A Parameter–Uniform Finite Difference Method for a Singularly Perturbed Linear Dynamical System

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Abstract

A coupled system of two singularly perturbed ordinary differential equations of first order with the prescribed initial values are considered. The leading term of each equation is multiplied by a small positive parameter and the parameters may differ. The solution exhibits overlapping layers. A Shishkin mesh is constructed. A classical finite difference scheme applied on this mesh (which is piecewise uniform) is proved to be uniformly first order accurate in both the parameters. Numerical results are presented in support of the theory.

Keywords: System of differential equations, Initial Value Problems(IVPs), Singular Perturbation Problems(SPPs), boundary layers, Classical finite difference scheme, Fitted mesh.

Subject Classification: AMS 65L10 CR G1.7

1 Introduction

In many fields of applied mathematics we often come across initial/boundary value problems with small positive parameter(s). In particular, system of singularly perturbed first order ordinary differential equations occur in chemical reactor theory. Related works are found in [Shi92], [Shi89]. Parabolic and regular layers are typical for such problems [Sch79], [OS99], Matthews et.al [SMS00], [SMS02] have suggested parameter-robust numerical methods for system of singularly perturbed second order ordinary differential equations with one or two small parameters. Possible approaches to the construction of such methods and also some special schemes are given in [DMS80], [Shi92], [MRS96], [RST96]. In [MS02] a coupled system of two singularly perturbed linear reaction-diffusion two-point boundary value problems is examined. They have constructed a piecewise-uniform mesh that is a variant of the Shishkin mesh. They have suggested a central differencing method on this mesh. This method is proved to be almost first order accurate uniformly in both small parameters. We consider the singularly perturbed linear dynamical system

$$L\vec{u}(t) = \left\{ \begin{array}{lcll} (L\vec{u})_1(t) & = & \varepsilon_1 {u_1}'(t) + a_{11}(t)u_1(t) + a_{12}(t)u_2(t) = & f_1(t) \\ (L\vec{u})_2(t) & = & \varepsilon_2 {u_2}'(t) + a_{21}(t)u_1(t) + a_{22}(t)u_2(t) = & f_2(t), \quad t \in (0, T_0], T_0 > 0 \end{array} \right\} \tag{1.1}$$

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$$u_i(0) = u_i^0, \qquad \text{for } i = 1, 2 \tag{1.2}$$

where $\vec{u} = (u_1, u_2)^T$, $u_i \in \mathcal{C}^{(1)}(\Omega_0)$, $i = 1, 2$ with $\Omega_0 = (0, T_0], \overline{\Omega} = [0, T_0]$ and t denotes the time variable. The functions a_{ij}, $f_i \in \mathcal{C}^{(2)}(\overline{\Omega})$, $i, j = 1, 2$, satisfy the following inequalities

$$\left.\begin{array}{ll} (i) & a_{11}(t) > |a_{12}(t)| \;\;,\;\; a_{22}(t) > |a_{21}(t)| \\ (ii) & \qquad a_{12}(t) \;,\; a_{21}(t) \leq 0 \end{array}\right\} \qquad \forall\, t \in [0, T_0] \tag{1.3}$$

We introduce the positive number

$$\alpha < \min_{t \in \overline{\Omega}} \{a_{11}(t) + a_{12}(t) \;\;,\;\; a_{21}(t) + a_{22}(t)\} \tag{1.4}$$

and the singular perturbation parameters $\varepsilon_1, \varepsilon_2$ satisfy $0 < \varepsilon_1, \varepsilon_2 \leq 1$. Without loss of generality we take $T_0 = 1$ and hence

$$\Omega_0 = (0, 1], \;\; \Omega = (0, 1) \;\text{ and }\; \overline{\Omega} = [0, 1].$$

Linear systems of the above form may arise whenever a nonlinear singularly perturbed dynamical system is linearised about a steady state solution.

We assume that $A(t)$ is nonsingular for all t. The above problem is singularly perturbed in the following sense. First we introduce the reduced problem by putting $\varepsilon = 0$ in the system (1.1), (1.2). This gives the linear algebraic system

$$A(t)\vec{v}(t) = \vec{f}(t) \tag{1.5}$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \quad \vec{v}(t) = (v_1(t), v_2(t))^T \text{ and } \vec{f}(t) = (f_1(t), f_2(t))^T$$

Notice that this has a unique solution for each value of t, and hence the arbitrary initial condition (1.2) cannot be imposed. This shows that there are initial layers in the components of the solution in the neighbourhood of $t = 0$.

Our goal is to construct a numerical method for solving the system (1.1), (1.2), which yields numerical approximations converging in the maximum norm to the exact solution. We also require that this convergence is parameter-uniform, in the sense that the numerical approximations satisfy maximum norm error estimates having error constants that are independent of the singular perturbation parameters. It is important to note that standard numerical methods do not have the above property, even in the case of a single equation; see, for example, [MRS96] and [FHM$^+$00] for various discussions of this point.

A numerical method of the required type was constructed in [HV06] for the simpler problem in which all of the singular perturbation parameters are equal. In this case all of the solution components have an initial layer of the same width, which simplifies the construction of the numerical method. In the general case each component of the solution has its own initial layer and, as we shall see, a more complicated numerical method is required.

We remark here that the numerical method we construct here uses an *a priori* piecewise–uniform mesh, which is constructed in advance of the solution process. This is a completely different from the standard approaches to solve such problems numerically, which normally use adaptive mesh methods; see, for example, [RST96]. Our reason for using a numerical method with an *a priori* piecewise–uniform mesh is mainly theoretical; we can prove that our method is parameter–uniform in the above sense. We are not aware of any adaptive mesh method for solving this problem, that as been proved theoretically to be parameter–uniform. The main interest of this paper is therefore theoretical, and we do not claim that the constructed method is either more or, for that matter, less efficient that the well–tried standard adaptive mesh methods.

It is worth remarking that, in general, parameter–uniform methods based on *a priori* piecewise–uniform meshes are usually easy to construct. However, the proofs that they are parameter–uniform are usually difficult, requiring delicate analysis. This is true even in the case of a single equation, see, for example, [MRS96].

For the case $\varepsilon_1 < \varepsilon_2$, the solution $\vec{u} = (u_1, u_2)^T$ has the following layer pattern. Both components u_1 and u_2 exhibits an initial layer of width $O(\varepsilon_2)$, while the component u_1 has an additional layer of width $O(\varepsilon_1)$. For each real valued function $\varphi \in \mathcal{C}(S)$, (where S is a closed set in $\overline{\Omega}$), we define the maximum norm by

$$\| \varphi \|_S = \max_{t \in S} |\varphi(t)|$$

For each vector valued function $\vec{v}(t) = (v_1(t), v_2(t))^T \in \mathcal{C}(S) \times \mathcal{C}(S)$, set $|\vec{v}(t)| = (|v_1(t)|, |v_2(t)|)^T$, and

$$\| \vec{v} \|_S = \max \{ \| v_1 \|_S, \| v_2 \|_S \}$$

Given two vector valued funcitons $\vec{v}(t)$ and $\vec{w}(t)$, we write $\vec{v}(t) \le \vec{w}(t)$ if $v_1(t) \le w_1(t)$ and $v_2(t) \le w_2(t)$ for all $t \in \overline{\Omega}$. A mesh $\overline{\Omega}^N = \{t_i\}_{i=0}^N$ is a set of points satisfying $0 = t_0 < t_1 < \cdots < t_N = 1$. A mesh funtion $V = \{V(t_i)\}_{i=0}^N$ is a real valued function defined on $\overline{\Omega}^N$. The discrete maximum norm for the above functions is defined by

$$\| V \|_{\overline{\Omega}^N} = \max_{i=0,1,\dots,N} |V(t_i)|$$

and we define

$$\| \vec{V} \|_{\overline{\Omega}^N} = \max \left\{ \| V_1 \|_{\overline{\Omega}^N}, \| V_2 \|_{\overline{\Omega}^N} \right\}$$

where the vector mesh functions $\vec{V} = (V_1, V_2)^T = \{V_1(t_i), V_2(t_i)\}_{i=0}^N$. Throughout the paper, C denotes a generic constant, independent of t, ε_1 and ε_2.

2 Analytical results

Suppose that the parameter ν satisfies $0 < \nu \le 1$ and the function $a(.)$ satisfies $a(t) > a_0 > 0$ for all $t \in \overline{\Omega}$, where a_0 is some positive constant. For each $y \in \mathcal{C}^{(1)}(\overline{\Omega})$, define the scalar differential operator

$$L_\nu y(t) := \nu y'(t) + a(t) y(t).$$

This operator has the following properties:

Lemma 2.1 *(Comparision principle for scalar problem) If* $y(0) \ge |z(0)|$ *and* $L_\nu y(t) \ge |L_\nu z(t)|$ *for all* $t \in \Omega_0$, *then* $y(t) \ge |z(t)|$ *for all* $t \in \overline{\Omega}$.

Proof : It can be easily deduced from the maximum principle satisfied by L_ν [DMS80] on considering the functions $y(t) \pm z(t)$. ∎

It is to be noted that Lemma 2.1 can be applied to each one of the above operators. An analogous result is given for the operator L of the coupled system in the following lemma.

Lemma 2.2 *(Comparision principle for the coupled system) If* $\vec{\Psi}(0) \ge |\vec{\varphi}(0)|$ *and* $L\vec{\Psi} \ge |L\vec{\varphi}|$ *on* Ω_0, *then* $\vec{\Psi}(t) \ge |\vec{\varphi}(t)|$ *for all* $t \in \overline{\Omega}$.

Proof : We prove the Lemma for case $\vec{\varphi} = \vec{0}$ and the general case follows by considering the functions $\vec{\Psi} \pm \vec{\varphi}$. Suppose that the Lemma is false. Then there exists at least one $t^* \in \overline{\Omega}$ such that

$$min_{i=1,2}\{\Psi_i(t^*)\} = min_{i=1,2}\left\{min_{t\in\overline{\Omega}}(\Psi_i(t))\right\} < 0$$

Then $t^* \neq 0$. Without loss of generality, let us assume that $\Psi_1(t^*) \leq \Psi_2(t^*)$. Then $\varepsilon_1\Psi_1'(t^*) \leq 0$, because $0 \leq -a_{12}(t) \leq a_{11}(t)$ on $\overline{\Omega}$, we have $a_{11}(t^*)\Psi_1(t^*) < -a_{12}(t^*)\Psi_2(t^*)$. Hence $\varepsilon_1\Psi_1'(t^*) + a_{11}(t^*)\Psi_1(t^*)+a_{12}(t^*)\Psi_2(t^*) < 0$, this contradicts the hypothesis of the lemma. ▪

A Shishkin decomposition of $\vec{u}$ is given by $\vec{u} = \vec{v} + \vec{w}$ where $\vec{v} = (v_1, v_2)^T$ is the solution to $L\vec{v} = \vec{f}$ on Ω_0 and $\vec{v} = A^{-1}\vec{f}$ at $t = 0$ and $\vec{w} = (w_1, w_2)^T$ satisfies $L\vec{w} = \vec{0}$ on Ω_0 and $\vec{w}(0) = \vec{u}(0) - \vec{v}(0)$. Here, $\vec{v}$ is the smooth component of $\vec{u}$ and $\vec{v}$ has bounded derivative uniformly for $0 < \varepsilon_1, \varepsilon_2 \leq 1$. The function $\vec{w}$ is the layer part of the solution $\vec{u}$.

Lemma 2.3 *There exists a constant C such that $\| \vec{v}^{(k)} \| \leq C$ for $k = 0, 1$ and $\| v_1'' \| \leq C\varepsilon_1^{-1}$ and $\| v_2'' \| \leq C\varepsilon_2^{-1}$.*

Proof : Let $C_1 = \| A(0)^{-1}\vec{f}(0) \| + \dfrac{\| \vec{f} \|}{\alpha}$ and $\vec{\Psi}$ be a constant barrier function $C_1(1,1)^T$.

$$L\vec{\Psi} = C_1 \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix} > C_1\alpha(1,1)^T$$

$$|L\vec{v}| = |\vec{f}| = (|f_1|, |f_2|)^T \leq \| \vec{f} \| (1,1)^T \leq C_1\alpha\,(1,1)^T \leq L\vec{\Psi}$$
$$|\vec{v}(0)| = |A(0)^{-1}\vec{f}(0)| \leq \| A(0)^{-1}\vec{f}(0) \| (1,1)^T \leq \vec{\Psi}(0)$$

Hence by Lemma 2.2, $|\vec{v}(t)| \leq \vec{\Psi}(t)$ in $\overline{\Omega}$.
Therefore,

$$\| \vec{v} \| \leq C_1 \tag{2.1}$$

At $t = 0$, $\varepsilon_1 v_1' = 0$ and $\varepsilon_2 v_2' = 0$ or $v_1'(0) = 0 = v_2'(0)$.
This implies that

$$\vec{v}\,'(0) = \vec{0}.$$

Differentiating $L\vec{v} = \vec{f}$ once

$$L\vec{v}\,' = \vec{f}\,' - \begin{pmatrix} a_{11}'v_1 + a_{12}'v_2 \\ a_{21}'v_1 + a_{22}'v_2 \end{pmatrix}$$

$$\left|L\vec{v}\,'\right| \leq (C + 2C_1C_2)\,(1,1)^T \leq C(1,1)^T$$

where $C \geq \| \vec{f}\,' \|$ and $C_2 = max\{\,|a_{11}'|, |a_{12}'|, |a_{21}'|, |a_{22}'|\,\}$.
Define $\vec{\Psi} = \frac{1}{\alpha}(C + 2C_1C_2)\,(1,1)^T$.

$$\begin{aligned} L\vec{\Psi} &= \frac{C}{\alpha}\begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix} + \frac{2C_1C_2}{\alpha}\begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix} \\ &\geq C(1,1)^T + 2C_1C_2(1,1)^T \\ &= (C + 2C_1C_2)(1,1)^T \\ &\geq \left|L\vec{v}\,'\right| \end{aligned}$$

Therefore by Lemma 2.2,

$$\| \vec{v}\,' \| \leq C \tag{2.2}$$

Consider $(L\vec{v})_2 = \varepsilon_2 v_2' + a_{21}v_1 + a_{22}v_2 = f_2$
Differentiating once, we have

$$\varepsilon_2 v_2'' = f_2' - a_{21}v_1' - a_{21}'v_1 - a_{22}v_2' - a_{22}'v_2$$

Therefore,

$$\left|v_2''(t)\right| \le C\varepsilon_2^{-1} \tag{2.3}$$

Similarly

$$\left|v_1''(t)\right| \le C\varepsilon_1^{-1} \tag{2.4}$$

Equations $(2.1) - (2.4)$ completes the proof. ∎

We now find bounds on the layer component $\vec{w}$ of $\vec{u}$. Consider the layer functions

$$B_1(t) = e^{-\alpha t/\varepsilon_1}$$
$$B_2(t) = e^{-\alpha t/\varepsilon_2}$$

Lemma 2.4 *The solution $\vec{w}(t) = (w_1(t), w_2(t))^T$ of the problem $L\vec{w} = \vec{0}$, $\vec{w}(0) = \vec{u}(0) - \vec{v}(0)$ satisfies*

$$\begin{array}{ll}
|w_1(t)| \le CB_2(t) & |w_2(t)| \le CB_2(t) \\
|w_1'(t)| \le C\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right] & \left|w_2'(t)\right| \le C\varepsilon_2^{-1}B_2(t) \\
|w_1''(t)| \le C\varepsilon_1^{-1}\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right] & \left|w_2''(t)\right| \le C\varepsilon_2^{-1}\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right]
\end{array}$$

Proof : We have $\| \vec{w}(0) \| \le C$.
Define

$$\vec{\Psi}(t) = C\begin{pmatrix} B_2(t) \\ B_2(t) \end{pmatrix}$$

$$\begin{aligned}
(L\vec{\Psi})_1 &= C\left\{\varepsilon_1\left(\frac{-\alpha}{\varepsilon_2}B_2(t)\right) + a_{11}(t)B_2(t) + a_{12}(t)B_2(t)\right\} \qquad \text{by (1.4)} \\
&\ge CB_2(t)\left(-\alpha + a_{11}(t) + a_{12}(t)\right) \ge 0 = (L\vec{w})_1
\end{aligned}$$

$$\begin{aligned}
(L\vec{\Psi})_2 &= C\left\{\varepsilon_2\left(\frac{-\alpha}{\varepsilon_2}B_2(t)\right) + a_{21}(t)B_2(t) + a_{22}(t)B_2(t)\right\} \qquad \text{by (1.4)} \\
&\ge CB_2(t)\left(a_{21}(t) + a_{22}(t) - \alpha\right) > 0 = (L\vec{w})_2
\end{aligned}$$

Thus, $L\vec{\Psi}(t) \ge |L\vec{w}(t)|$ on Ω. Also, $\vec{\Psi}(0) \ge |\vec{w}(0)|$, by choosing C sufficiently large. Thus, by Lemma 2.2, we have $\vec{\Psi}(t) \ge |\vec{w}(t)|$. Hence, we have

$$|w_1(t)| \le CB_2(t) \tag{2.5}$$
$$|w_2(t)| \le CB_2(t) \tag{2.6}$$

To bound the first order derivatives, we consider $(L\vec{w})_1 = 0$,

$$\varepsilon_1 w_1' + a_{11}w_1 + a_{12}w_2 = 0$$

which implies that

$$\left|w_1'(t)\right| \le C\varepsilon_1^{-1}B_2(t) \tag{2.7}$$

In particular,

$$\left|w_1'(0)\right| \leq C\varepsilon_1^{-1} \tag{2.8}$$

Similarly, from the second equation $(L\vec{w})_2 = 0$

$$\varepsilon_2 w_2' + a_{21}w_1 + a_{22}w_2 = 0$$

we get

$$\left|w_2'(t)\right| \leq C\varepsilon_2^{-1}B_2(t) \tag{2.9}$$

To find the sharp bound for $w_1'(t)$, consider the scalar equation

$$L_1 w_1 = \varepsilon_1 w_1' + a_{11}w_1 = -a_{12}w_2$$

and hence

$$L_1 w_1' = (-a_{12}w_2)' - a_{11}'w_1$$

$$\left|L_1 w_1'(t)\right| \leq C\varepsilon_2^{-1}B_2(t) \tag{2.10}$$

Define a barrier function $\Psi(t) = C\left(\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right)$, then it is easy to check that

$$\Psi(0) \geq |w_1'(0)| \tag{2.11}$$

Further,

$$\begin{aligned}
L_1\Psi(t) &= C\left\{\varepsilon_1\left[\varepsilon_1^{-1}\left(\frac{-\alpha}{\varepsilon_1}\right)B_1(t) + \varepsilon_2^{-1}\left(\frac{-\alpha}{\varepsilon_2}\right)B_2(t)\right] + a_{11}\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right]\right\} \\
&\geq C\left\{\varepsilon_1^{-1}B_1(t)(-\alpha + a_{11}) + \varepsilon_2^{-1}B_2(t)(-\alpha + a_{11})\right\} \qquad \text{as } \varepsilon_1 < \varepsilon_2 \\
&\geq C(-\alpha + a_{11})\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right] \\
&\geq C(-\alpha + a_{11})\varepsilon_2^{-1}B_2(t) \\
&\geq \left|L_1 w_1'(t)\right| \qquad \text{in comparison with (2.10)}
\end{aligned} \tag{2.12}$$

Applying Lemma 2.1 to $\Psi(t)$, (2.11) & (2.12) imply that,$\Psi(t) \geq |w_1'(t)|$.
Therefore,

$$\left|w_1'(t)\right| \leq C\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right]. \tag{2.13}$$

Differentiating $(L\vec{w})_1 = 0$ and $(L\vec{w})_2 = 0$ once and using the estimates of $w_1'(t)$ and $w_2'(t)$, we have

$$\left|w_1''(t)\right| \leq C\varepsilon_1^{-1}\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right] \tag{2.14}$$

$$\left|w_2''(t)\right| \leq C\varepsilon_2^{-1}\left[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)\right] \tag{2.15}$$

This completes the proof. ∎

To prove the parameters-uniform convergence of the discrete solution, the cases $\frac{\varepsilon_2}{2} \leq \varepsilon_1 \leq \varepsilon_2$ and $\varepsilon_1 < \frac{\varepsilon_2}{2}$ are considered separately. In the former case, the bounds of the Lemma 2.4 suffices, but for the latter case, the decomposition of $\vec{w}(t)$ is needed. To find the decomposition of $\vec{w}(t)$ and their estimates we apply the following lemma:

Lemma 2.5 *If* $\varepsilon_2 \in \left(2\varepsilon_1, \dfrac{\alpha}{2}\right)$, *then there exists a unique point*

$$t^* = t^*(\varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1 \varepsilon_2}{\alpha(\varepsilon_2 - \varepsilon_1)} \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right) \in (0, 1)$$

such that

$$\varepsilon_1^{-1} B_1(t^*) = \varepsilon_2^{-1} B_2(t^*). \tag{2.16}$$

Also, the following inequalities are found true:

$$\varepsilon_1^{-1} B_1(t) > \varepsilon_2^{-1} B_2(t) \qquad \text{on } [0, t^*) \tag{2.17}$$

$$\varepsilon_1^{-1} B_1(t) < \varepsilon_2^{-1} B_2(t) \qquad \text{on } (t^*, 1] \tag{2.18}$$

Proof : Consider

$$\begin{aligned}
\frac{B_1(t^*)}{B_2(t^*)} &= e^{-\alpha t^* (\varepsilon_1^{-1} - \varepsilon_2^{-1})} \\
&= e^{-\alpha t^* \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2}\right)} \\
&= e^{-\alpha \frac{\varepsilon_1 \varepsilon_2}{\alpha(\varepsilon_2 - \varepsilon_1)} \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2}\right)} \\
&= \frac{\varepsilon_1}{\varepsilon_2} \\
\varepsilon_1^{-1} B_1(t^*) &= \varepsilon_2^{-1} B_2(t^*).
\end{aligned}$$

and hence

$$t^* = \frac{\varepsilon_1 \varepsilon_2}{\alpha(\varepsilon_2 - \varepsilon_1)} \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right).$$

As $\varepsilon_1 < \varepsilon_2$, obviously we have $t^* > 0$ and

$$t^* = \frac{\varepsilon_1 \varepsilon_2}{\alpha(\varepsilon_2 - \varepsilon_1)} \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right) < \frac{\varepsilon_2}{\alpha(1 - (\varepsilon_1/\varepsilon_2))}$$

As $\varepsilon_1 < \dfrac{\varepsilon_2}{2}$, we have $\dfrac{\varepsilon_1}{\varepsilon_2} < \dfrac{1}{2}$.
Therefore, $1 - \dfrac{\varepsilon_1}{\varepsilon_2} > \dfrac{1}{2}$ implies that $\dfrac{1}{1 - \dfrac{\varepsilon_1}{\varepsilon_2}} < 2$.
Hence, $t^* < \dfrac{2\varepsilon_2}{\alpha} < 1$ if $\varepsilon_2 < \dfrac{\alpha}{2}$.
To prove the uniqueness of t^*, let us assume that there exists $t' \in (0, 1]$ such that

$$\varepsilon_1^{-1} B_1(t') = \varepsilon_2^{-1} B_2(t'). \tag{2.19}$$

On dividing (2.16) by (2.19), we get

$$B_1(t^* - t') = B_2(t^* - t')$$

and hence

$$e^{-\alpha(t^* - t')/\varepsilon_1} = e^{-\alpha(t^* - t')/\varepsilon_2}$$

$$-\alpha(t^* - t')\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) = 0$$

which implies that $t^* = t'$.
Next, consider the case when $t < t^*$,

$$\begin{aligned}
t &< \frac{\varepsilon_1\varepsilon_2}{\alpha(\varepsilon_2-\varepsilon_1)}\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right) \\
\frac{\alpha t(\varepsilon_2-\varepsilon_1)}{\varepsilon_1\varepsilon_2} &< \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right) \\
-\alpha t\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right) &< \ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right) \\
\frac{B_2(t)}{B_1(t)} &< \frac{\varepsilon_2}{\varepsilon_1} \\
\varepsilon_1^{-1}B_1(t) &> \varepsilon_2^{-1}B_2(t) \qquad \text{on } [0, t^*)
\end{aligned}$$

Similarly, when $t > t^*$, the inequality is reversed. This completes the proof of the lemma. ∎

Lemma 2.6 *Suppose that* $\varepsilon_2 \in \left(2\varepsilon_1, \frac{\alpha}{2}\right)$. *Then, there are functions*

$$w_{1,1}(t), w_{1,2}(t), w_{2,1}(t) \text{ and } w_{2,2}(t)$$

such that

$$w_1(t) = w_{1,1}(t) + w_{1,2}(t), \qquad w_2(t) = w_{2,1}(t) + w_{2,2}(t)$$

and

$$\begin{aligned}
\left|w'_{1,1}(t)\right| &\le C\varepsilon_1^{-1}B_1(t), & \left|w''_{1,2}(t)\right| &\le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_2(t) \\
\left|w'_{2,1}(t)\right| &\le C\varepsilon_2^{-1}B_1(t), & \left|w''_{2,2}(t)\right| &\le C\varepsilon_2^{-2}B_2(t) \quad \forall\, t \in \overline{\Omega}.
\end{aligned}$$

Proof : Since $\varepsilon_1 < \varepsilon_2$, by Lemma 2.5, we define a function $w_{1,2}(t)$ on $\overline{\Omega}$ as follows:

$$w_{1,2}(t) = \begin{cases} w_1(t), & \text{for } t \in [t^*, 1] \\ \displaystyle\sum_{k=0}^{2} \frac{(t-t^*)^k}{k!} w_1^{(k)}(t^*), & \text{for } t \in [0, t^*) \end{cases}$$

For $t \in [0, t^*)$, from the above construction, we have

$$\begin{aligned}
\left|w''_{1,2}(t)\right| &= \left|w''_1(t^*)\right| \\
&\le C\varepsilon_1^{-1}[\varepsilon_1^{-1}B_1(t^*) + \varepsilon_2^{-1}B_2(t^*)] && \text{using (2.14)} \\
&\le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_2(t^*) && \text{by the definition of } t^* \qquad (2.20) \\
&\le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_2(t)] && (2.21)
\end{aligned}$$

And, on $[t^*, 1]$, we have

$$\begin{aligned}
\left|w''_{1,2}(t)\right| &= \left|w''_1(t)\right| \\
&\le C\varepsilon_1^{-1}[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)] && \text{using (2.14)} \\
&\le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_2(t) && \text{using (2.18)} \qquad (2.22)
\end{aligned}$$

In summary,

$$\left|w''_{1,2}(t)\right| \;\leq\; C\varepsilon_1^{-1}\varepsilon_2^{-1}B_2(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.23}$$

The construction of $w_{1,2}$ was inspired by Madden and Stynes[MS02]. Set $w_{1,1} = w_1 - w_{1,2}$. Then, $w_{1,1} \equiv 0$ on $[t^*, 1]$, hence on $[t^*, 1]$ we have

$$|w'_{1,1}| \equiv 0 \tag{2.24}$$

and

$$|w''_{1,1}| \equiv 0 \tag{2.25}$$

Also, the inequalities (2.17) and (2.18) holds and so on $[0, t^*)$

$$\begin{aligned}
\left|w''_{1,1}(t)\right| &\leq \left|w''_1(t)\right| + \left|w''_{1,2}(t)\right| \\
&\leq C\varepsilon_1^{-1}[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)] && \text{using (2.14) and (2.23)} \\
&\leq C\varepsilon_1^{-2}B_1(t) && \text{using (2.17)}
\end{aligned} \tag{2.26}$$

In summary,

$$\left|w''_{1,1}(t)\right| \;\leq\; C\varepsilon_1^{-2}B_1(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.27}$$

Integrating over (t, t^*), and using (2.24) on $[t^*, 1]$, as $t^* \in [t^*, 1]$,

$$w'_{1,1}(t^*) - w'_{1,1}(t) = \int_t^{t^*} w''_{1,1}(s)ds.$$

Now,

$$\begin{aligned}
\left|w'_{1,1}(t)\right| &\leq \int_t^{t^*} \left|w''_{1,1}(s)\right| ds \\
&\leq \int_t^{t^*} C\varepsilon_1^{-2}B_1(s)ds \\
&\leq \left|C\varepsilon_1^{-1}[B_1(t^*) - B_1(t)]\right| \\
&\leq C\varepsilon_1^{-1}[B_1(t^*) + B_1(t)]
\end{aligned}$$

Since $B_1(t^*) \;<\; B_1(t)$ for all $t \in (0, t^*)$, we have

$$\left|w'_{1,1}(t)\right| \;\leq\; C\varepsilon_1^{-1}B_1(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.28}$$

The above analysis is applied to w_2 also. Define a function $w_{2,2}(t)$ on $\overline{\Omega}$ as follows:

$$w_{2,2}(t) = \begin{cases} w_2(t), & \text{for } t \in [t^*, 1] \\ \displaystyle\sum_{k=0}^{2} \frac{(t-t^*)^k}{k!} w_2^{(k)}(t^*), & \text{for } t \in [0, t^*) \end{cases}$$

Continuing as for $w_{1,2}$ leads to

$$\left|w''_{2,2}(t)\right| \;\leq\; C\varepsilon_2^{-2}B_2(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.29}$$

Set $w_{2,1} = w_2 - w_{2,2}$. Then, $w_{2,1} \equiv 0$ on $[t^*, 1]$, hence on $[t^*, 1]$ we have

$$|w'_{2,1}| \equiv 0 \tag{2.30}$$

and

$$|w''_{2,1}| \equiv 0 \tag{2.31}$$

Also, the inequalities (2.17) and (2.18) holds and hence

$$\begin{aligned} \left|w''_{2,1}(t)\right| &\le \left|w''_2(t)\right| + \left|w''_{2,2}(t)\right| \\ &\le C\varepsilon_2^{-1}[\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-1}B_2(t)] \\ &\le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_1(t) \qquad \text{on } [0, t^*) \end{aligned} \tag{2.32}$$

In summary,

$$\left|w''_{2,1}(t)\right| \le C\varepsilon_1^{-1}\varepsilon_2^{-1}B_1(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.33}$$

Integrating over (t, t^*), we get

$$0 - w'_{2,1}(t) = \int_t^{t^*} w''_{2,1}(s)ds$$

Since $t^* \in [t^*, 1]$, we have $w'_{2,1}(t^*) = 0$, as before.
Now,

$$\begin{aligned} \left|w'_{2,1}(t)\right| &\le \int_t^{t^*} \left|w''_{2,1}(s)\right| ds \\ &\le \int_t^{t^*} \left|C\varepsilon_1^{-1}\varepsilon_2^{-1}B_1(s)\right| ds \\ &\le \left|C\varepsilon_2^{-1}[B_1(t^*) - B_1(t)]\right| \\ &\le C\varepsilon_2^{-1}[B_1(t^*) + B_1(t)] \end{aligned}$$

Since $B_1(t^*) < B_1(t)$ for all $t \in (t, t^*)$, we have

$$\left|w'_{2,1}(t)\right| \le C\varepsilon_2^{-1}B_1(t) \qquad \text{on} \qquad \overline{\Omega}. \tag{2.34}$$

This completes the proof. ▪

3 The discrete problem

The IVP (1.1),(1.2) is discretized using a fitted mesh method composed of a classical finite difference operator on a piecewise uniform fitted mesh. The discrete problem is

$$L^N\vec{U}(t_j) = \begin{cases} (L^N\vec{U})_1(t_j) = \varepsilon_1 D^- U_1(t_j) + a_{11}(t_j)U_1(t_j) + a_{12}(t_j)U_2(t_j) = f_1(t_j) \\ (L^N\vec{U})_2(t_j) = \varepsilon_2 D^- U_2(t_j) + a_{21}(t_j)U_1(t_j) + a_{22}(t_j)U_2(t_j) = f_2(t_j), j = 1(1)N \end{cases} \tag{3.1}$$

$$U_i(0) = u_i(0), \qquad \text{for } i = 1, 2 \tag{3.2}$$

where

$$D^- U_i(t_j) = \frac{U_i(t_j) - U_i(t_{j-1})}{t_j - t_{j-1}}, \qquad j = 1(1)N, \ i = 1, 2$$

and the fitted mesh $\overline{\Omega}^N_{\sigma_1,\sigma_2}$ is given by $\{t_j\}_0^N$ where

$$\begin{aligned} t_j &= \frac{4j\sigma_1}{N} \quad , j = 0(1)\frac{N}{4} \\ t_{\frac{N}{4}+j} &= \sigma_1 + \frac{4j\,(\sigma_2-\sigma_1)}{N} \quad ,j = 1(1)\frac{N}{4} \\ t_{\frac{N}{2}+j} &= \sigma_2 + \frac{2j\,(1-\sigma_2)}{N} \quad ,j = 1(1)\frac{N}{2} \end{aligned}$$

The transition parameters σ_1 and σ_2 are given by

$$\sigma_1 = min\left\{ \frac{\sigma_2}{2}, \frac{\varepsilon_1}{\alpha} lnN \right\} \quad , \quad \sigma_2 = min\left\{ \frac{1}{2}\, , \frac{\varepsilon_2}{\alpha} lnN \right\} \tag{3.3}$$

Theorem 3.1 *(Discrete maximum principle) If* $\left\{\vec{\Psi}(t_i)\right\}_{i=0}^N$ *be any vector mesh function such that* $\vec{\Psi}(t_0) \geq \vec{0}$ *and* $L^N\vec{\Psi}(t_i) \geq \vec{0}$ *for* $1 \leq i \leq N$, *then* $\vec{\Psi}(t_i) \geq \vec{0}$ *for* $0 \leq i \leq N$.

Proof : If the conclusion of the theorem is false, one can choose k such that

$$\min_{j=1,2} \{\Psi_j(t_k)\} = \min_{j=1,2} \{min_i\,(\Psi_j(t_i))\} < 0, \qquad 1 \leq i \leq N.$$

Clearly $t_k \neq 0$. Without loss of generality, let us assume that $\Psi_1(t_k) < \Psi_2(t_k)$. Then

$$D^-\,(\Psi_1(t_k)) = \frac{\Psi_1(t_k) - \Psi_1(t_{k-1})}{h_k} \leq 0.$$

As $0 \leq -a_{12}(t_k) < a_{11}(t_k)$ on $\overline{\Omega}^N_{\sigma_1,\sigma_2}$, we have

$$a_{11}(t_k)\Psi_1(t_k) < -a_{12}(t_k)\Psi_1(t_k) < -a_{12}(t_k)\Psi_2(t_k),$$

and so

$$a_{11}(t_k)\Psi_1(t_k) + a_{12}(t_k)\Psi_2(t_k) < 0.$$

Hence

$$\varepsilon_1 D^-\Psi_1(t_k) + a_{11}(t_k)\Psi_1(t_k) + a_{12}(t_k)\Psi_2(t_k) < 0,$$

which contradicts the hypothesis that $L^N\vec{\Psi}(t_i) \geq \vec{0}$. ∎

An immediate consequence of this theorem is the following discrete stability result.

Theorem 3.2 *Let* $\left\{\vec{U}(t_i)\right\}_{i=0}^N$ *be any vector mesh function such that* $\vec{U}(t_0) \geq \vec{0}$. *Then*

$$\| \vec{U}(t_i) \| \leq C \max\left\{ \| \vec{U}(0) \|, \frac{1}{\alpha} \| L^N\vec{U}(t_i) \| \right\}, \qquad i = 0(1)N,$$

where C *is a constant independent of* i, ε_1 *and* ε_2.

Proof : Let $M = \max\left\{ \| \vec{U}(0) \|, \frac{1}{\alpha} \| L^N\vec{U}(t_i) \| \right\}$. Define the barrier function

$$\vec{\Psi}_i^{\pm} = M(1,1)^T \pm \vec{U}(t_i).$$

Then

$$\vec{\Psi}_0^{\pm} \;=\; M(1,1)^T \;\pm\; \vec{U}(t_0) \;\geq\; \vec{0}.$$

Also, for $i = 1(1)N$, we have

$$L^N \vec{\Psi}_i^{\pm} \;=\; M \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix} \pm\; L^N \vec{U}(t_i) \;>\; M\alpha(1,1)^T \;\pm\; \vec{f} \;>\; \vec{0}.$$

Then by Theorem 3.1, the result follows. ∎

To estimate the error, we use the following standard results.

$$|(D^- - D)\psi(t_i)| \leq \max_{[t_i - t_{i-1}]} |\psi''(s)| \,\frac{(t_i - t_{i-1})}{2} \tag{3.4}$$

$$|(D^- - D)\psi(t_i)| \leq 2 \max_{[t_i - t_{i-1}]} |\psi'(s)| \tag{3.5}$$

Lemma 3.3 *If the transition points are such that* $\sigma_1 = \dfrac{\sigma_2}{2}$ *and* $\sigma_2 = \dfrac{1}{2}$, *then for* $t_i \in \overline{\Omega}^N_{\sigma_1,\sigma_2}$,

$$|L^N(\vec{W} - \vec{w})(t_i)| \;\leq\; CN^{-1} \ln N(1,1)^T$$

where $\vec{w}, \vec{W}$ *are the solutions of* (1.1)-(1.2), (3.1)-(3.2)*respectively.*

Proof : Here, the mesh is uniform and we have $t_i - t_{i-1} = N^{-1}$. As $\sigma_1 = \dfrac{1}{4}$, we have $\dfrac{1}{4} \;\leq\; \dfrac{\varepsilon_1}{\alpha} \ln N$ and hence $\varepsilon_1^{-1} \;\leq\; C \ln N$. Similarly, when $\sigma_2 = \dfrac{1}{2}$, we have $\dfrac{1}{2} \;\leq\; \dfrac{\varepsilon_2}{\alpha} \ln N$ and hence $\varepsilon_2^{-1} \;\leq\; C \ln N$. Now,

$$\begin{aligned}
|L^N(\vec{W} - \vec{w})(t_i)| \;&=\; \begin{pmatrix} \varepsilon_1 (D^- - D) w_1(t_i) \\ \varepsilon_2 (D^- - D) w_2(t_i) \end{pmatrix} \\
&\leq\; \begin{pmatrix} \varepsilon_1 \displaystyle\max_{[t_{i-1},t_i]} |w_1''(s)| \,\frac{(t_i - t_{i-1})}{2} \\ \varepsilon_2 \displaystyle\max_{[t_{i-1},t_i]} |w_2''(s)| \,\frac{(t_i - t_{i-1})}{2} \end{pmatrix} \quad \text{using (3.4)} \\
&\leq\; CN^{-1} \begin{pmatrix} \varepsilon_1 \, \varepsilon_1^{-1} \, [\varepsilon_1^{-1} B_1(t_{i-1}) + \varepsilon_2^{-1} B_2(t_{i-1})] \\ \varepsilon_2 \, \varepsilon_2^{-1} \, [\varepsilon_1^{-1} B_1(t_{i-1}) + \varepsilon_2^{-1} B_2(t_{i-1})] \end{pmatrix} \quad \text{using the Lemma 2.4} \\
&\leq\; CN^{-1} \ln N(1,1)^T \qquad \text{for all } \; t_i \;\in\; \overline{\Omega}^N_{\sigma_1,\sigma_2}
\end{aligned}$$

∎

Lemma 3.4 *If the transition points are such that* $\sigma_1 = \dfrac{\sigma_2}{2}$ *and* $\sigma_2 = \dfrac{\varepsilon_2}{\alpha} \ln N$, *then for* $t_i \in \overline{\Omega}^N_{\sigma_1,\sigma_2}$,

$$|L^N(\vec{W} - \vec{w})(t_i)| \;\leq\; CN^{-1} \ln N(1,1)^T$$

where $\vec{w}, \vec{W}$ *are the solutions of* (1.1)-(1.2), (3.1)-(3.2)*respectively.*

Proof : Here, the mesh is piecewise uniform. As $\sigma_1 = \dfrac{\sigma_2}{2}$, $\dfrac{\sigma_2}{2} < \dfrac{\varepsilon_1}{\alpha}\ln N$. This implies that $\dfrac{\varepsilon_2}{2\,\alpha}\ln N < \dfrac{\varepsilon_1}{\alpha}\ln N$ or $\varepsilon_1 > \dfrac{\varepsilon_2}{2}$, the Lemma 2.4 suffices. Also $\sigma_2 - \sigma_1 = \sigma_1$

Case 1 : $t_i \in (0, \sigma_1]$
Now,

$$\begin{aligned}
|L^N(\vec{W}-\vec{w})(t_i)| &\leq C\begin{pmatrix} \varepsilon_1\,\varepsilon_1^{-1}\,[\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{\sigma_1}{N/4} \\ \varepsilon_2\,\varepsilon_2^{-1}\,[\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{\sigma_1}{N/4} \end{pmatrix} \quad \text{using (3.4)} \\
&\leq CN^{-1}\begin{pmatrix} [\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{\sigma_2}{2} \\ [\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{\sigma_2}{2} \end{pmatrix} \\
&\leq CN^{-1}\ln N(1,1)^T \qquad \text{for all } \ t_i \in (0,\sigma_1] \text{ as } \varepsilon_1 > \frac{\varepsilon_2}{2}.
\end{aligned}$$

Case 2 : $t_i \in (\sigma_1, \sigma_2]$

$$\begin{aligned}
|L^N(\vec{W}-\vec{w})(t_i)| &\leq C\begin{pmatrix} \varepsilon_1\,\varepsilon_1^{-1}\,[\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{(\sigma_2-\sigma_1)}{N/4} \\ \varepsilon_2\,\varepsilon_2^{-1}\,[\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\dfrac{(\sigma_2-\sigma_1)}{N/4} \end{pmatrix} \quad \text{using (3.4)} \\
&\leq CN^{-1}\begin{pmatrix} [\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\sigma_1 \\ [\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})]\,\sigma_1 \end{pmatrix} \\
&\leq CN^{-1}\ln N(1,1)^T \qquad \text{for all } \ t_i \in (\sigma_1,\sigma_2]
\end{aligned}$$

Case 3 : $t_i \in (\sigma_2, 1]$

$$\begin{aligned}
|L^N(\vec{W}-\vec{w})(t_i)| &= \begin{pmatrix} \varepsilon_1 \max\limits_{[t_{i-1},t_i]} \left|w_1'(s)\right| \\ \varepsilon_2 \max\limits_{[t_{i-1},t_i]} \left|w_2'(s)\right| \end{pmatrix} \quad \text{using (3.5)} \\
&\leq \begin{pmatrix} \varepsilon_1[\varepsilon_1^{-1}B_1(t_{i-1})+\varepsilon_2^{-1}B_2(t_{i-1})] \\ \varepsilon_2\varepsilon_2^{-1}B_2(t_{i-1}) \end{pmatrix} \qquad \text{as } \varepsilon_1 \leq \varepsilon_2 \\
&\leq CB_2(t_{i-1})(1,1)^T \\
&\leq Ce^{-\alpha\sigma_2/\varepsilon_2}(1,1)^T \\
&= Ce^{-lnN}(1,1)^T \\
&= CN^{-1}(1,1)^T \qquad \text{for all } \ t_i \in (\sigma_2, 1]
\end{aligned}$$

This completes the proof of the lemma. ∎

Lemma 3.5 *If the transition points are such that* $\sigma_1 = \dfrac{\varepsilon_1}{\alpha}\ln N$ *and* $\sigma_2 = \dfrac{1}{2}$, *then for* $t_i \in \overline{\Omega}^N_{\sigma_1,\sigma_2}$,

$$|L^N(\vec{W}-\vec{w})(t_i)| \leq CN^{-1}\ln N(1,1)^T$$

where $\vec{w}$, $\vec{W}$ *are the solutions of* (1.1)-(1.2), (3.1)-(3.2)*respectively.*

Proof : Here, the mesh is piecewise uniform. As $\sigma_2 = \frac{1}{2}$, we have $\frac{1}{2} < \frac{\varepsilon_2}{\alpha}\ln N$ and $\frac{\sigma_2}{2} > \frac{\varepsilon_1}{\alpha}\ln N$, and so we get $\varepsilon_1 < \frac{\varepsilon_2}{2}$. Also $\varepsilon_2^{-1} \leq C\ln N$, as $\sigma_2 = \frac{1}{2}$. Here, we use the Lemma 2.4 and the Lemma 2.6 appropriately.

Case 1 : $t_i \in (0, \sigma_1]$
Now,

$$\begin{aligned}
|L^N(\vec{W} - \vec{w})(t_i)| &= \begin{pmatrix} \varepsilon_1 \max\limits_{[t_{i-1},t_i]} |w_1''(s)| \, \dfrac{(t_i - t_{i-1})}{2} \\ \varepsilon_2 \max\limits_{[t_{i-1},t_i]} |w_2''(s)| \, \dfrac{(t_i - t_{i-1})}{2} \end{pmatrix} \quad \text{using the Lemma 2.4 and (3.4)} \\
&\leq C \begin{pmatrix} \varepsilon_1 \, \varepsilon_1^{-1} \, [\varepsilon_1^{-1} B_1(t_{i-1}) + \varepsilon_2^{-1} B_2(t_{i-1})] \, \dfrac{\sigma_1}{N/4} \\ \varepsilon_2 \, \varepsilon_2^{-1} \, [\varepsilon_1^{-1} B_1(t_{i-1}) + \varepsilon_2^{-1} B_2(t_{i-1})] \, \dfrac{\sigma_1}{N/4} \end{pmatrix} \\
&\leq CN^{-1}\ln N(1,1)^T \qquad \text{for all } \ t_i \ \in \ (0, \sigma_1]
\end{aligned}$$

Case 2 : $t_i \in (\sigma_1, \sigma_2]$
As $\varepsilon_1 < \frac{\varepsilon_2}{2}$, we recall the decomposition of $\vec{w}$ in Lemma 2.6, and hence,

$$\begin{aligned}
|L^N(\vec{W} - \vec{w})(t_i)| &\leq \begin{pmatrix} \varepsilon_1 (D^- - D)[w_{1,1}(t_i) + w_{1,2}(t_i)] \\ \varepsilon_2 (D^- - D)[w_{2,1}(t_i) + w_{2,2}(t_i)] \end{pmatrix} \\
&\leq \begin{pmatrix} [\varepsilon_1 \max\limits_{[t_{i-1},t_i]} \left|w_{1,1}'(s)\right| + [\varepsilon_1 \max\limits_{[t_{i-1},t_i]} \left|w_{1,2}''(s)\right| \, \dfrac{(t_i - t_{i-1})}{2}] \\ [\varepsilon_2 \max\limits_{[t_{i-1},t_i]} \left|w_{2,1}'(s)\right| + [\varepsilon_1 \max\limits_{[t_{i-1},t_i]} \left|w_{2,2}''(s)\right| \, \dfrac{(t_i - t_{i-1})}{2}] \end{pmatrix} \quad \text{using (3.5) and (3.4)} \\
&\leq \begin{pmatrix} [\varepsilon_1 C\varepsilon_1^{-1} B_1(t_{i-1})] + [\varepsilon_1 C\varepsilon_1^{-1}\varepsilon_2^{-1} B_2(t_{i-1}) \dfrac{\sigma_2}{N/4}] \\ [\varepsilon_2 C\varepsilon_2^{-1} B_1(t_{i-1})] + [\varepsilon_2 C\varepsilon_2^{-2} B_2(t_{i-1}) \dfrac{\sigma_2}{N/4}] \end{pmatrix} \\
&\leq [CB_1(t_{i-1}) + CN^{-1}\ln N](1,1)^T
\end{aligned}$$

Since, $t_i > \sigma_1$, we have $t_{i-1} \geq \sigma_1$, and hence

$$B_1(t_{i-1}) \;=\; e^{-(\alpha t_{i-1})/\varepsilon_1} \;\leq\; e^{-(\alpha\sigma_1)/\varepsilon_1} \;\leq\; e^{(-\varepsilon_1\alpha\ln N)/\varepsilon_1\alpha} \;=\; N^{-1}.$$

$$\begin{aligned}
|L^N(\vec{W} - \vec{w})(t_i)| &\leq [CN^{-1} + CN^{-1}\ln N](1,1)^T \\
&\leq CN^{-1}\ln N(1,1)^T \qquad \text{for all } \ t_i \ \in \ (\sigma_1, \sigma_2]
\end{aligned}$$

Case 3 : $t_i \in (\sigma_2, 1]$
Proceeding as in the previous case, we have,

$$\begin{aligned}
|L^N(\vec{W} - \vec{w})(t_i)| &\leq \begin{pmatrix} [\varepsilon_1 C\varepsilon_1^{-1} B_1(t_{i-1})] + [\varepsilon_1 C\varepsilon_1^{-1}\varepsilon_2^{-1} B_2(t_{i-1}) \dfrac{(1-\sigma_2)}{N/2}] \\ [\varepsilon_2 C\varepsilon_2^{-1} B_1(t_{i-1})] + [\varepsilon_2 C\varepsilon_2^{-2} B_2(t_{i-1}) \dfrac{(1-\sigma_2)}{N/2}] \end{pmatrix} \\
&\leq [CB_1(t_{i-1}) + CN^{-1}\ln N](1,1)^T
\end{aligned}$$

Since, $t_i > \sigma_2$, we have $t_{i-1} \geq \sigma_2$, and hence

$$B_1(t_{i-1}) \;=\; e^{-(\alpha t_{i-1})/\varepsilon_1} \;\leq\; e^{-(\alpha\sigma_2)/\varepsilon_1} \;\leq\; e^{(-\varepsilon_1\alpha\ln N)/\varepsilon_1\alpha} \;=\; N^{-1},$$

as $\sigma_2 > \dfrac{2\varepsilon_1}{\alpha}\ln N > \dfrac{\varepsilon_1}{\alpha}\ln N$.

$$\begin{aligned} |L^N(\vec{W}-\vec{w})(t_i)| &\leq [CN^{-1} + CN^{-1}\ln N](1,1)^T \\ &\leq CN^{-1}\ln N(1,1)^T \text{ for all } t_i \in (\sigma_2, 1] \end{aligned}$$

This completes the proof of the lemma. ∎

Lemma 3.6 *If the transition points are such that* $\sigma_1 = \dfrac{\varepsilon_1}{\alpha}\ln N$ *and* $\sigma_2 = \dfrac{\varepsilon_2}{\alpha}\ln N$ *then for* $t_i \in \overline{\Omega}^N_{\sigma_1,\sigma_2}$,

$$|L^N(\vec{W}-\vec{w})(t_i)| \;\leq\; CN^{-1}\ln N(1,1)^T$$

where $\vec{w}, \vec{W}$ *are the solutions of* (1.1)-(1.2), (3.1)-(3.2) *respectively.*

Proof : Here, the mesh is piecewise uniform and we use the Lemma 2.4 and the Lemma 2.6 appropriately as $\varepsilon_1 < \dfrac{\varepsilon_2}{2}$

Case 1 : $t_i \in (0, \sigma_1]$

$$\begin{aligned} |L^N(\vec{W}-\vec{w})(t_i)| &\leq \begin{pmatrix} \dfrac{\sigma_1}{N/4}\varepsilon_1\varepsilon_1^{-1}[\varepsilon_1^{-1}B_1(t_{i-1}) + \varepsilon_2^{-1}B_2(t_{i-1})] \\ \dfrac{\sigma_1}{N/4}\varepsilon_2\varepsilon_2^{-1}[\varepsilon_1^{-1}B_1(t_{i-1}) + \varepsilon_2^{-1}B_2(t_{i-1})] \end{pmatrix} \quad \text{using the Lemma 2.4 and (3.4)} \\ &\leq CN^{-1}\begin{pmatrix} \varepsilon_1\ln N[\varepsilon_1^{-1}B_1(t_{i-1}) + \varepsilon_2^{-1}B_2(t_{i-1})] \\ \varepsilon_1\ln N[\varepsilon_1^{-1}B_1(t_{i-1}) + \varepsilon_2^{-1}B_2(t_{i-1})] \end{pmatrix} \\ &\leq CN^{-1}\ln N(1,1)^T \end{aligned}$$

Hence, $\parallel L^N(\vec{W}-\vec{w}) \parallel \leq CN^{-1}\ln N$.

Case 2 : $t_i \in (\sigma_1, \sigma_2]$
Repeating the proof of the Case 2 of the Lemma 3.5 we get the desired result.

Case 3 : $t_i \in (\sigma_2, 1]$

$$\begin{aligned} |L^N(\vec{W}-\vec{w})(t_i)| &\leq \begin{pmatrix} \varepsilon_1[\varepsilon_1^{-1}B_1(t_{i-1}) + \varepsilon_2^{-1}B_2(t_{i-1})] \\ \varepsilon_2\varepsilon_2^{-1}B_2(t_{i-1}) \end{pmatrix} \quad \text{using the Lemma 2.4 and (3.5)} \\ &\leq CB_2(t_{i-1})(1,1)^T \\ &\leq C\,e^{-\alpha\sigma_2/\varepsilon_2}(1,1)^T \\ &= C\,e^{-\ln N}(1,1)^T \\ &= CN^{-1}(1,1)^T \end{aligned}$$

This concludes the proof of the lemma. ∎
The main theoretical result, a parameters-uniform error estimate is given in the following theorem.

Theorem 3.7 *Let $\vec{u}$ be the solution to* (1.1), (1.2) *and $\vec{U}$ the solution to the discrete problem* (3.1), (3.2) *on the mesh $\overline{\Omega}^N_{\sigma_1,\sigma_2}$. Then there exists a constant C such that*

$$\| \vec{u} - \vec{U} \|_{\overline{\Omega}^N_{\sigma_1,\sigma_2}} \leq CN^{-1} \ln N,$$

where C is a constant independent of N, ε_1 and ε_2.

Proof : Consider a decomposition of $\vec{U}$ given by,

$$\vec{U} = \vec{V} + \vec{W}$$

where $\vec{V}$ is the solution of the problem

$$L^N\vec{V} = \vec{f}, \qquad \vec{V}(0) = \vec{v}(0)$$

and $\vec{W}$ is the solution of the problem

$$L^N\vec{W} = 0, \qquad \vec{W}(0) = \vec{w}(0).$$

The error due to the discretization can be written in the form

$$(\vec{U} - \vec{u}) = (\vec{V} - \vec{v}) + (\vec{W} - \vec{w})$$

and the errors in the smooth and singular components of the solution can be estimated separately. We estimate the error in the smooth component first. We have

$$(L^N(\vec{V} - \vec{v}))(t_i) = \begin{pmatrix} \varepsilon_1(D^- - D)v_1(t_i) \\ \varepsilon_2(D^- - D)v_2(t_i) \end{pmatrix}$$

and hence

$$\begin{aligned} |L^N(\vec{V} - \vec{v})(t_i)| &\leq C \begin{pmatrix} \varepsilon_1 \max\limits_{[t_{i-1},t_i]} \left|v_1''(s)\right| \\ \varepsilon_2 \max\limits_{[t_{i-1},t_i]} \left|v_2''(s)\right| \end{pmatrix} \frac{(t_i - t_{i-1})}{2} \\ &< CN^{-1}(1,1)^T \qquad \text{using Lemma 2.3.} \end{aligned}$$

Since $\vec{V}(0) - \vec{v}(0) = \vec{0}$, using Theorem 3.2 we obtain,

$$\| \vec{V} - \vec{v} \|_{\overline{\Omega}^N} \leq CN^{-1}. \tag{3.6}$$

We now estimate the error in the singular component. For all the possibilities of σ_1 and σ_2, we have from Lemma 3.3 - Lemma 3.6 that

$$\| L^N(\vec{W} - \vec{w}) \| \leq CN^{-1} \ln N \qquad \text{in } \Omega_0.$$

Also $(\vec{W} - \vec{w})(0) = \vec{0}$ and hence by Theorem 3.2,

$$\| \vec{W} - \vec{w} \| \leq CN^{-1} \ln N \qquad \text{in } \overline{\Omega}. \tag{3.7}$$

Combining the estimates for smooth and singular components, we have

$$\| \vec{u} - \vec{U} \|_{\overline{\Omega}^N_{\sigma_1,\sigma_2}} \leq C\, N^{-1} \ln N,$$

This concludes the proof of the theorem. ∎

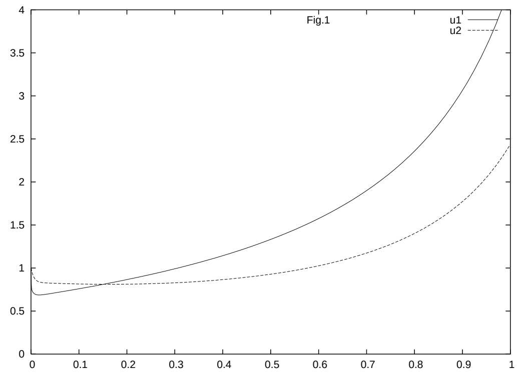
4 Numerical Example

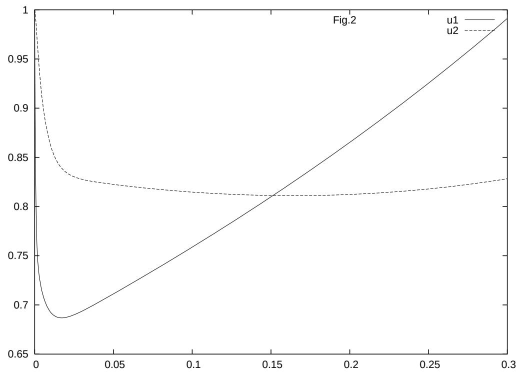
In order to show the applicability of the present method the following problem is considered

Example 4.1 *Consider the following system of singularly perturbed Initial Value Problem:*

$$\left.\begin{array}{l} \varepsilon_1 {u_1}'(t) + (1+e^{-t})u_1(t) - (1+t)u_2(t) = 0.5(1+t), \\ \varepsilon_2 {u_2}'(t) - (1+t^2)u_1(t) + 2(1+t)u_2(t) = 1 + \frac{t}{4} \end{array}\right\} \quad \forall\, t \in (0,1] \tag{4.1}$$

$$u_1(0) = 1, \quad u_2(0) = 1. \tag{4.2}$$





The Fitted Mesh Method suggested in Chapter 3 is applied to the above problem. In addition, the parameter-robust orders of convergence and error constant are calculated by the general algorithm[FHM+00] and are presented in Table 1. The solution of the discrete problem with $\varepsilon_1 = 2^{-10}$, $\varepsilon_2 = 2^{-7}$ and $N = 128$ on Shishkin mesh is displayed in Fig 1 and Fig 2.

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Table 1: *Values of* D_ε^N, D^N, p^N, p^*, *and* $C_{p^*}^N$ *generated by the general algorithm of[FHM+ 00] for Example 4.1*

$\varepsilon_2 = 2^{-7}$ &	Number of mesh points N								
ε_1	8	16	32	64	128	256	512	1024	2048
2^{-3}	0.112+0	0.549-1	0.242-1	0.963-2	0.498-2	0.305-2	0.187-2	0.107-2	0.610-3
2^{-5}	0.652-1	0.428-1	0.241-1	0.125-1	0.629-2	0.311-2	0.185-2	0.106-2	0.603-3
2^{-7}	0.304-1	0.224-1	0.138-1	0.768-2	0.471-2	0.289-2	0.178-2	0.102-2	0.579-3
2^{-9}	0.191-1	0.145-1	0.925-2	0.525-2	0.280-2	0.144-2	0.844-3	0.479-3	0.269-3
2^{-11}	0.161-1	0.123-1	0.789-2	0.450-2	0.241-2	0.124-2	0.632-3	0.318-3	0.160-3
2^{-13}	0.153-1	0.118-1	0.823-2	0.552-2	0.340-2	0.199-2	0.110-2	0.597-3	0.317-3
2^{-15}	0.151-1	0.116-1	0.944-2	0.718-2	0.454-2	0.285-2	0.165-2	0.932-3	0.517-3
2^{-17}	0.150-1	0.117-1	0.983-2	0.756-2	0.499-2	0.310-2	0.183-2	0.105-2	0.589-3
2^{-19}	0.150-1	0.118-1	0.995-2	0.767-2	0.511-2	0.318-2	0.189-2	0.108-2	0.608-3
2^{-21}	0.150-1	0.118-1	0.999-2	0.769-2	0.514-2	0.320-2	0.190-2	0.109-2	0.613-3
2^{-23}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.190-2	0.109-2	0.614-3
2^{-25}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-27}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-29}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-31}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-33}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-35}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-37}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
2^{-39}	0.150-1	0.118-1	0.100-1	0.770-2	0.515-2	0.321-2	0.191-2	0.109-2	0.615-3
D^N	0.112+0	0.549-1	0.242-1	0.125-1	0.629-2	0.321-2	0.191-2	0.109-2	0.615-3
p^N	0.752+0	0.118+1	0.950+0	0.993+0	0.970+0	0.752+0	0.805+0	0.827+0	
$C_{0.752}^N$	0.132+1	0.109+1	0.806+0	0.703+0	0.594+0	0.511+0	0.511+0	0.493+0	0.468+0

The order of convergence = 0.752
The error constant = 1.320

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