LECTURE 2

Topological Gauge Theories:

Flat Connections and Higgs Bundles

FLAT CONNECTIONS

- Σ_h - Riemann surface of genus h
- $\mathcal{P}$ - principal $\mathbf{G}$-bundle over Σ; E associated vector bundle
- $\mathbf{G}$ - compact group (**important**), $SU(N)$, in the rest
- $\mathcal{G}_\Sigma = \mathrm{Map}(\Sigma, \mathbf{G})$ - gauge group
- $\underline{\mathbf{g}} = \mathrm{Lie}(\mathbf{G})$
- $\mathcal{A}$ - space of all connections in $\mathcal{P}$
- $A \in \mathcal{A}$ - connection in $\mathcal{P}$, $[A]$ - its gauge equivalence class
- F-curvature of A: $F = dA + A^2$
- $\varphi \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}})$ - zero-form in $\mathrm{ad}_{\underline{\mathbf{g}}}$ representation
- $\psi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}})$ - odd one-form taking values in $\mathrm{ad}_{\underline{\mathbf{g}}}$

Infinitesimal gauge transformations:

$$\mathcal{L}_\alpha A = d\varphi + [A, \varphi]$$
$$\mathcal{L}_\alpha \psi = -[\varphi, \psi]$$
$$\mathcal{L}_\alpha \varphi = -[\alpha, \varphi]$$

In the space of fields A, φ, ψ **BRST** transformation δ exists such that it squares to zero up to infinitesimal gauge transformations with $\alpha = \varphi$:

$$\delta^2 = -i\mathcal{L}_\varphi$$

$$\delta\, A = i\psi, \quad \delta\, \psi = -(d\varphi + [A, \varphi]), \quad \delta\, \varphi = 0,$$

Action functional:

$$\mathcal{S}_0 = \frac{1}{2\pi} \int_\Sigma Tr(i\varphi F + \frac{1}{2}\psi \wedge \psi)$$

Flat Connections - critical points of $\mathcal{S}_0$ with respect to φ.

Consider the (path)integral over all fields with this action functional:

$$Z_{YM}(\Sigma) = \frac{1}{\mathrm{Vol}(\mathcal{G}_\Sigma)} \int D\varphi\, DA\, D\psi\; e^{\frac{1}{2\pi}\int_\Sigma (i\mathrm{Tr}\,\varphi\, F(A) + \frac{1}{2}\mathrm{Tr}\,\psi\wedge\psi)}$$

$\mathrm{Vol}(\mathcal{G}_\Sigma)$ - volume of the gauge group $\mathcal{G}_\Sigma = \mathrm{Map}(\Sigma, G)$

The measure $DA\, D\psi$ - a canonical flat measure

The measure $D\varphi$ - defined using the standard normalization of the Killing form on $\underline{\mathbf{g}}$

In this integral one also sums over all topological classes of the principal G-bundle over Σ_h.

This theory is called Topological 2d YM theory. The path integral is invariant under δ.

• **Observables - descend:** Given homogeneous invariant polynomial of $\varphi(P)$, $I(\varphi)$, local observable is given by:

$$o_I^{(0)}(P) = \mathrm{Tr}\, I(\varphi) \Rightarrow \mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi)$$

since $o^{(0)}(P)$ is independent of position of P up to δ because:

$$do^{(k)} = -i\delta(o^{(k+1)}).$$

Non-local observables $\mathcal{O}^i, \quad \delta\mathcal{O}^i = 0$, are given by:

$$\mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi)$$

$$\mathcal{O}_I^{(1)} = \int_\Gamma o^1 = \int_\Gamma \sum_{a=1}^{\dim(\underline{\mathbf{g}})} \frac{\partial I(\varphi)}{\partial \varphi^a} \psi^a$$

$$\mathcal{O}_I^{(2)} = \int_\Sigma o^2 = \int_\Sigma \frac{1}{2} \int_\Sigma \sum_{a,b=1}^{\dim(\underline{\mathbf{g}})} \frac{\partial^2 I(\varphi)}{\partial \varphi^a \, \partial \varphi^b} \psi^a \wedge \psi^b +$$

$$+i \int_\Sigma \sum_{a=1}^{\dim(\underline{\mathbf{g}})} \frac{\partial I(\varphi)}{\partial \varphi^a} F(A)^a$$

$\Gamma \in \Sigma$ - a closed curve on a two-dimensional surface Σ.

Physical obesrvables $\Leftrightarrow$ δ-equivariant cohomology classes.

Q: compute the correlators of these Observables

$$< \mathcal{O}^{i_1} ... \mathcal{O}^{i_n} >= \frac{1}{\mathrm{Vol}(\mathcal{G}_\Sigma)} \int DAD\psi D\varphi \mathcal{O}^{i_1} ... \mathcal{O}^{i_n} e^{S_0}$$

• Consider the moduli space of flat connections $\mathcal{M} : F = 0$, subspace in $\mathcal{A}$.

• Consider the product $\mathcal{M} \times \Sigma$ and form so called *universal bundle* $\mathcal{E}$, whose restriction onto $[A] \times \Sigma \subset \mathcal{M} \times \Sigma$ coincides with E.

• $\mathbf{a}$ - the *universal* connection in $\mathcal{E}$, $F_{\mathbf{a}}$ - curvuture of $\mathbf{a}$.

• $\{I_k\}$ - additive basis in the space of invariants: $\mathrm{Fun}(\underline{\mathbf{g}})^{\mathbf{G}} \approx \mathrm{Fun}(\underline{\mathbf{t}})^W$; W - Weyl group.

d_k - degree of I_k, and e_α cycle in Σ (points, closed curve Γ or Σ itself)

• Restrict $I_k(\frac{F_{\mathbf{a}}}{2\pi i})$ to $\mathcal{M} \times e_\alpha$ and integrate over fibres of projection $\mathcal{M} \times e_\alpha \to \mathcal{M}$:

$$\Omega_k^\alpha = \int_{e_\alpha} I_n(\frac{F_{\mathbf{a}}}{2\pi i}) \in \mathrm{H}^{2d_n - \mathrm{dim} e_\alpha}(\mathcal{M})$$

Examples: $I_1 = Tr\varphi^2, d_1 = 2; I_2 = Tr\varphi^3, d_2 = 3$

Intersection theory on $\mathcal{M}$ - compute:

$$\int_{\mathcal{M}} \Omega_{n_1}^{\alpha_1} \wedge \ldots \wedge \Omega_{n_k}^{\alpha_k}$$

• There is one to one correspondence between $\mathcal{O}$'s and Ω's: δ-cohomology $\Leftrightarrow H^*(\mathcal{M})$; basically through:

$$F_{\mathbf{a}} = F_A + \psi + \varphi$$

• Coerrelators of $\mathcal{O}$'s in 2d Topological YM theory compute $\left\langle \Omega_{n_1}^{\alpha_1} \dots \Omega_{n_k}^{\alpha_k} \right\rangle$

Correlators can be combined, as in 2d topological sigma models, in generating function:

$$Z(\Sigma_h; t_i, ...t_n) = \frac{1}{\mathrm{Vol}(\mathcal{G}_\Sigma)} \int DAD\psi D\varphi e^{S(t_1,...,t_n)}$$

$$S(t_1, ..., t_n) = S_0 + \sum_{i=1}^{n} t_i \mathcal{O}^i$$

Intuitive argument, which turns out to be correct is following:

Consider the case when Topological YM theory, S_0, is perturbed with local observables only, and only degree 2 operator is used - $t_1 = -\epsilon$:

$$S = S_0 + t_1 \int_\Sigma (\mathrm{vol}_\Sigma) Tr\varphi^2 = \frac{1}{2\pi} \int_\Sigma (i\varphi F + \frac{1}{2}\psi \wedge \psi) - \epsilon \int_\Sigma (\mathrm{vol}_\Sigma) Tr\varphi^2$$

This is QFT which is equivalent to physical 2d YM theory - φ enters quadratically and can be integrated out (eliminated) giving $\varphi = \frac{i}{2\epsilon} F$, thus theory with the action:

$$-\frac{1}{4\epsilon^2} \int_\Sigma F \wedge *F$$

Topological YM theory corresponds to action S for zero coupling $\epsilon = 0 : S(t_1 = 0) = S_0$, but in above action this means that in the remanning integral over gauge fields only flat connections contribute, very similar to the argument in A and B topological sigma models.

• $\epsilon = 0$, Topological YM theory $\Rightarrow$ integral over the moduli space of flat connections $\Rightarrow$ volume.

• As in Donaldson theory the physical approach of computing the correlators - **Abelianization**

After all non-abelain components are integrated out the path integral reduces to the integral over abelian fields from Cartan sub-algebra.

$$S = \int_{\Sigma} \Big[\sum_k \left(\frac{\partial I(\varphi)}{\partial \varphi_k} \right) F_k + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 I(\varphi)}{\partial \varphi_i \partial \varphi_j} \right) \psi_i \wedge \psi_j \Big] +$$

$$+ \int_{\Sigma} T(\varphi) \frac{1}{8\pi} \mathcal{R}^{(2)}$$

$\psi_A^{ab} = \mathrm{diag}(\psi_1, \ldots, \psi_N)$, F_k is the curvature, corresponding to the k'th entry of A^{ab}, and $\mathcal{R}^{(2)}$ is the two dimensional scalar curvature.

One needs to find two functions of abelian fields $I(\varphi)$ and $T(\varphi)$. Plus - explicit form of observables in terms of abelian fields.

For 2d topological YM theory: $I(\varphi) = \sum_k \varphi_k^2, T(\varphi) = 0$

• Abelian theory reduces to finite-dimentional integral!

• Localization technique is a major tool in computing correlators in all topological quantum field theories.

• The generating function for deformation with local observables only can be computed explicitly:

$$Z_{YM}(\Sigma_h; t_1, ..., t_n) =$$

$$= \left(\frac{\mathrm{Vol}(G)}{(2\pi)^{\dim(G)}}\right)^{2h-2} \sum_{\mu \in P_{++}} (\dim\, V_\mu)^{2-2h}\, e^{-\sum_{k=1}^{\infty} t_k\, p_k(\mu+\rho)}$$

• V_μ - unitary irreducible representation of $G = SU(N)$

• $\mu \in \mathcal{Z}_+$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_N$-highest weight

• $p_k \in \mathbf{C}[\underline{\mathbf{h}}^*]^{\mathbf{W}}$ is the bases of invariant polynomials on the dual $\underline{\mathbf{h}}^*$ to Cartan subalgebra $\underline{\mathbf{h}}$.

• $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ is a half-sum of positive roots

• P_{++} is a subset of the dominant weights of G

• **Hamiltonian approach**: consider Riemann surface with boundary - a cylinder $R \times S^1$.

σ - coordinate along S^1 and t coordinate along R.

A_σ - component of gauge field along S^1, A_t - along R.

Action (after integrating out fermions):

$$S(\varphi, A) = \frac{1}{2\pi} \int dt d\sigma \; \mathrm{Tr}\left(\varphi \partial_t A_\sigma + A_t(\partial_\sigma \varphi + [A_\sigma, \varphi])\right)$$

This has a form $\int dt (\sum_i p_i \dot{q}_i - \lambda \Phi(p, q))$ with:

• λ: lagrange multiplier $\to A_t$; $\sum_i \to \int_R$, $p_i \to \varphi(\sigma)$, $q_i \to A_\sigma(\sigma)$ and Φ - constraint:

$$\Phi(\varphi, A_\sigma) = \partial_\sigma \varphi + [A_\sigma, \varphi] = 0$$

So, infinite-dimensional phase space M with coordinates φ, A_σ is reduced with respect to constraint $\Phi = 0$

Action functional after this infinite-dimensional Hamiltonian reduction

$$\int_R dt \sum_i \tilde{p}_i \dot{\tilde{q}}_i = \int d^{-1}\omega$$

ω - symplectic form on reduced phase space $\tilde{M}$; $\tilde{p}_i, \tilde{q}_i$ - Darboux coordinates on $\tilde{M}$.

$\tilde{M}$ - finite-dimensional:

$$\tilde{M} = (T^* H)/W$$

H - Cartan sub-group, W - Weyl group.

Choose polarization associated with projection $\pi : T^* H \to H$.

The Hilbert space of the theory can be realized as a space of Ad_G-invariant functions on G. The paring is defined by the integration with a bi-invariant normalized Haar measure:

$$< \Psi_1, \Psi_2 >= \int_G dg\, \overline{\Psi}_1(g)\, \Psi_2(g)$$

Being restricted to the subspace of Ad_G-invariant functions it descends to the integral over Cartan torus H:

$$< \Psi_1, \Psi_2 >= \frac{1}{|W|} \int_H dx\, \Delta_G^2(e^{2\pi i x}) \overline{\Psi}_1(x)\, \Psi_2(x)$$

Denote $H_0 = H \cap G^{reg}$ is an intersection of the Cartan torus $H \subset G$ with a subset G^{reg} of regular elements of G. In the case of $G = U(N)$ the set $H/H_0 = \cup_{j<k}\{e^{2\pi i x_j} = e^{2\pi i x_k}\}$ is the main diagonal.

$$\Delta_G^2(e^{2\pi i x}) = \prod_{\alpha \in R^+} (e^{i\pi\alpha(x)} - e^{-i\pi\alpha(x)})^2,$$

$x = \sum_{j=1}^{\mathrm{rank}(\underline{\mathbf{g}})} x_j e^j$ and $\{e^j\}$ is an orthonormal bases of $\underline{\mathbf{h}}$, and R^+ is a set of positive roots of $\underline{\mathbf{g}}$.

The set of invariant operators descending onto $\tilde{M} \Leftrightarrow \mathrm{Ad}_G$-invariant polynomials of φ (commuting set)

$$\mathcal{O}^{(0)\,k}(\varphi) = \frac{1}{(2\pi)^k} \mathrm{Tr}\, \varphi^k.$$

trace is taken in the N-dimensional representation.

Bases of wave-functions:

$$\mathcal{O}^{(0)\,k}(\varphi)\Psi_\lambda(x_1,\cdots,x_N) = p_k(\lambda)\Psi_\lambda(x_1,\cdots,x_N),$$

$$\Psi_\lambda(x_{w(1)},\cdots,x_{w(N)}) = \Psi_\lambda(x_1,\cdots,x_N), \quad w \in W.$$

$\lambda = (\lambda_1,\cdots,\lambda_N)$ are elements of the weight lattice P of $\underline{\mathbf{g}}$

$p_k \in \mathbf{C}[\underline{\mathbf{h}}^*]^{\mathbf{W}}$ is the bases of invariant polynomials on the dual $\underline{\mathbf{h}}^*$ to Cartan subalgebra $\underline{\mathbf{h}}$.

Redefine

$$\Phi_i(x) = \Delta_G(e^{2\pi i x})\Psi_i(x)$$

Integration measure becomes a flat measure on H:

$$< \Psi_1, \Psi_2 >= \frac{1}{|W|}\int_H dx\,\overline{\Phi}_1(x)\,\Phi_2(x),$$

Then:

$$\mathcal{O}^{(0)\,k} = \frac{1}{(2\pi i)^k}\sum_{i=1}^{N}\frac{\partial^k}{\partial x_i^k}.$$

These wave functions are skew-symmetric with respect to the action of W. To get the symmetric: $\Phi_i(x) = |\Delta_G(e^{2\pi i x})|\Psi_i(x)$.

• Thus $\Rightarrow$ representation theory of G enters. The bases of W-skew-invariant eigenfunctions - characters of the finite-dimensional irreducible representations of G:

$$\mathrm{ch}_\mu(g) = \mathrm{Tr}_{V_\mu}\, g$$

$$\Phi_{\mu+\rho}(g) = \Delta_G(e^{2\pi i x}) \, \mathrm{ch}_\mu(g), \qquad \mu \in P_{++},$$

$$\Phi_{\mu+\rho}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \, w(\mu+\rho)(x)},$$

$l(w)$ is a length of a reduced decomposition of $w \in W$.

Generating function for Σ being cylinder thus is the sum over whole spectrum:

$$G(x, x'|t_1, ..., t_n) = \sum_\lambda \Phi_\lambda(x) e^{-\sum_k t_k \mathcal{O}^{0\,k}} \Phi_\lambda(x') =$$

$$= \sum_\lambda \Phi_\lambda(x) e^{-\sum_k t_k p_k(\lambda)} \Phi_\lambda(x')$$

For $\Sigma = T^2$ we need to set $x = x'$ and integrate over x:

$$Z_{T^2} = \int dx G(x, x) = \sum_\lambda e^{-\sum_k t_k p_k(\lambda)}$$

Relation with representation theory of compact Lie group allows to evaluate the generating function $\Leftrightarrow$ intersection form on moduli space of flat connections, for any Riemann surface with boundaries.

Gauged WZW Model - Flat connections continues

The Topological Yang-Mills theory allows the following non-trivial generalization to the G/G gauged WZW model.

$$Z_{GWZW}(\Sigma_h) = \frac{1}{\mathrm{Vol}(\mathcal{G}_{\Sigma_h})} \int Dg\, DA\, D\psi \;\; e^{kS(g,A,\psi)},$$

$$S(g,A,\psi) = S_{WZW}(g) - \frac{1}{2\pi} \int_{\Sigma_h} d^2z\, \mathrm{Tr}\,(A_z g^{-1} \bar{\partial}_{\bar{z}} g + g\partial_z g^{-1} A_{\bar{z}} +$$

$$g A_z g^{-1} A_{\bar{z}} - A_z A_{\bar{z}}) + \frac{1}{4\pi} \int_{\Sigma_h} d^2z\, \mathrm{Tr}(\psi \wedge \psi),$$

$S_{WZW}(g)$ - action functional for Wess-Zumino-Witten model:

$$S_{WZW} = -\frac{1}{8\pi} \int_{\Sigma_h} d^2z\, \mathrm{Tr}(g^{-1}\partial_z g \cdot g^{-1} \partial_{\bar{z}} g) - i\Gamma(g),$$

$$\Gamma(g) = \frac{1}{12\pi} \int_B d^3y\, \epsilon^{ijk}\, \mathrm{Tr}\, g^{-1}\partial_i g \cdot g^{-1}\partial_j g \cdot g^{-1} \partial_k g.$$

k - a positive integer and $\partial B = \Sigma_h$.

$$\delta A = i\psi, \quad \delta\psi^{(1,0)} = i(A^g)^{(1,0)} - iA^{(1,0)}$$

$$\delta\psi^{(0,1)} = -i(A^{g^{-1}})^{(0,1)} + iA^{(0,1)}, \quad \delta g = 0$$

$$\mathcal{L}_g\, A^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)} \quad \mathcal{L}_g\, A_A^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)},$$

$$\mathcal{L}_g\, \psi^{(1,0)} = -g\psi^{(1,0)} g^{-1} + \psi^{(1,0)}, \quad \mathcal{L}_g\, \psi^{(0,1)} = g^{-1}\psi^{(0,1)} g - \psi^{(0,1)}$$

$$\mathcal{L}_g\, g = 0.$$

$A^g = g^{-1}dg + g^{-1}Ag$ is a gauge transformation, and:

$$Q^2 = \mathcal{L}_g$$

In contrast to YM theory these transformations include the finite (from the gauge group) symmetries of the action instead of those from Lie algebra.

In the limit $g \to 1 + i\epsilon\varphi$, $\epsilon \to 0$ one recovers topological YM theory.

Local observables - functions of g. Deform as in YM theory:

$$\Delta S = \sum_{\mu \in P_{++}} t_\mu \int_{\Sigma_h} d^2 z \, \mathrm{Tr}_{V_\mu} g \ \mathrm{vol}_{\Sigma_h}.$$

$t_\mu = 0$ for all but finite subset of P_{++} to make the path integral well defined.

As in 2d YM theory the generating function can be exactly computed and is related to representation theory of quantum groups for q root of unity and representations of Kac-Moody algebras at level k:

$$Z_{GWZW}(\Sigma_h) = |Z(G)|^{2h-2}(\frac{k+c_v}{4\pi^2})^{\frac{1}{2}\dim \mathcal{M}_G(\Sigma_h)}\mathrm{Vol}_q(G)^{2h-2}\times$$

$$\times \sum_{\mu\in P^k_{++}} (\dim_q V_\mu)^{2-2h}\, e^{-\sum_{\mu\in P^k_{++}} t_\mu \mathrm{ch}_{V_\mu}(e^{2\pi i\hat{\lambda}})} \quad .$$

$\dim \mathcal{M}_G(\Sigma_h) = \dim G(2h-2)$ - the dimension of the moduli space of flat G-bundles on Σ_h, $|Z(G)|$ is a dimension of the center of G and:

$$\dim_q V_\mu = \mathrm{Tr}_{V_\mu} q^{-\hat{\rho}} = \prod_{\alpha\in R_+} \frac{(q^{\frac{1}{2}(\mu+\rho,\alpha)} - q^{-\frac{1}{2}(\mu+\rho,\alpha)})}{(q^{\frac{1}{2}(\rho,\alpha)} - q^{-\frac{1}{2}(\rho,\alpha)})}$$

$$\mathrm{Vol}_q(G) =$$

$$= (2\pi)^{\dim G}(k+c_v)^{-\frac{1}{2}(\dim G-\dim H)} \prod_{\alpha\in R_+} \left(q^{\frac{1}{2}(\rho,\alpha)} - q^{-\frac{1}{2}(\rho,\alpha)}\right)^{-1}$$

the sum is over the set P^k_{++} of integrable representations of the affine group $\widehat{LG}_k$ at the level k.

The same set also enumerates irreducible representations of $U_q(g)$ for $q = \exp(2\pi i/(k+c_v))$. Define:

$$\exp(2\pi i\hat{\lambda}) = \exp(2\pi i \sum_j \lambda_j\, e_j) \in H$$

$\mathrm{ch}_{V_\mu}(e^{2\pi i\hat{\lambda}})$ is a character of the element $\exp(2\pi i\hat{\lambda})$ taken in the representation V_μ.

The expressions $\dim_q V_\mu$ are known as quantum dimensions of the representations of the quantum group. For $g = gl_N$ we have:

$$\dim_q V_\mu = \prod_{i<j}^{N} \frac{\left(q^{\frac{1}{2}(\mu_i - \mu_j + j - i)} - q^{\frac{1}{2}(\mu_j - \mu_i + i - j)}\right)}{\left(q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(i-j)}\right)}.$$

and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$, $\mu_i \in \mathbf{Z}_+$.

The q-analog of the character is given by:

$$\Psi_\mu(x) = \mathrm{ch}_\mu q^{-\rho+x}$$

Yang-Mills-Higgs Theory: Intersection Theory on Moduli Space of Higgs Bundles

Higgs bundles were defined by Hitchin, more than 20 years ago, through:

$$F(A) - \Phi \wedge \Phi = 0$$
$$\nabla_A^{(1,0)} \Phi^{(0,1)} = 0$$
$$\nabla_A^{(0,1)} \Phi^{(1,0)} = 0$$

Here new "field" enters, Φ, one-form in adjoint representation of gauge group.

These are dimensional reduction of instanton equations $F^+ = 0$ in four dimensional space down to two dimensions.

We will need later its super-partner, Ψ, odd Grassman variable, also one-form in adjoint representation:

$$(\Phi, \psi_\Phi) : \qquad \Phi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}}), \qquad \psi_\Phi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}})$$

the decompositions

$$\Phi = \Phi^{(1,0)} + \Phi^{(0,1)}$$
$$\psi_\Phi = \psi_\Phi^{(1,0)} + \psi_\Phi^{(0,1)}$$

will correspond to the decomposition of the space of one-forms $\mathcal{A}^1(\Sigma_h) = \mathcal{A}^{(1,0)}(\Sigma_h) \oplus \mathcal{A}^{(0,1)}(\Sigma_h)$ defined in terms of a fixed complex structure on Σ_h.

The space of the solutions has a natural hyperkähler structure and admits compatible $U(1)$ action.

• Write the QFT similar to topological Yang Mills theory where the flat connection condition is replaced by Hitchin equations for Higgs bundle.

• More general question - write the integral representation for integration over moduli space of linear hyperkähler quotients. Hitchin equation defines infinite-dimensional hyperkähler quotient.

Answer to both was given by MNS (Moore, Nekrasov and S. Sh.) in mid 90's.

Introduce extra fields φ_0 similar to φ in case of flat connections:

$$\varphi_0 \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}})$$

In addition one needs to introduce more fields, φ_+, φ_-, 0-forms in adjoint representation, and their super-partners χ, odd Grassman variables also in adjoint representation:

$$(\varphi_\pm, \chi_\pm): \qquad \varphi_\pm \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}}), \qquad \chi_\pm \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\underline{\mathbf{g}}})$$

MNS defined Action functional and δ-operator acting on the space of these fields:

$$S = S_0 + S_1$$

such that:

$$S_0(\varphi_0,\varphi_\pm,A,\Phi,\psi_A,\psi_\Phi,\chi_\pm)=\frac{1}{2\pi}\int_{\Sigma_h}d^2z\,\mathrm{Tr}(i\varphi_0\,(F(A){-}\Phi{\wedge}\Phi){-}$$

$$-c\Phi\wedge *\Phi+\varphi_+\nabla_A^{(1,0)}\,\Phi^{(0,1)}+\varphi_-\nabla_A^{(0,1)}\,\Phi^{(1,0)})$$

$$S_1(\varphi_0,\varphi_\pm,A,\Phi,\psi_A,\psi_\Phi,\chi_\pm)=\frac{1}{2\pi}\int_{\Sigma_h}d^2z\,\mathrm{Tr}(\frac{1}{2}\psi_A\wedge\psi_A+$$

$$\frac{1}{2}\psi_\Phi{\wedge}\psi_\Phi{+}{+}\chi_+[\psi_A^{(1,0)},\Phi^{(0,1)}]{+}\chi_-[\psi_A^{(0,1)},\Phi^{(1,0)}]{+}\chi_+\nabla_A^{(1,0)}\psi_\Phi^{(0,1)}{+}$$

$$\chi_-\nabla_A^{(0,1)}\psi_\Phi^{(1,0)})$$

δ-cohomology is defined through:

$$\delta A=i\psi_A,\quad \delta\psi_A=-\nabla_A\varphi_0,\quad \delta\varphi_0=0,\quad \delta\Phi=i\psi_\Phi$$

$$\delta\psi_\Phi^{(1,0)}=-[\varphi_0,\Phi^{(1,0)}]{+}c\Phi^{(1,0)},\quad \delta\psi_\Phi^{(0,1)}=-[\varphi_0,\Phi^{(0,1)}]{-}c\Phi^{(0,1)},$$

$$\delta\chi_\pm=i\varphi_\pm,\quad \delta\varphi_\pm=-[\varphi_0,\chi_\pm]\pm c\chi_\pm.$$

and as before one has the action of vector field:

$$\mathcal{L}_v\varphi_\pm=\mp\varphi_\pm,\quad \mathcal{L}_v\chi_\pm=\pm\chi_\pm$$

$$\mathcal{L}_{\varphi_0}\,A=-\nabla_A\varphi_0,\quad \mathcal{L}_{\varphi_0}\,\psi_A=-[\varphi_0,\psi_A]$$

$$\mathcal{L}_{\varphi_0}\,\Phi=-[\varphi_0,\Phi],\quad \mathcal{L}_{\varphi_0}\,\psi_\Phi=-[\varphi_0,\psi_\Phi]$$

$$\mathcal{L}_{\varphi_0}\,\varphi_0=0,\quad \mathcal{L}_{\varphi_0}\,\varphi_\pm=-[\varphi_0,\varphi_\pm]$$

$$\mathcal{L}_{\varphi_0}\,\chi_\pm=-[\varphi_0,\chi_\pm],$$

But now, due to circle action of Hitchin there is one more vector field:

$$\mathcal{L}_v \Phi^{(1,0)} = +\Phi^{(1,0)}, \quad \mathcal{L}_v \Phi^{(0,1)} = -\Phi^{(1,0)},$$
$$\mathcal{L}_v \psi_\Phi^{(1,0)} = +\psi_\Phi^{(1,0)}, \quad \mathcal{L}_v \psi_\Phi^{(0,1)} = -\psi_\Phi^{(0,1)},$$

δ squares to zero up to action of these two vector fields:

$$\delta^2 = i\mathcal{L}_{\varphi_0} + c\mathcal{L}_v$$

The action for YMH theory is sum of action of YM theory and δ-exact term:

$$S_{YMH}(\varphi_0, \varphi_\pm, \chi_\pm, A, \psi_A, \Phi, \psi_\Phi) = S_{YM}(\varphi_0, A, \psi_A) +$$
$$+\delta[\int_{\Sigma_h} d^2z \, \mathrm{Tr}\,(\frac{1}{2}\Phi \wedge \psi_\Phi + \varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)})]$$

so it is obviously δ-invariant, as well as $\mathcal{L}_{\varphi_0}$ and $\mathcal{L}_v$ invariant.

• **Observables:** not surprisingly local δ-cohomology is generated by same operators as in YM theory:

$$o_I^{(0)}(P) = \mathrm{Tr}\, I(\varphi_0) \Rightarrow \mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi_0)$$

1-observable and 2-observable is defined exactly same way as in YM theory via descend procedure, but now with new operator δ.

Add to the action quadratic combination of scalars $\varphi_0, \varphi_\pm$:

$$S_{YMH} + \tau_1 \int_\Sigma Tr\varphi_0^2 + \tau_2 \int_\Sigma Tr\varphi_- \varphi_+$$

Bosonic part of the action after integration (elimination) of $\varphi_0, \varphi_\pm$ is:

$$\frac{1}{\tau_1^2}\int_\Sigma Tr|F(A)-\Phi\wedge\Phi|^2+\frac{1}{\tau_2^2}\int_\Sigma Tr\nabla_A^{(1,0)}\Phi^{(0,1)}\nabla_A^{(0,1)}\Phi^{(1,0)}-$$

$$-c\int_\Sigma Tr\Phi\wedge *\Phi$$

YMH theory corresponds to $\tau_1 = \tau_2 = 0$ and we see that contributions come only from solutions of:

$$F(A)-\Phi\wedge\Phi=0, \quad \nabla_A^{(0,1)}\Phi^{(1,0)}=0$$

Higgs bundle equations of Hitchin. We call the theory with $\tau_1 = \tau_2 = 0$ topological YMH theory.

Deform the action by adding the observables:

$$\tilde{S} = S + t_i\mathcal{O}^i$$

and evaluate the integral over all fields:

$$Z_\Sigma(t) =< e^{-\tilde{S}(t)} >$$

This integral in topological YMH theory is (equivariant) integral over moduli space of Higgs bundles and for $t_i = 0$ *corresponds to the "regularized volume" of moduli space. Parameter* c *is equivariant* $\Rightarrow$ *regularization paramter.*

Moduli space of Higgs bundles equivalently can be described and space of solutions of complexified flat connection:

$$F^c(A^c = A + i\Phi) = 0$$

This compexified equation is gauge invariant under complexified gauge transformations and the quitient is same as space of solutions to Hitchin equations modulo real gauge transformations.

Topological YMH theory depends on one important, equivariant parameter, c. There are two key limiting cases - $c \to \infty$ and $c \to 0$.

• $c - \infty$. c enters bosonic part of action through:

$$-c \int_{\Sigma} Tr|\Phi|^2$$

Thus for $c \to \infty$ only $\Phi = 0$ contributes and Φ together with its super-partner ψ_Φ drops out. We are left with topological YM theory for real group - case already studied.

• $c = 0$. Oposite limit, $c \to 0$, corresponds to flat connections for complexefied group thus if considerations of topological YM would apply to complexified (non-compact) groups answer should be described by representation theory of complex group. Unfortunately these are infinite-dimensional and answer should diverge - and it does, the theory is not regularized since $c = 0$.

• Topological YMH theory interpolates between representations of compact group and its complexification giving proper treatment of latter in $c \to 0$ limit.

Computation - intersection form, the deformed partition function, for topological YMH with group $U(N)$ was computed by MNS: via abelianization and hyperkähler localization technique developed there.

Abelianized action descends from functional wich coincides with so called Yang functional introduced by C. N. Yang for Bethe Ansatz of NLS equation!

$$I(\lambda) = \sum_{j=1}^{N} (\frac{1}{2}\lambda_i^2 - 2\pi n_j \lambda_j) + \sum_{k,j=1}^{N} \int_0^{\lambda_j - \lambda_k} \text{ arctg } \lambda/c \, d\lambda$$

$\lambda \equiv \varphi_0$. Last term can be written is $\lambda log \lambda$ and in that form sometimes is called Veneziano-Yankielovich superpotential.

Explicit form of the action also has non-zero Gauss-Bonet term. Bosonic part of action is:

$$S = \int_{\Sigma_h} \sum_{i=1}^{N} [(\varphi_0)_i + \sum_{j=1}^{N} \log \left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic} \right)] F(A)^i +$$

$$+ \frac{1}{2} \int_{\Sigma_h} \sum_{i,j=1}^{N} \log \left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic} \right) R^{(2)} \sqrt{g}$$

As already explained the observables in topological YMH theory are defined by same invariant polynomials as in topological YM theory (flat connection). Zero observables are:

$$\mathcal{O}_k^0 = \sum_{i=1}^{N} {(\varphi_0)_i}^k$$

$$\tilde{S}(t) = S + \sum_k t_k \mathcal{O}_k^0$$

$$Z_{\Sigma_h}(t) = < e^{-\tilde{S}(t)} >$$

The generating function $Z_{\Sigma_h}(t)$ is still expressed in terms of infinite-dimensional integral over abelian fields but luckily this integral is exactly computable because of localization technique.

Computing the Gaussian integral over fermions and summing over non-trivial topological classes of connection leads to finite-dimensional integral representation ($\int_\Sigma F^k = 2\pi i n_k$):

$$Z_{\Sigma_h}(t) = \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbf{R^N}} d^N\lambda$$

$$\mu(\lambda)^h \sum_{(n_1,\cdots,n_N)\in \mathbf{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{k\neq j} (\lambda_k - \lambda_j)^{n_k - n_j + 1 - h} \times$$

$$\prod_{k,j} (\lambda_k - \lambda_j - ic)^{n_k - n_j + 1 - h} \, e^{-\sum_{k=1}^\infty t_k \, p_k(\lambda)}$$

$a(c)$ - h indpendent constant

$p_k(\lambda)$ - S_N-invariant polynomial functions of degree k on $\mathbf{R^N}$

$$\mu(\lambda) = \det \| \frac{\partial^2 I(\lambda)}{\partial\lambda_i \, \partial\lambda_j} \|$$

Collect n-depends part in the sum:

$$Z_{\Sigma_h}(t) = \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbf{R^N}} d^N\lambda \; \mu(\lambda)^h \sum_{(n_1,\cdots,n_N)\in \mathbf{Z}^N} e^{2\pi i \sum_j n_j \alpha_j(\lambda)}$$

$$\times \prod_{k\neq j} (\lambda_k - \lambda_j)^{1-h} \prod_{k,j} (\lambda_k - \lambda_j - ic)^{1-h} \, e^{-\sum_{k=1}^\infty t_k \, p_k(\lambda)}$$

$$e^{2\pi i \alpha_j(\lambda)} = \mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k\neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic}$$

Use identity:

$$\mu(\lambda) \sum_{(n_1,\cdots,n_N)\in \mathbf{Z}^N} e^{2\pi i \sum_j n_j \alpha_i(\lambda)} =$$

$$= \mu(\lambda) \sum_{(m_1,\cdots,m_N)\in \mathbf{Z}^N} \prod_j \delta(\alpha_j(\lambda) - m_j) =$$

$$= \sum_{(\lambda_1^*,\cdots,\lambda_N^*)\in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*)$$

where $\mathcal{R}_N$ denotes the set of colutions of $\alpha_k(\lambda) = m_k$ or the same as set of colutions to $\mathcal{F}_j(\lambda) = 1$:

$$e^{2\pi i \lambda_j} \prod_{k\neq j} \frac{\lambda_k - \lambda_j + ic}{\lambda_k - \lambda_j - ic} = 1, \qquad k = 1, \cdots, N$$

This is a set of transcedental equations on N real numbers λ_i - it is the **Bethe Ansatz** equation for the N-particle sector of quantum theory of **Nonlinear Schrödinger** equation. Finally:

$$Z_\Sigma(t) = e^{(1-h)a(c)} \sum_{\lambda\in\mathcal{R}_N} D_\lambda^{2-2h}\, e^{-\sum_{k=1}^{\infty} t_k\ p_k(\lambda)}$$

$$D_\lambda = \mu(\lambda)^{1/2} \prod_{i<j} (\lambda_i - \lambda_j)(c^2 + (\lambda_i - \lambda_j)^2)^{1/2}$$

$\mathcal{R}_N$ - C. N. Yang proved that this set can be enumerated by the multiplets of the integer numbers $(p_1, \cdots, p_N) \in \mathbf{Z}^N$ such that $p_1 \geq p_2 \geq \cdots \geq p_N,\ p_i \in \mathbf{Z}$. Thus, the sum in is the sum over the same set of partitions as in the case of flat connections.

• *Intersection forms on moduli space of Higgs bundels are one parameter, c, defomrations of those for the connections.*

• *c - equivariant parameter - regularization.*

• *Bethe Ansatz for N-particle sector of NLS enters instead of highest weight representations of compact group.*

A. Gerasimov, S.Sh., 2006

• Is there a nice representation theory interpretation of intersection numbers for Higgs bundles, similar to that for flat connections?

• What are wave-functions - ortho-normal bases in Hilbert space, eigenfunctions of quantum Hamiltonians $\Leftrightarrow$ observables? Eigenvalues known from partition function computation. Should follow for exact computations from Rieman surface with boundaries - unknown.

• Bethe Ansatz for NLS - is the bases of wave-functions in YMH related to wave-functions in N-particle sector of NLS?

• What is the meaning of $D(\lambda)$?

Hamiltonian picture

• Phase space of YMH theory for $c \neq 0$ is same as for YM theory. Bases of wave functions - one parameter deformation of latter. For $c = 0$ local δ-cohomologies contain additional operators - phase space is T^*H^c/W instead of T^*H/W since $c = 0$ is complexified flat connection.

Choose same polarization as before $\pi : T^*H/W \to H$.

• Wave functions - S_N invariant functions on a torus H or equivalently the functions on $\mathbf{R}^\mathbf{N}$ invariant under action of the semidirect product of the lattice $P_0 = \pi_1(H)$ and Weyl $W = S_N$ group $\Rightarrow$ under the action of the affine Weyl group W^{aff}).

• The lattice P_0 can be interpreted as a lattice of the $\mathbf{R}^\mathbf{N}$-valued constant connections on S^1 which are gauge equivalent to the zero connection.

The corresponding gauge transformations act on the wave functions by the shifts $x_j \to x_j + n_j$, $n_j \in \mathbf{Z}$ of the argument of the wave functions in the chosen polarization.

It simple known fact that the wave functions in two-dimensional Yang-Mills theory (and we will see also in YMH) can be obtained by the averaging over this gauge transformations and global gauge transformations by the nontrivial elements of the normalizer of Cartan torus $W = N(H)/H$.

This infinte sum over π_1's is same as infinite sum in MNS partition function - sum over topological classes of YM connection.

Thus we conclude:

• **Wave-functions - functions on Cartan $\Phi_\lambda^{(c)}(x)$, periodic functions of N variables.**

• **λ's - labels parametrizing the state in the spectrum; appear as eigenvalues of φ_0 in the integral representation.**

• **Hamiltonians H_k- operators from δ-cohomology, local observables $\mathcal{O}^k$.**

• **Equation for eigenfunctions from flat connections case gets deformed but eigenvalues - same.**

$$H_k \Phi_\lambda^{(c)}(x_1, \cdots, x_N) = p_k(\lambda) \Phi_\lambda^{(c)}(x_1, \cdots, x_N),$$

$$\Phi_\lambda^{(c)}(x_{w(1)}, \cdots, x_{w(N)}) = \Phi_\lambda^{(c)}(x_1, \cdots, x_N), \quad w \in W.$$

Latter follows from simple argument: in YMH one can consider same generating function for the manifold with boundary, e. g. cylinder, and then glue the cylinder in torus. From general principles of QM:

$$Z_{cyl}(t) = G(t; x, x') = \sum_{spectrum} \bar{\Phi}_\lambda^{(c)}(x) e^{-t_k H_k} \Phi_\lambda^{(c)}(x') =$$

$$= \sum_{spectrum} \bar{\Phi}_\lambda^{(c)}(x) e^{-t_k E_k(\lambda)} \Phi_\lambda^{(c)}(x')$$

$$Z_{T^2}(t) = \int d^N x G(t; x, x) = \sum_{\lambda \in spectrum} e^{-t_k E_k(\lambda)}$$

So we see that: $H_k \to \mathcal{O}^k; \quad E_k(\lambda) = p_k(\lambda); \quad \lambda \in \mathcal{R}.$

Since explicit computations for manifold with boundary is not available we can use indirect way of deducing the bases of wave-functions in Hilbert space:

- Relax periodicity condition on Φ^c_λ. This corresponds to QM problem, dimensional reduction of YMH to one-dimension. This is gauge theory - QM on trivalent graph Γ_h.
- Phase space of such theory is almost same: $T^*\underline{\mathbf{h}}/W$.

Hamiltonian equations on eigenfunctions still remain because operators from local cohomology have same form and the answers for dimensionally reduced theory to one-dimension (YMH theory on Γ_h, trivalent graph) are known for $Z_{\Gamma_h}(t)$:

$$Z_{\Gamma_h}(t) \sim \int_{R^N} D_\lambda^{2-2h}\, e^{-\sum_{k=1}^{\infty} t_k\; p_k(\lambda)}$$

for exactly same D_λ and $E_k(\lambda) = p_k(\lambda)$. For $S^1 \to h = 1$.

This need to be compared to answers for generating function on interval I:

$$Z_I(t) = G_0(t; x, x') = \sum_{spectrum} \bar{\Phi}^0_\lambda(x) e^{-t_k H_k} \Phi^0_\lambda(x') =$$

$$= \sum_{spectrum} \bar{\Phi}^0_\lambda(x) e^{-t_k E_k(\lambda)} \Phi^0_\lambda(x')$$

Φ^0 denotes wave-function on interval; and for partition function:

$$Z_{S^1}(t) = \int d^N x G_0(t; x, x) = \sum_{spectrum} e^{-t_k E_k(\lambda)}$$

We conclude that restriction on spectrum disappears, now $\lambda \in R^N$, but eigenvalues are the same: $E_k(\lambda) = p_k(\lambda)$. When imposing periodicity condition on wave-functions:

• Spectrum gets projected to the subspace in $\mathbf{R^N}$ given by solutions to BA equations: $\lambda \in \mathcal{R}^N$. And of course wave-functions are periodic.

There is another important relation between Green function of dimensionally reduced theory and Green function of original theory - cylinder partition function:

$$G(t;x,x') = \int_{\lambda \in \mathbf{R^N}/S^N} d^N\lambda \bar{\Phi}^0_\lambda(x) P(\lambda) e^{-t_k H_k} P(\lambda) \Phi^0_\lambda(x') =$$

$$= \sum_{(k_1,\ldots,k_n)\in \mathbf{Z}^N} G_0(t;x,x'+k)$$

where $P(\lambda)$ projects the bases of wave-functions in QM problem to bases of periodic wave-functions in YMH theory.

This procedure is frequently used in QM and in particular in topological YM theory.

All the properties described above are satisfied by bases of wave-functions in NLS theory. Latter is known explicitly both for interval I and for S^1 - periodic.

N-particle wave-functions in Nonlinear Schrödinger theory

The Hamiltonian of Nonlinear Schrödinger theory with a coupling constant c is given by:

$$\mathcal{H}_2 = \int dx\,(\frac{1}{2}\frac{\partial\phi^*(x)}{\partial x}\frac{\partial\phi(x)}{\partial x} + c(\phi^*(x)\phi(x))^2)$$

with the following Poisson structure for bosonic fields ϕ - function on infinite line $\mathbf{R}$ or on a circle S^1:

$$\{\phi^*(x), \phi(x')\} = \delta(x - x').$$

The operator of the number of particles is:

$$\mathcal{H}_0 = \int dx\,\phi^*(x)\phi(x)$$

and it commutes (has zero Poisson bracket) with Hamiltonian - conserved charge.

Equation for eigenfunction in fixed particle number sector $\mathcal{H}_0 = N$ is:

$$(-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{1\le i<j\le N}\delta(x_i - x_j))\Phi^0_\lambda(x) = 2\pi^2(\sum_{i=1}^{N}\lambda_i^2)\Phi^0_\lambda(x)$$

Quantum integrability implies the existence of higher Hamiltonians - H_k; their eigenvalues are symmetric polynomials $p_k(\lambda)$ from previous part.

Finite-particle sub-sectors of the Nonlinear Schrödinger theory is described in terms of the representation theory of a particular kind of Hecke algebra

$R = \{\alpha_1, \cdots, \alpha_l\}$ - root system, W - corresponding Weyl group and P - a weight lattice.

Degenerate affine Hecke algebra $\mathcal{H}_{R,c}$ associated to R is defined as an algebra with the basis S_w, $w \in W$ and $\{D_\lambda, \lambda \in P\}$ such that S_w $w \in W$ generate subalgebra isomorphic to group algebra $\mathbf{C[W]}$ and the elements D_λ, $\lambda \in P$ generate the group algebra $\mathbf{C[P]}$ of the weight lattice P.

In addition one has the relations:

$$S_{s_i} D_\lambda - D_{s_i(\lambda)}\, S_{s_i} = c\,\frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \qquad i = 1, \cdots, n.$$

s_i - the generators of the Weyl algebra corresponding to the reflection with respect to the simple roots α_i.

The center of $\mathcal{H}_{R,c}$ is isomorphic to the algebra of W-invariant polynomial functions on $R \otimes \mathbf{C}$. The degenerate affine Hecke algebras were introduced by Drinfeld and independently by Lusztig.

For $U(N)$ YMH theory we are interested in case of gl_N root system and thus we have $W = S_N$.

Introduce the following differential operators (Dunkle/Lax operators):

$$\mathcal{D}_i = -i\frac{\partial}{\partial x_i} + i\frac{c}{2}\sum_{j=i+1}^{N}(\epsilon(x_i - x_j) + 1)s_{ij}$$

$\epsilon(x)$ - sign-function and $s_{ij} \in S_N$ is a transposition (ij). These operators together with the action of the symmetric group provide a representation of the degenerate affine algebra $\mathcal{H}_{N,c}$ for $g = gl(N)$:

$$S_{s_i} \to s_i, \quad D_i \to \mathcal{D}_i, \;\; i = 1, \cdots, N$$

The image of the quadratic element of the center is given by:

$$\frac{1}{2}\sum_{i=1}^{N} \mathcal{D}_i^2 = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + c \sum_{1\leq i<j\leq N} \delta(x_i - x_j)$$

which is the Hamiltonian on the N-particle sector of Nonlinear Schrödinger theory. Higher Hamiltonians: $H_k = \sum_{i=1}^{N} \mathcal{D}_i^k$.

We need S_N-invariant solutions. They play the role of spherical vectors (with respect to the spherical subalgebra $\mathbf{C}[\mathbf{W}] \in \mathcal{H}_{\mathbf{N},\mathbf{c}}$) in the representation theory of degenerate affine Hecke algebra.

Normalized eigenfunctions are:

$$\Phi_\lambda^0(x) = \sum_{w\in W} (-1)^{l(w)} \prod_{i<j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)} - ic\epsilon(x_i - x_j)} \right)^{\frac{1}{2}} \times$$

$$\times \exp(2\pi i \sum_k \lambda_{w(k)} x_k)$$

Periodic case is given by Hamiltonian equation:

$$(-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+c\sum_{n\in\mathbf{Z}}\sum_{1\le i<j\le N}\delta(x_i-x_j+n))\Phi_\lambda(x)=$$

$$=2\pi^2(\sum_{i=1}^{N}\lambda_i^2)\Phi_\lambda(x)$$

with conditions:

$$\Phi_\lambda(x_1,\cdots,x_j+1,\cdots,x_N)=\Phi_\lambda(x_1,\cdots,x_N)$$

$$\Phi_\lambda(x_{w(1)},\cdots,x_{w(N)})=\Phi_\lambda(x_1,\cdots,x_N),\quad w\in S_N$$

Solution to this problem is given by same wave-functions as in case of infinite line but with restriction on λ's - Bethe Ansatz equations:

$$\mathcal{F}_j(\lambda)\equiv e^{2\pi i\lambda_j}\prod_{k\neq j}\frac{\lambda_k-\lambda_j-ic}{\lambda_k-\lambda_j+ic}=1$$

The normalized wave functions in the periodic case are:

$$\Phi_\lambda^{norm}(x)=\left(\det\|\frac{\partial\log\mathcal{F}_j(\lambda)}{\partial\lambda_k}\|\right)^{-1/2}\Phi_\lambda(x)=\mu(\lambda)^{-1/2}\Phi_\lambda(x)$$

• $c \to \infty$: **Representation theory of compact Lie groups**

In this case we must get topological YM theory for unitary group from YMH theory. Indeed Bethe Ansatz becomes:

$$(-1)^{N-1} e^{2\pi i \lambda_k} = 1$$

with solutions:

$$\lambda_j = \frac{N-1}{2} + m_i$$

YMH partition function becomes identical to that for YM theory.

At the same time the normalized wave-functions of NLS theory:

$$\Phi_\lambda^{c=\infty}(x) = |\Delta(e^x)| \mathrm{ch}_\lambda(x)$$

thus giving the wave-function of YM theory.

• $c \to 0$: **Representation theory of complex Lie groups**

This limit is more complicated both in MNS computation in YMH theory and in NLS theory.

Limiting Bethe Ansatz equation is:

$$e^{2\pi i \lambda_k} = 1$$

The interpretation of the limit in YMH theory is not so obvious because the localization technique does not straightforwardly applicable for $c = 0$. But we shall get answers in YM theory for complexified group G^c.

In the Yang-Mills theory for G^c one expects to have a sum (integral and the sum) over the set of unitary representations arising in the decomposition of the regular representation of G in $L^2(G)$, i.e. over the principal series of unitary representations.

The simplest example is a representation of $GL(N, \mathbf{C})$ obtained by quantization of the regular coadjoint orbit generalizing two-sheet hyperboloid for $GL(2, \mathbf{C})$. The corresponding character is given by:

$$\mathrm{ch}_\lambda(e^x) = \frac{1}{|\Delta_G(e^x)|} \sum_{w \in S_N} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} x_j}$$

$\lambda_j = m_j + i\rho_j$ and S_N is a Weyl group of $GL(N, \mathbf{C})$.

In the case of the finite-dimensional representations the dimension of the representation is given by the value of the corresponding character at the unit element of the group. But in case of infinite dimensional representations this doesn't work because value at unit element diverges.

The correct definition of the dimension $\mathcal{D}_\lambda$ of the principal series unitary representations:

$$\delta_e^{(G)}(g) = \sum_{\lambda \in \widehat{G}} \mathcal{D}_\lambda \, \mathrm{ch}_\lambda(g)$$

$\delta_e^{(G)}(g)$ - delta-function with the support at the unit element $e \in G$ of the group, $\mathrm{ch}_\lambda(g)$ is a character and $\widehat{G}$ is a unitary dual to G.

$\mathcal{D}_\lambda$ - formal degree of representation.

Both in YMH ad NLS in the limit $c \to 0$ the subset of the principal series of representations corresponding to $\lambda_k = m_k \in \mathbf{Z}$ (i.e. $\rho_k = 0$) enters.

• $c \neq 0, \infty$: **Representation theory of p-adic Lie groups**

Definition - Hall-Littlewood polynomials, special case of Macdonald polynomials:

$\{\Lambda_i\}$, $i = 1, \cdots, N$ - set of formal variables; $\mu = (\mu_1, \cdots, \mu_N)$ be a partition of length N.

$$P_\mu(\Lambda_1, \cdots, \Lambda_N | t) = \frac{1}{v_\mu(t)} \sum_{w \in S_N} w \left(\Lambda_1^{\mu_1} \cdots \Lambda_N^{\mu_N} \prod_{i<j} \frac{\Lambda_i - \Lambda_j t}{\Lambda_i - \Lambda_j} \right) =$$

$$= \frac{1}{v_\mu(t)\, \Delta(\Lambda)} \sum_{w \in S_N} (-1)^{l(w)}\, w\, (\Lambda_1^{X_1} \cdots \Lambda_N^{X_N} \prod_{i<j} (\Lambda_i - \Lambda_j t))$$

where for the partition $\mu = (1^{m_1}, 2^{m_2}, \cdots, r^{m_r}, \cdots)$:

$$v_\mu = \prod_{j=1}^{N} \prod_{i=1}^{m_j} \frac{1-t^i}{1-t}, \qquad \Delta(\Lambda) = \prod_{i<j} (\Lambda_i - \Lambda_j).$$

The spherical functions for $G = GL(N, Q_p)$, $K = GL(N, \mathbf{Z}_p)$ (here $\mathbf{Z}_p$ is a ring of p-adic integers) has the following representation in terms Hall-Littlewood polynomials:

$$\omega_s(p^{\mu_1}, \cdots, p^{\mu_N}) = p^{-\sum_{i=1}^{N} (n-i)\mu_i} \frac{v_\mu(p^{-1})}{v_N(p^{-1})} P_\mu(p^{-s_1}, \cdots, p^{-s_N} | p^{-1})$$

where $s = (s_1, \cdots, s_N) \in \mathbf{Z}^N$ and

$$v_N(t) = \prod_{i=1}^{N} \frac{1 - t^i}{1 - t}$$

Now we are ready to check whether normalized wave-function in NLS on infinity line and on circle lead to correct properties for cylinder partition function in YMH theory.

As explained we need to check in NLS:

$$G(t;x,x') = \sum_{(\lambda_1,\ldots,\lambda_N)\in\mathcal{R}^N} \bar{\Phi}_\lambda(x) e^{-t_k H_k} \Phi_\lambda(x') =$$

$$= \int_{\lambda\in\mathbf{R^N}/S^N} d^N\lambda \bar{\Phi}^0_\lambda(x) P(\lambda) e^{-t_k H_k} P(\lambda) \Phi^0_\lambda(x') =$$

$$= \sum_{(k_1,\ldots,k_n)\in\mathbf{Z}^N} G_0(t;x,x'+k) =$$

$$= \sum_{k\in\mathbf{Z}} \int_{\lambda\in\mathbf{R^N}/S^N} d^N\lambda \bar{\Phi}^0_\lambda(x) e^{-t_k H_k} \Phi^0_\lambda(x'+k)$$

with $\Phi^0_\lambda(x)$ normalized wave-function on infinite line, $\Phi_\lambda(x)$ normalized periodic wave-function and projector $P(\lambda)$ is:

$$P(\lambda) = \mu(\lambda) \sum_{m\in\mathbf{Z}^N} \prod_{j=1}^{N} \delta(\alpha_j(\lambda) - m_j) =$$

$$= \sum_{(\lambda^*_1,\cdots,\lambda^*_N)\in\mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda^*_j)$$

and verify if:

$$Z^{YMH}_{T^2}(t) = \int dx_1 dx_2 \ldots dx_N G(t;x,x)$$

Explicit computations confirm each of these conditions.

• **Topological YMH theory for $U(N)$ is completely equivalent to N-particle sector of NLS.**

• : G/G **gauged WZW model**

Set of fields: $(A, \psi_A, \Phi, \psi_\Phi, \chi_\pm, \varphi_\pm, g)$

Equivariant parameter: $t \in \mathbf{R}^*$

Odd and even symmetries, with $A^g = g^{-1}Ag + g^{-1}dg$:

$$\mathcal{L}_{(g,t)}\, A^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)}$$

$$\mathcal{L}_{(g,t)}\, A_A^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)},$$

$$\mathcal{L}_{(g,t)}\, \psi_A^{(1,0)} = -g\psi_A^{(1,0)}g^{-1} + \psi_A^{(1,0)}$$

$$\mathcal{L}_{(g,t)}\, \psi_A^{(0,1)} = g^{-1}\psi_A^{(0,1)}g - \psi_A^{(0,1)}, \quad \mathcal{L}_{(g,t)}\, g = 0,$$

$$\mathcal{L}_{(g,t)}\, \Phi^{(1,0)} = tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}$$

$$\mathcal{L}_{(g,t)}\, \Phi^{(0,1)} = -t^{-1}g^{-1}\Phi^{(0,1)}g + \Phi^{(0,1)},$$

$$\mathcal{L}_{(g,t)}\, \psi_\Phi^{(1,0)} = tg\psi_\Phi^{(1,0)}g^{-1} - \psi_\Phi^{(1,0)}$$

$$\mathcal{L}_{(g,t)}\, \psi_\Phi^{(0,1)} = -t^{-1}g^{-1}\psi_\Phi^{(0,1)}g + \psi_\Phi^{(0,1)},$$

$$\mathcal{L}_{(g,t)}\, \chi_+ = tg\chi_+g^{-1} - \chi_+, \quad \mathcal{L}_{(g,t)}\chi_- = -t^{-1}g^{-1}\chi_-g + \chi_-$$

$$\mathcal{L}_{(g,t)}\, \varphi_+ = t^{-1}g\varphi_+g^{-1} - \varphi_+, \quad \mathcal{L}_{(g,t)}\varphi_+ = -tg^{-1}\varphi_+g + \varphi_+$$

$$\delta\, A = i\psi_A, \quad \delta\, \psi_A^{(1,0)} = i(A^g)^{(1,0)} - iA^{(1,0)}$$

$$\delta\, \psi_A^{(0,1)} = -i(A^{g^{-1}})^{(0,1)} + iA^{(0,1)},$$

$$\delta\Phi = i\psi_\Phi,\ \delta\psi_\Phi^{(1,0)} = tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}$$

$$\delta\psi_\Phi^{(0,1)} = -t^{-1}g^{-1}\Phi^{(0,1)}g - \Phi^{(0,1)}$$

$$\delta\chi_\pm = i\varphi_\pm, \quad \delta\varphi_+ = tg\chi_+g^{-1} - \chi_+, \quad \delta\varphi_- = -t^{-1}g^{-1}\chi_-g + \chi_-.$$

$$\delta g = 0$$

Action:

$$S = S_{GWZW} + \delta(\int_{\Sigma_h} d^2z \, \mathrm{Tr}\,(\frac{1}{2}\Phi \wedge \psi_\Phi +$$

$$+\tau_1\,(\varphi_+ \nabla_A^{(1,0)}\Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)}\Phi^{(1,0)}) + \tau_2(\chi_+\varphi_- + \chi_-\varphi_+)\mathrm{vol}_{\Sigma_h})$$

Applying localization technique we get for partition function $Z =< e^{-S} >$:

$$Z_{\Sigma_h} \sim \frac{1}{|W|}\int_H d^N\lambda\ \mu_q(\lambda)^h \sum_{(n_1,\cdots,n_N)\in \mathbf{Z}^N} e^{2\pi i \sum_{m=1}^N \beta_m(\lambda) n_m} \times$$

$$\times \prod_{j<k} (e^{i\pi(\lambda_j-\lambda_k)} - e^{i\pi(\lambda_k-\lambda_j)})^{2-2h} \prod_{j<k} |te^{i\pi(\lambda_j-\lambda_k)} - e^{i\pi(\lambda_k-\lambda_j)}|^{2-2h}$$

Integral is over Cartan torus H,

$$\mu_q(\lambda) = \det \| \frac{\partial \beta_j(\lambda)}{\partial \lambda_k} \|$$

$$e^{2\pi i \beta_j(\lambda)} = e^{2\pi i \lambda_j (k+c_v)} \prod_{k\neq j} \frac{te^{2\pi i(\lambda_j-\lambda_k)} - 1}{te^{2\pi i(\lambda_k-\lambda_j)} - 1}$$

In partition function summation over integers leads to the restriction on the integration parameters $(\lambda_1, ..., \lambda_N) \in \mathcal{R}_q^N$:

$$e^{2\pi i \lambda_j (k+c_v)} \prod_{k\neq j} \frac{sin(i\pi(\lambda_j - \lambda_k + ic))}{sin(i\pi(\lambda_j - \lambda_k - ic))} = 1 \quad , \quad t = e^c$$

This is the $s \to -i\infty$ limit of Bethe equations for XXZ quantum integrable spin chain:

$$\left(\frac{sin(i\pi(\lambda_j - isc))}{sin(i\pi(\lambda_j + isc))}\right)^{(k+c_v)} \prod_{k\neq j} \frac{sin(i\pi(\lambda_j - \lambda_k + ic))}{sin(i\pi(\lambda_j - \lambda_k - ic))} = 1$$

Finally:

$$Z_{\Sigma_h} = \sum_{\lambda_i \in \mathcal{R}_q^N} (D_\lambda^q)^{2-2h}$$

$$D_\lambda^q = \mu_q(\lambda)^{1/2} \prod_{i<j} (q^{\frac{1}{2}(\lambda_i-\lambda_j)} - q^{\frac{1}{2}(\lambda_j-\lambda_i)}) \prod_{i<j} |tq^{\frac{1}{2}(\lambda_i-\lambda_j)} - q^{\frac{1}{2}(\lambda_j-\lambda_i)}|$$

$$q = \exp(2\pi i/(k + c_v))$$

• XXZ spin chain is solved using the representation of double affine Hecke algebra and has full affine quantum group symmetry.

• Hall-Littlewood polynomials are replaced by Macdonald polynomials.

• Since partition function (correlators) in G/G WZW model gives partition function for CS theory on $\Sigma \times S^1$ (3-manifold with boundary Σ) above model can be considered as defining the partition function for yet unknown CS theory for complexified group.

• The meaning of D_λ (D_λ^q), replacing the dimension of irreducible representation of compact group (quantum group) for case of flat connection - still not known but obviousely it shall be some natural object for Yangian in YMH theory, described in terms of degenerate double affine Hecke algebra (affine quantum group for G/G model, described in terms of double affine Hecke algebra).