, in the time direction, the number of points is reduced by one. Denoting the temporal length of the dual lattice by T'/a one then has $T' = L$. A further justification may be obtained by looking at the fermionic degrees of freedom: the four Dirac spinors are reconstructed from the 2^4 one-component spinors of the corners of a hypercube and may be assumed to live on a lattice with nodes in the center of each hypercube. This is nothing but the dual lattice where every second node is eliminated in each direction. In order to get an integer multiple of 4 Dirac spinor fields, it is thus necessary that both L/a and T'/a are even. Finally, the case $T = L - a$ can be treated the same way, if one imagines a modified dual lattice overlapping by $a/2$ in both time directions. In conclusion, lattices with $T = L \pm a$ are interpreted as having physical time extent T' with $T' = T + sa$ with $s = \pm 1$, and it is then possible to set $T' = L$. This modification affects even the pure gauge theory, so that we have to revisit the $O(a)$ effects there before turning to the fermionic contributions.

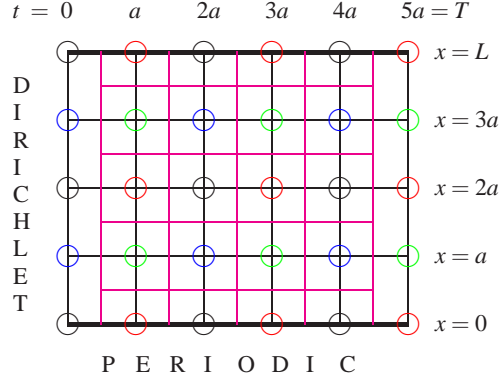


Figure 1: 2^d fermionic degrees of freedom on a $d = 2$ dimensional lattice.

4. Pure gauge theory

On the lattice we choose the usual Wilson plaquette gauge action,

$$S[U] = \frac{1}{g_0^2} \sum_p w(p) \text{tr} \{1 - U(p)\}, \quad (4.1)$$

where the sum runs over all oriented plaquettes p , and $U(p)$ denotes the parallel transporter around p . The coefficients $w(p)$ are weight factors to be specified shortly. Due to the abelian nature of the boundary fields C and C' the lattice version of Eqs. (2.2) for the link variables $U(x, \mu)$ reads

$$U(0, \mathbf{x}, k) = \exp(aC_k), \quad U(T, \mathbf{x}, k) = \exp(aC'_k). \quad (4.2)$$

With these boundary conditions it is known that the $O(a)$ effects generated by the presence of the boundary are encoded in a single operator,

$$a \int_{x_0=0, T} d^3 \mathbf{x} \text{tr} \{F_{0k} F_{0k}\}. \quad (4.3)$$

An $O(a)$ improved lattice action can thus be obtained from Eq. (4.1), by setting $w(p)$ equal to 1 except for the time-like plaquettes attached to one of the time boundaries, where one sets $w(P) = c_t(g_0)$. The coefficient c_t then multiplies a lattice version of (4.3), and, if chosen appropriately, the $O(a)$ effects in observables are cancelled.

4.1 Tree-level considerations

In perturbation theory, c_t is expanded

$$c_t(g_0) = c_t^{(0)} + c_t^{(1)} g_0^2 + O(g_0^4). \quad (4.4)$$

In the standard framework with $T = L$ one has $c_t^{(0)} = 1$ and the next two coefficients are known, too [7]. However, when taking the continuum limit at fixed $T'/L = 1$, a modification of the tree-level coefficient $c_t^{(0)}$ is required. To calculate it, we first need to determine the minimal action configuration as a function of $c_t^{(0)}$.

4.2 Equations of motion

In order to be able to write the equations of motion concisely, we follow [1] and introduce the covariant divergence of the plaquette, slightly modified by the inclusion of the weight factors,

$$d_w^* P(x, \mu) = \sum_{\nu=0}^3 \left\{ w[P_{\mu\nu}(x)] P_{\mu\nu}(x) - w[P_{\mu\nu}(x - \hat{\nu})] U^\dagger(x - \hat{\nu}, \nu) P_{\mu\nu}(x - \hat{\nu}) U(x - \hat{\nu}, \nu) \right\}. \quad (4.5)$$

The lattice action will be stationary if and only if the traceless antihermitian part of $d_w^* P(x, \mu)$ vanishes,

$$d_w^* P(x, \mu) - d_w^* P^\dagger(x, \mu) - \frac{1}{N} \text{tr} \left\{ d_w^* P(x, \mu) - d_w^* P^\dagger(x, \mu) \right\} = 0. \quad (4.6)$$

We make an ansatz of the form,

$$V(x, \mu) = \exp(a B_\mu(x_0)), \quad (4.7)$$

with a spatially constant and Abelian field $B_\mu(x_0)$. Up to gauge equivalence, the equations of motion are then solved by,

$$B_0(x_0) = 0, \quad B_k(x_0) = \begin{cases} (x_0 - \frac{T}{2}) f + \frac{C_k + C'_k}{2} & x_0 \in [a, T - a] \\ C_k & x_0 = 0 \\ C'_k & x_0 = T \end{cases} \quad (4.8)$$

where f can be computed either numerically or as a power series in a . To check whether the above ansatz really leads to the absolute minimum gauge configuration, a cooling procedure has been applied to random gauge configurations. While this does not constitute a proof, the apparent absence of configurations with lower action for various lattice sizes lends further support to our assumptions.

4.3 Choice of $c_t^{(0)}$

The tree-level coefficient $c_t^{(0)}$ is to be chosen such that $O(a)$ effects in observables are cancelled. To this order we may just require the lattice action to coincide with its continuum counterpart up to $O(a^2)$ terms. Expanding the lattice action to $O(a)$

$$S_{latt} = S_{cont} \left\{ 1 + \frac{a}{L} \left[-2 + s + \frac{4c_t^{(0)} - 2}{c_t^{(0)}} \right] + O(a^2) \right\}. \quad (4.9)$$

we thus conclude

$$c_t^{(0)} = \frac{2}{2 + s}. \quad (4.10)$$

A closer look then reveals that this choice even removes the lattice artefacts in the action up to order a^4 .

4.4 One-loop calculation

Working in a renormalisable gauge, the first two terms in the effective action, Eq. (2.5), are given by

$$\Gamma_0[B] = g_0^2 S[V], \quad \Gamma_1[B] = \frac{1}{2} \ln \det \Delta_1 - \ln \det \Delta_0, \quad (4.11)$$

where Δ_0 and Δ_1 are the fluctuation operators for the ghost fields and gauge fields respectively. The SF coupling to this order then becomes

$$\bar{g}^2(L) = g_0^2 + m_1(L/a)g_0^4 + \dots, \quad m_1(L/a) = -\Gamma'_1/\Gamma'_0. \quad (4.12)$$

To compute the quantities,

$$\partial_\eta (\ln \det \Delta_j) / \Gamma'_0, \quad j = 0, 1, \quad (4.13)$$

we have followed the strategy used in [1]. One expects that $m_1(L/a)$ has an asymptotic expansion of the form,

$$m_1(L/a) \stackrel{a/L \rightarrow 0}{\sim} \sum_{n=0}^{\infty} (a/L)^n (B_n + A_n \ln(L/a)). \quad (4.14)$$

The results obtained for $m_1(L)$ have been confirmed by an independent calculation performed by S. Takeda and U. Wolff [8]. The (preliminary) results obtained for B_0 and B_1 are shown in Table 1, where we have set A_0 and A_1 to their expected values (after having confirmed them numerically).

s	A_0	B_0	A_1	B_1
-1	$22/(4\pi)^2$	0.368283(1)	0	-0.2318(3)
0	$22/(4\pi)^2$	0.3682817(7)	0	-0.1779(3)
1	$22/(4\pi)^2$	0.3682818(7)	0	0.1232(4)

Table 1: Preliminary results for the coefficients in the asymptotic expansion (in collaboration with S. Takeda and U. Wolff).

4.5 Determination of $c_t^{(1)}$

To determine the one-loop coefficient, we expand the lattice action as a Taylor series about $c_t = c_t^{(0)}$,

$$S_{latt} = S_{latt}|_{c_t=c_t^{(0)}} + g_0^2 \frac{\partial S_{latt}}{\partial c_t} \Big|_{c_t=c_t^{(0)}} c_t^{(1)}. \quad (4.15)$$

Inserting this expansion in the definition of the coupling we arrive at,

$$\bar{g}^2(L) = g_0^2 + \left(m_1(L/a) - c_t^{(1)} \partial_{c_t} \Gamma'_0 \Big|_{c_t=c_t^{(0)}} / \Gamma'_0 \Big|_{c_t=c_t^{(0)}} \right) g_0^4 + \mathcal{O}(g_0^6) \quad (4.16)$$

The factor multiplying $c_t^{(1)}$ behaves like $\mathcal{O}(a/L)$, and can thus remove the contribution of B_1 , if we adjust $c_t^{(1)}$ properly. We obtain

$$c_t^{(1)} = \frac{B_1}{2} \left(c_t^{(0)} \right)^2. \quad (4.17)$$

5. Staggered fermion action

The fermionic part of the action takes the form,

$$S_f = a^4 \sum_{\mathbf{x}} \sum_{x_0=a}^{T-a} \frac{1}{2a} \eta_\mu(x) \bar{\chi}(x) [\lambda_\mu U(x, \mu) \chi(x + \hat{\mu}) - \lambda_\mu^\dagger U^\dagger(x - \hat{\mu}, \mu) \chi(x - \hat{\mu})] + S_B^{(0)} + S_B^{(T)}, \quad (5.1)$$

where the last two terms encode fermionic boundary terms [6]. The coefficient $c_t^{(1)}$ also receives a contribution from the fermionic part of the action. We have obtained preliminary results for the contribution to $m_1(L/a)$; in particular, we find that A_0 and B_0 coincide with the results obtained by Heller [6]. However, before we can quote a value for the staggered one-loop contribution to $c_t^{(1)}$, a more detailed analysis of the fermionic $O(a)$ boundary counterterm needs to be carried out.

6. Conclusions

First steps have been taken in a study of the SF running coupling in QCD with four quark flavours. Staggered fermions are a natural choice but require some revision of the standard framework, due to the fact that $L = T$ can only be imposed up to $O(a)$ terms. Our proposal to take the continuum limit at fixed $T' = L$ with $T' = T \pm a$ modifies the $O(a)$ improvement counterterm proportional to c_t , which we then determined perturbatively at the tree and the one-loop level. As a byproduct this yields an alternative definition of the pure gauge SF which is currently being tested in simulations (in collaboration with S. Takeda and U. Wolff). In the near future, this will hopefully be followed by numerical simulations including staggered fermions.

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