Topological Quantum Field Theories

and

Some Modern Problems of Mathematics

PATHWAYS LECTURE SERIES IN MATHEMATICS

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Plan

- Formulate mathematical, enumerative, problem
- Define physical, QFT, problem which computes same thing
- Explain physics role of this QFT and relation to Math
- Formulate answers (in some cases but not all)
- Discuss the results

Examples

LECTURE I

1. Gromov-Witten theory - Top. Sigma Model, type A

2. Deformation theory - complex structures: Top. B model

3. Donaldson Theory - twisted N=2 SYM on four manifold.

LECTURE II

4. Intersection theory on moduli space of flat connections over Riemann surface - Topological YM theory on Riemann surface $\Rightarrow$ CS and G/G WZW.

5. Intersection theory on moduli space of Higgs Bundles (Hitchin system) - Topological YM-Higgs theory $\Leftrightarrow$ Nonlinear Schrödinger theory and representation theory of Quantum Groups $\Rightarrow$ CS and G/G WZW for complexified G.

LECTURE 1

2d Topological Sigma Models : A & B

Main Math application can be summarized by:

Mirror formula

- Type A sigma model on V = Type B sigma model on $\tilde{V}$
- Relates GW on V to (gen)deformations of cmplx str on $\tilde{V}$
- Manifolds V and $\tilde{V}$ are called mirrors.
- *For Kähler manifolds:* $h^{p,q}(V) = h^{-p,q}(\tilde{V})$
- The concept extends to: V symplectic and $\tilde{V}$ complex.
- Mirror exchanges *kähler* (A) and *complex* (B) deformations.

$$\sum_{n;\{k_1,\ldots,k_n\}} \frac{T^{k_1}\ldots T^{k_n}}{n!} \left\langle \mathcal{O}_a^{(0)}\mathcal{O}_b^{(0)}\mathcal{O}_c^{(0)} \int_\Sigma \mathcal{O}_{k_1}^{(2)} \cdots \int_\Sigma \mathcal{O}_{k_n}^{(2)} \right\rangle_A$$

$$= \frac{\partial^3 \mathcal{F}_B\,(T)}{\partial T^a \partial T^b \partial T^c}$$

Type A sigma models : **Gromov – Witten theory**

Two dimensional sigma model - maps

$$\Phi : \Sigma \to V$$

Σ - two dimensional manifold, world-sheet

V - some Riemannian manifold.

Let V be complex manifold.

• Mathematical reformulation of what physicists call partition function in the topological type A sigma model:

Given a set of submanifolds $C_1, \ldots, C_k$, $C_i \subset V$, compute the number $N_{C_1,\ldots,C_k;\beta}$ of rigid genus g holomorphic curves $\Sigma \subset V$, $[\Sigma] = \beta \in \mathrm{H}_2(V; \mathbf{Z})$ passing through them

The cycles in $\mathrm{H}_*(V)$ represented by $C_1, \ldots, C_k$ are Poincare dual to some cohomology classes $\omega_1, \ldots, \omega_k \in \mathrm{H}^*(V)$.

Physical picture

(Supersymmetric) Sigma model - defined through action functional $\Leftrightarrow$ functional of "fields": $\phi^I, \psi^I_+, \psi^I_-$.

Φ - a map: (Σ - Riemann surface) $\rightarrow$ (V - Riemannian manifold of metric g_{IJ}).

Pick local coordinates: on Σ - $z, \bar{z}$, on V - ϕ^I.

- ϕ^I: Map Φ - locally described by $\phi^I(z, \bar{z})$.

K ($\overline{K}$) - the canonical (anti-canonical) line bundles of Σ (the bundle of one forms of types $(1, 0)$ ($(0, 1)$))

TV - complexified tangent bundle of V.

to get supersymmetry $\Rightarrow$ add Grassmann variables:

- ψ^I_+ - a section of $K^{1/2} \otimes \Phi^*(TV)$

- ψ^I_- - a section of $\overline{K}^{1/2} \otimes \Phi^*(TV)$.

Physical Sigma Model:

$$\mathcal{S}_0(\phi,\psi_-,\psi_+) = \frac{1}{f^2}\int_\Sigma \left(\frac{1}{2}g_{IJ}(\phi)\partial_z\phi^I\partial_{\bar z}\phi^J + \frac{i}{2}g_{IJ}\psi_-^I D_z\psi_-^J\right) +$$
$$+\left(\frac{i}{2}g_{IJ}\psi_+^I D_{\bar z}\psi_+^J + \frac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L\right)$$

f^2 - **coupling constant**

R_{IJKL} - Riemann tensor of V.

$D_{\bar z}$ - $\bar\partial$ operator on $K^{1/2}\otimes\Phi^*(TV)$ constructed using the pullback of the Levi-Civita connection on TV.

• **Now suppose** V is Kähler

Sigma model has extended SUSY: $\mathcal{N}=2$.

Map $\Phi \to$ local coordinates: $\phi^i, \phi^{\bar i} = \overline{\phi^i}$.

Decompose: $TV = T^{1,0}V \oplus T^{0,1}V$.

ψ_+^i ($\psi_+^{\bar i}$) - the projection of ψ_+ in:

$$K^{1/2}\otimes\Phi^*(T^{1,0}V) \qquad (K^{1/2}\otimes\Phi^*(T^{0,1}V))$$

ψ_-^i ($\psi_-^{\bar i}$) - the projections of ψ_- in:

$$\bar K^{1/2}\otimes\Phi^*(T^{1,0}V) \qquad (\bar K^{1/2}\otimes\Phi^*(T^{0,1}V))$$

Action has more parameters:

$$\mathcal{S}_0 = i\theta \int_\Sigma \frac{1}{2} g_{i\bar{j}} \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} \right) + \frac{1}{f^2} \int_\Sigma \frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J +$$

$$+ i\psi_-^{\bar{i}} D_z \psi_-^i g_{\bar{i}i} + i\psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i g_{\bar{i}i} + R_{i\,\bar{i}j\,\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}}$$

θ-another parameter, **theta-angle**.

Twist:

$+$: ${\psi_+}^i$ and ${\psi_+}^{\bar{i}}$ - sections of $\Phi^*(T^{1,0}X)$ and $K \otimes \Phi^*(T^{0,1}X)$.

$-$: ψ_+^i and $\psi_+^{\bar{i}}$ - sections of $K \otimes \Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$.

A Model: $+$ twist of ψ_+ and a $-$ twist of ψ_-.

B Model: $-$ twists of both ψ_+ and ψ_-

Locally the twisting does nothing at all, since locally K and $\overline{K}$ are trivial.

- χ - section of $\Phi^*(TX)$ ($\chi^i = \psi^i_+$, and $\chi^{\bar{i}} = \psi^{\bar{i}}_-$);
- $\psi^{\bar{i}}_+$ - $(1,0)$ form on Σ with values in $\Phi^*(T^{0,1}X)$; $\psi^{\bar{i}}_+ = \psi^{\bar{i}}_z$.
- ψ^i_- - $(0,1)$ form with values in $\Phi^*(T^{1,0}X)$; $\psi^i_- = \psi^i_{\bar{z}}$.

Topological transformation laws:

$$\begin{aligned}
\delta\Phi^I &= i\chi^I \\
\delta\chi^I &= 0 \\
\delta\psi^{\bar{i}}_z &= -\partial_z\phi^{\bar{i}} - i\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_z \\
\delta\psi^i_{\bar{z}} &= -\partial_{\bar{z}}\phi^i - i\chi^j\Gamma^i_{jm}\psi^m_{\bar{z}}.
\end{aligned}$$

$\delta^2 = 0$ - on the space of solutions of equations of motion (minimizing the action). Can be made "off-shell" by introducing auxiliary fields.

Let $t = \theta + \frac{i}{f^2}$.

Action:

$$\mathcal{S}_0 = \frac{1}{f^2}\int_\Sigma d^2z\ \delta R + t\int_\Sigma \Phi^*(\omega)$$

$$R = g_{i\bar{j}}\left(\psi^{\bar{i}}_z\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{i}}\psi^j_{\bar{z}}\right),$$

$$\int_\Sigma \Phi^*(\omega) = i\int_\Sigma d^2z\ \left(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}g_{i\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}g_{i\bar{j}}\right)$$

– the integral of the pullback of the Kähler form $\omega = -ig_{i\bar{j}}dz^idz^{\bar{j}}$.

$\int\Phi^*(\omega)$ - depends only on the cohomology class of ω and the homology class $\beta \in \mathrm{H}_2(V)$ of the image of the map Φ.

In physics one computes correlation functions of some operators (observables) in given theory.

Definition. Observable $\{\mathcal{O}_i\}$ – a functional of the fields, s.t. $\delta\mathcal{O}_i = 0$.

Definition. Physical observable = a δ - cohomology class, $\mathcal{O}_i \sim \mathcal{O}_i + \delta\Psi_i$.

Definition. Correlator - path integral:

$$\langle\prod_a \mathcal{O}_a\rangle_\beta = e^{-2\pi t\int_\beta \omega} \int_{\mathcal{B}_\beta} D\phi\ D\chi\ D\psi\ e^{-\frac{1}{f^2}\delta\int R}\cdot\prod_a \mathcal{O}_a.$$

$\mathcal{B}_\beta$ - the component of the field space for maps of degree $\beta = [\Phi(\Sigma)] \in \mathrm{H}_2(V, \mathbf{Z})$, and $\langle\ \ \rangle_\beta$ - degree β contribution to the expectation value.

Correlators of the observables depend only on their δ-cohomology class, in particular — independent of the complex structure of Σ and V, and depend only on the cohomology class of the Kähler form ω.

Standard argument: $\delta \sim$ exterior derivative on the field space $\mathcal{B} \to \langle \delta\Psi \rangle_\beta = 0$ for any reasonable Ψ. Thus, the $\mathcal{O}_i$ should be considered as representatives of the δ-cohomology classes.

Thus, correlator is independent of f^2. If $f^2 \to 0$ - Gaussian model.

Bosonic part of the *Action*

$$it \int \Phi^*(\omega) + \frac{1}{f^2} \int_\Sigma g_{i\bar{j}}(\phi) \partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i$$

for given β is minimized by holomorphic map:

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = 0.$$

The entire path integral, for maps of degree β, reduces to an integral over the space of degree β holomorphic maps $\mathcal{M}_\beta$.

• **Descend procedure**

Pick an n-form $W = W_{I_1 I_2 \ldots I_n}(\phi) d\phi^{I_1} \wedge d\phi^{I_2} \wedge \ldots \wedge d\phi^{I_n}$ on $V \Rightarrow$ a local functional

$$\mathcal{O}_W(P) = W_{I_1 I_2 \ldots I_n}(\Phi(P)) \chi^{I_1} \ldots \chi^{I_n}(P).$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{dW},$$

d the exterior derivative on V.

$\Rightarrow W \mapsto \mathcal{O}_W$ - natural map from the **de Rham cohomology** of V to the space of physical observables, **δ-cohomology**, of quantum field theory $A(V)$. For local operators - isomorphism.

Let d - be the DeRham differential on Σ. We have **descend equations**:

$\mathrm{d}\mathcal{O}_W = \delta \mathcal{O}_W^{(1)}$, $\oint_C \mathcal{O}_W^{(1)}$ - 1-observable. The physical observable depends on the homology class of C in $\mathrm{H}_1(\Sigma)$.

$\mathrm{d}\mathcal{O}_W^{(1)} = \delta \mathcal{O}_W^{(2)}$, $\int_\Sigma \mathcal{O}_W^{(2)}$ - 2-observable.

Deformations of the theory: change the action as follows:

$$\mathcal{S}_A(T) = \mathcal{S}_0 + T^a \int_\Sigma \mathcal{O}_{W_a}$$

T^a are the formal parameters (nilpotent). The path integral with the action $\mathcal{S}_T$ computes the *generating function* $\mathcal{F}_A(T)$ of the correlation functions of the two-observables:

$$\mathcal{F}_A(T) = \langle e^{-\int_\Sigma \mathcal{S}(T)} \rangle$$

$$\mathcal{S}(0) = \mathcal{S}_0, \qquad \frac{\partial \mathcal{S}}{\partial T^a}|_{T=0} = \int_\Sigma \mathcal{O}_{W_a}$$

Reduction to the enumerative problem

C - submanifold of V (only its homology class matters).

The "Poincaré dual" to C - cohomology class that counts intersections with C. Represent by a differential form $W(C)$ that has delta function support on C:

$$W(C) = \delta_C$$

<u>*Conclude*</u>:

Correlators of topological observables $\mathcal{O}_{W(C_1)} \dots \mathcal{O}_{W(C_k)}$ are integrals over $\mathcal{M}_\beta$ of the products of delta functions which pick out the holomorphic maps whose image intersects the submanifolds $C_1, \dots, C_n$:

Let $C_1, \dots, C_k \subset V$ - complex submanifolds, $\dim C_l = d_l$.

$\omega_m = W(C_m) \in H^*(V)$ - their Poincare duals.

Let $z_1, \dots, z_m \in \Sigma$, $m \leq k$ be the marked points.

For a complex submanifold $C \subset V$ and for $1 \leq l \leq m$ define the following submanifolds $\mathcal{M}^0_{C,l} \subset \mathcal{M}$, $\mathcal{M}^2_C \subset \mathcal{M}$:

Definition. $\mathcal{M}^0_{C,l} = \{\Phi : \Sigma \to V | \, \Phi \in \mathcal{M}, \, \Phi(z_l) \in C\}$

Definition. $\mathcal{M}^2_C = \{\Phi : \Sigma \to V | \, \Phi\,(\Sigma) \cap C \neq \emptyset\}$

The correlation functions in the type A sigma model are simply the intersection numbers:

$$\langle \mathcal{O}^{(0)}_{C_1}(z_1) \ldots \mathcal{O}^{(0)}_{C_m}(z_m) \int_\Sigma \mathcal{O}^{(2)}_{C_{m+1}} \cdots \int_\Sigma \mathcal{O}^{(2)}_{C_k} \rangle =$$

$$\# \mathcal{M}^0_{C_1,1} \cap \ldots \mathcal{M}^0_{C_m,m} \cap \mathcal{M}^2_{C_{m+1}} \cap \ldots \cap \mathcal{M}^2_{C_k}$$

$$\sum \mathrm{dim} \mathcal{M}^0_{C_i,i} + \sum \mathrm{dim} \mathcal{M}^2_{C_i} = \mathrm{dim} \mathcal{M}_\beta$$

otherwise $\langle \ldots \rangle$ vanishes,

$$\mathrm{dim} \mathcal{M}_\beta = \int_\beta c_1(V) + (1-g) dim V$$

Problem: $\mathcal{M}_\beta$ is non-compact. Need to compactify it in order to get a nice intersection theory.

Compactification is not unique.

Option I. Kontsevich stable maps.

Option II. Freckleds – in case where V is a symplectic quotient of a G-equivariant submanifold of a vector (affine) symplectic space A: $V \subset A//G$.

Compactification of $\mathcal{M}$ - Regularization

Non-compactness of $\mathcal{M}$ comes from ultraviolet non-compactness of the fields space $\mathcal{B}$. (UV $= \|d\Phi\|^2 \to \infty$)

Physical picture

Option I = coupling to topological gravity $\approx$ averaging over conformal structures on Σ.

Option II = gauged linear sigma model with target A and gauge group G (and perhaps superpotential).

Type B sigma models: Kodaira-Spencer theory.

Consider the space S of generalized (in the sense of Kontsevich-Witten) deformations of complex structures of variety $\tilde{V}$ ($\tilde{V}$ - mirror to V).

The tangent space to S at some point s represented by a variety V_s' is given by:

$$T_s S = \bigoplus_{p,q} \mathrm{H}^p \left(\tilde{V}_s, \Lambda^q \mathcal{T}_{V_s} \right) \equiv \bigoplus_{p,q} \mathrm{H}^{-q,p}(\tilde{V}_s)$$

Let T denote special coordinates on this space.

The right-hand side of the mirror formula - essentially a partition function in type B sigma model expressed in terms of special coordinates, whose choice is *absolutely necessary* for the formulation of mirror symmetry.

Note: genus dependence doesn't enter in this definition. Precise mathematical definition of $\mathcal{F}_g^B(T)$ is not known.

Physical Picture

$\psi^{\bar{i}}_{\pm}$ - sections of $\Phi^*(T^{0,1}\tilde{V})$

ψ^{i}_{+} - section of $K \otimes \Phi^*(T^{1,0}\tilde{V})$

ψ^{i}_{-} - section of $\overline{K} \otimes \Phi^*(T^{1,0}\tilde{V})$.

ρ - one form with values in $\Phi^*(T^{1,0}\tilde{V})$; $\rho^i_z = \psi^i_+$, $\rho^i_{\bar{z}} = \psi^i_-$.

all fields above are valued in Grassmann algebra

Denote:

$$\eta^{\bar{i}} = \psi^{\bar{i}}_+ + \psi^{\bar{i}}_-$$
$$\theta_i = g_{i\bar{i}}\left(\psi^{\bar{i}}_+ - \psi^{\bar{i}}_-\right).$$

Transformations:

$$\delta\phi^i = 0$$
$$\delta\phi^{\bar{i}} = i\eta^{\bar{i}}$$
$$\delta\eta^{\bar{i}} = \delta\theta_i = 0$$
$$\delta\rho^i = -\, d\phi^i.$$

nilpotent symmetry: $\delta^2 = 0$ on-shell, on the solutions of the equations of motion (minimizing the action functional). Can be made off-shell by introducing extra fields.

Action:

$$\mathcal{S} = \frac{1}{f^2} \int_\Sigma d^2 z \Big(g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i \eta^{\bar{i}} (D_z \rho^i_{\bar{z}} + D_{\bar{z}} \rho^i_z) g_{i\bar{i}}$$
$$+ i\theta_i (D_{\bar{z}} \rho_z{}^i - D_z \rho_{\bar{z}}{}^i) + R_{i\bar{i}j\bar{j}} \rho^i_z \rho^j_{\bar{z}} \eta^{\bar{i}} \theta_k g^{k\bar{j}} \Big) \,.$$

Again one can rewrite the action using δ:

$$\mathcal{S} = \frac{1}{f^2} \int \delta U + \mathcal{S}_0$$

$$U = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right)$$

$$\mathcal{S}_0 = \int_\Sigma \left(-\theta_i D \rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right) .$$

As in A model - define the observables as:

Definition. Observable $\{\mathcal{O}_i\}$ – a functional of the fields, s.t. $\delta \mathcal{O}_i = 0$.

Definition. Physical observable = a δ - cohomology class, $\mathcal{O}_i \sim \mathcal{O}_i + \delta \Psi_i$.

Correlators

$$\langle \prod_a \mathcal{O}_a \rangle = \int_{\mathcal{B}_\beta} D\phi\ D\rho\ D\eta\ e^{-\frac{1}{f^2}\delta \int U - \mathcal{S}_0} \cdot \prod_a \mathcal{O}_a.$$

B theory is independent of the complex structure of Σ and the Kähler metric of $\tilde{V}$. Change of complex structure of Σ or Kähler metric of $\tilde{V}$ - Action changes by irrelevant terms of the form $\delta(\ldots)$.

The theory depends on the complex structure of $\tilde{V}$, which enters δ

B model is independent of f^2; take limit $f^2 \to 0$; In this limit, one expands around minima of the bosonic part of the Action = constant maps $\Phi : \Sigma \to \tilde{V}$:

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0$$

The space of such constant maps is a copy of $\tilde{V}$; the path integral reduces to an integral over $\tilde{V}$.

All above can be demonstrated by considerations similar to those in A-model.

Observables:

Consider $(0,p)$ forms on $\tilde{V}$ with values in $\wedge^q T^{1,0}\tilde{V}$, the q^{th} exterior power of the holomorphic tangent bundle of $\tilde{V}$.

$$W = d\bar{z}^{i_1} d\bar{z}^{i_2} \dots d\bar{z}^{i_p} W_{\bar{i}_1\,\bar{i}_2 \dots \bar{i}_p}{}^{j_1 j_2 \dots j_q} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_q}}$$

W is antisymmetric in the j's as well as in the $\bar{i}$'s.

Form local operator:

$$\mathcal{O}_W = \eta^{\bar{i}_1} \dots \eta^{\bar{i}_p} W_{\bar{i}_1 \dots \bar{i}_p}{}^{j_1 \dots j_q} \psi_{j_1} \dots \psi_{j_q}.$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{\bar{\partial} W},$$

$\mathcal{O}_W$ is δ-invariant if $\bar{\partial} W = 0$ and δ-exact if $W = \bar{\partial} S$ for some S.

$W \mapsto \mathcal{O}_W$ - natural map from $\oplus_{p,q} H^p(V, \wedge^q T^{1,0} V)$ to the δ-cohomology of the B model. It is isomorphism for local operators.

The story of *Correlators in B model*, *Descend Equations*, *Deformation of the action by 2-observables*, *Generating function $\mathcal{F}_B(T)$* is completely parallel to that in A-model:

$$S_B(T) = S + T^a \int \mathcal{O}_{W_a}$$

$$\mathcal{F}_B(T) = < e^{-S_B(T)} >$$

• Interesting examples of the deformations:

$W = \bar{A}^{j}_{\bar{i}} \frac{\partial}{\partial z^j} d\bar{z}^{\bar{j}}$ - deformation of the complex structure of $\tilde{V}$

$W = W(z)$ - holomorphic function (for non-compact $\tilde{V}$)- singularity (Landau-Ginzburg in physical terminology) theory

$W = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$ - non-commutative deformation

• **Complex structure deformations:**

$\tilde{V}_s$ – family of d complex dimensional projective varieties with $c_1(\tilde{V}_s) = 0$ - CY.

Calibrated CY manifold - $(\tilde{V}, \Omega)$; $\tilde{V}$ - CY supplied with the holomorphic $(d, 0)$ form Ω. Holomorphic $(d, 0)$ form - unique up to the multiplication by a non-zero complex number.

$\mathcal{M}$ - moduli of cmplx structures $\tilde{V}_{s_0}$:

$$\mathcal{T}_{s_0}\mathcal{M} \approx \mathrm{H}^{d-1,1}(\tilde{V}_{s_0})$$

The moduli space $\widehat{\mathcal{M}}_{\tilde{V}_{s_0}}$ of the calibrated CY manifolds is a C^*-bundle over $\mathcal{M}_{\tilde{V}_{s_0}}$. The normalized holomorphic $(d, 0)$ from Ω_0 defines locally a section of the bundle.

The choice of the complex structure provides the decomposition of the external derivative $D = D^{1,0} + D^{0,1} = \partial + \bar{\partial}$.

Let $(z^i, \bar{z})$ be local coordinates on $\tilde{V}$ and let $\bar{A} \in \Omega^{-1,1}(\tilde{V})$ be a $(-1, 1)$ differential, locally: $\bar{A} = \sum \bar{A}^j_i d\bar{z}^i \frac{\partial}{\partial z^j}$.

The deformation of the complex structure may be described in terms of the deformation of the operator $D^{0,1} = \bar{\partial}$

$$\bar{\partial} \to \bar{\partial}_{\bar{A}} = \bar{\partial} + \bar{A} = \sum d\bar{z}^i \left(\frac{\partial}{\partial \bar{z}^i} + \bar{A}_{\bar{i}}^j \frac{\partial}{\partial z^j}\right)$$

subjected to the integrability condition $\bar{\partial}_{\bar{A}}^2 = 0$ (Kodaira-Spencer equation).

$I_{KS}(\bar{A})$ - functional with critical points KS-equation. For 3-complex dimensions can be written as function of $\Omega^{(3)}$ via identification $\Omega^{(2,1)} = A \vdash \Omega$.

Special coordinates on $\widehat{\mathcal{M}}$: $T^i, i = 0, \ldots, h^{d-1,1}(\tilde{V}_s)$:

Let $\alpha_I(s), \beta^I(s)$, $I = 0, \ldots, h^{d-1,1}(Y)$ be a symplectic basis in $\mathrm{H}^d(\tilde{V}_s, \mathbf{Z})$:

$$\alpha_I \cap \alpha_J = \beta^I \cap \beta^J = 0, \quad \alpha_I \cap \beta^J = \delta_I^J$$

On the $\widehat{\mathcal{M}}$ this basis is defined uniquely once it is chosen at some marked point $p_0 \in \widehat{\mathcal{M}}$.

$$A^I(s) = \int_{\alpha_I(s)} \Omega, \quad A_{D,I}(s) = \int_{\beta^I(s)} \Omega$$

Ω - defined uniquely up to a constant. Let us fix this freedom by choosing a distinguished cycle α_0 and demanding $A^0 = 1$. Then

$$T^i = A^i, \quad i = 1, \ldots, \mathrm{dim}\mathcal{M}$$

There exists a function ${\mathcal{F}_{(0)}}_B$ on $\widehat{\mathcal{M}}$ such that

$$d\mathcal{F}_{(0)} = \sum_i A_{D,i} dA^i$$

Locally $\mathcal{F}_0$ can be viewed as a function of T^i - generating function of Lagrangian sub-manifold in $H^d(\tilde{V}, \mathbf{C})$ which coincides with $\widehat{\mathcal{M}}$.

Form a function of one extra variable $\lambda \in H^{(d,0)}$ (normalization of $(d,0)$ - form - coordinate in fibre):

$$Z(\lambda, T) = e^{-\sum_g \lambda^{2g-2}\mathcal{F}_g(T)} = e^{-\mathcal{F}(\lambda,T)}$$

If we denote base complex structure as (t, t^*), one can show that $Z_{(t,t^*)}(\lambda, T)$ depends on base complex structure t^* which is captured by differential equation is of heat-kernel type, Holomorphic Anomaly equation.

A. Gerasimov & S.Sh. 2004: value of Kodaira-Spencer action $I(\bar{A})$ at critical points coincides with $\mathcal{F}_{(0)}$ - generating function of Lagrangian sub-manifold introduced above.

Higher genus corrections to $Z(\lambda, T)$ - quantization of symplectomorphizm relating polarization defined by Lagrangian submanifold $\widehat{\mathcal{M}}$ to linear polarization at given base point $(t, t^*) \to$ corrections in coupling constant λ (volume $\Leftrightarrow$ holomorphic three form).

Mirror symmetry: A=B

not only for CY, but more general

Special case of CY threefolds: physical intuition

As $\mathcal{N} = 2$ SCFT's the theories A and B don't differ (internal authomorphism of the $\mathcal{N} = 2$ algebra maps A to B and vice versa)

SCFT has different large volume limits - the same theory looks as different sigma models with different target spaces V and $\tilde{V}$ in different limits.

T-duality - the simplest example.

FOUR DIMENSIONAL THEORY A

DONALDSON-WITTEN THEORY

- X – 4 dimensional compact smooth Riemannian manifold
- $b_i = b_i(X)$ – Betti numbers.
- On $\mathrm{H}^*(X)$: intersection form $(,)$; metric $\langle , \rangle$:

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2, \quad \langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \star\omega_2$$

$\star$ - the Hodge star operation.

$b_2^{\pm}$ – dim's of the positive and negative subspaces of $\mathrm{H}^2(X)$.

$\omega \in \mathrm{H}^2(X)$: $\omega^{\pm}$ – orthogonal projections to the spaces of self- and antiselfdual classes: $\mathrm{H}^{2,\pm}(X)$ – $(\omega^{\pm}, \cdot) = \pm\langle \omega^{\pm}, \cdot \rangle$, $\omega = \omega^+ + \omega^-$.

$\chi = \sum_{i=0}^{4} (-1)^i b_i$, – the Euler characteristics of X

$\sigma = b_2^+ - b_2^-$ the signature of X

- e_α is a basis in $\mathrm{H}_*(X, \mathbf{C})$,
- e^α the dual basis in $\mathrm{H}^*(X, \mathbf{C})$:

$$(e^\alpha, \omega) = \int_{e_\alpha} \omega$$

for any $\omega \in \mathrm{H}^*(X)$.

$\mathbf{G}' = SU(r+1)$, $\mathbf{G} = \mathbf{G}'/Z$, $Z \approx \mathbf{Z}_{r+1}$, $\mathbf{g} = \mathrm{Lie}\mathbf{G}$.

$\mathbf{T} = U(1)^r$ – maximal torus of $\mathbf{G}$, $W = \mathcal{S}_{r+1}$ the Weyl group,

$\mathbf{g} = \mathrm{Lie}(\mathbf{G}), \mathbf{t} = \mathrm{Lie}(\mathbf{T})$.

$h = r + 1$ – dual Coxeter number.

$\ell = (w_2; k)$, $k \in \mathbf{Z}$, $w_2 \in \mathrm{H}^2(X, Z)$ – generalized Stiefel-Whitney class.

$\mathcal{P}_\ell$ - a principal $\mathbf{G}$ bundle over X and E_ℓ the associated vector bundle with $w_2(E_\ell) = w_2$,

$c_2(E_\ell) + \frac{1}{2} w_2 \cdot w_2 = k$.

$\mathcal{A}_\ell$ - the space of connections in $\mathcal{P}_\ell$.

$\mathcal{G}_\ell$ - the group of gauge transformations of $\mathcal{P}_\ell$.

The Lie algebra of $\mathcal{G}_\ell$ - the algebra of sections of the associated adjoint bundle $\mathbf{g}_\ell = \mathcal{P}_\ell \times_{\mathbf{Ad}} \mathbf{g}$. ϕ - an element of Lie$\mathcal{G}_\ell$.

For the connection A (= the gauge field) let F_A denote its curvature (it is a section of $\Lambda^2 T_X^* \otimes \mathbf{g}_\ell$).

Definition. **G**-instanton is the solution to the equation

$$F_A^+ = F + \star F = 0$$

where $+$ acts on the $\Lambda^2 T_X^*$ part of F_A.

Definition. a **G**-instanton A is called irreducible if there are no infinitesimal gauge transformations, preserving A. This condition is equivalent to the absence of the solutions to the equation

$$d_A \phi = 0, \quad 0 \neq \phi \in \Gamma(\mathbf{g}_\ell)$$

where d_A is the connection on $\mathbf{g}_\ell$ associated with A.

Definition. a **G**-instanton is called unobstructed if there are no solutions to the equation $(d_A^+)^* \chi = 0$, $0 \neq \chi \in \Gamma\left(\Lambda^{2,+} T_X^* \otimes \mathbf{g}_\ell\right)$.

Definition. The moduli space $\mathcal{M}_\ell$ of **G**-instantons is the space of all irreducible unobstructed **G**-instantons modulo action of $\mathcal{G}_\ell$. For the instanton A let $[A]$ denote its gauge equivalence class - a point in $\mathcal{M}_\ell$.

The tangent space to $\mathcal{M}_\ell$ at A is the middle cohomology group of the Atiyah-Hitchin-Singer (AHS) complex of bundles over X:

$$0 \to \Lambda^0 T^*_X \otimes \mathbf{g}_\ell \to \Lambda^1 T^*_X \otimes \mathbf{g}_\ell \to \Lambda^{2,+} T^*_X \otimes \mathbf{g}_\ell \to 0$$

the first arrow is d_A, the second is $d_A^+ = \mathrm{P}_+ d_A$.

P_+ - the projection $\Lambda^2 T^*_X \otimes \mathbf{g}_\ell \to \Lambda^{2,+} T^*_X \otimes \mathbf{g}_\ell$.
$d_A^+ \circ d_A = F_A^+ = 0 \to$ the sequence is the complex.

$H^0(AHS) = 0$ for irred. instantons. $H^2(AHS) = 0$ - obstruction space; absent for unobstructed instantons.

Lemma. The dimension of the moduli space $\mathcal{M}_\ell$:

$$\mathrm{dim}\mathcal{M}_\ell = 4hk - \mathrm{dim}\mathbf{G}\frac{\chi + \sigma}{2}$$

Proof: index theorem applied to the AHS complex.

Remark. $\mathcal{M}_\ell$ is non-compact. Sometimes it can be compactified (Donaldson-Uhlenbeck) by adding the point-like instantons:

$$\overline{\mathcal{M}}_\ell = \mathcal{M}_\ell \cup \mathcal{M}_{\ell-(0;1)} \times X \cup \ldots \cup \mathcal{M}_{\ell-(0;k)} \times S^k X$$

For A from class $[A] \in \mathcal{M}_\ell$ the space $T_{[A]}\mathcal{M}_\ell$ can be identified with the space of solutions α:

$$d_A^+ \alpha = 0, \quad d_A^* \alpha = 0$$

$\alpha \in \Gamma\left(\Lambda^1 T^* X \otimes \mathbf{g}_\ell\right)$.

Consider the product $\mathcal{M}_\ell \times X$ and form the *universal bundle* $\mathcal{E}_\ell$ - the bundle whose restriction onto $[A] \times X \subset \mathcal{M}_\ell \times X$ coincides with E_ℓ.

$\mathbf{d}$ be the differential in the DeRham complex on $\mathcal{M}_\ell \times X$ and d_m, d be its components along $\mathcal{M}_\ell$, X respectively.

Definition. The *universal* connection is the $\mathbf{G}$-connection $\mathbf{a}$ in $\mathcal{E}_\ell$ with the following properties:

1. $\mathbf{a}|_{[A]\times X} \in [A]$
2. $\mathbf{a}|_{\mathcal{M}_\ell \times \{x\}} = \frac{1}{\Delta_A} d_A^* d_m A$ with $\Delta_A = d_A^* d_A$

Lemma. The curvature of the universal connection can be expanded as:

$$\mathcal{F}_{\mathbf{a}} = F_A + \psi + \phi$$

ψ is the fundamental solution to the equations:

$$\boxed{d_A^+ \psi = 0, \quad d_A^* \psi = 0}$$

ϕ is given by:

$$\phi = \frac{1}{\Delta_A}[\psi, \star\psi]$$

Comments. We view ψ as the mixed $(\mathcal{M}_\ell, X)$ component of the curvature of $\mathbf{a}$. It means that locally we view ψ as one-form on $\mathcal{M}_\ell$ with values in $\mathbf{g}$. Using metric on X and the induced metric on $\mathcal{M}_\ell$ we identify $T_{[A]}\mathcal{M}_\ell$ with $T^*_{[A]}\mathcal{M}_\ell$.

Similarly ϕ is the $(\mathcal{M}_\ell, \mathcal{M}_\ell)$ component of the curvature of $\mathbf{a}$.

$\{I_k\}$ - additive basis in the space of invariants: $\mathrm{Fun}(\mathbf{g})^{\mathbf{G}} \approx \mathrm{Fun}(\mathbf{t})^W$.

d_k - the degree of I_k.

$\mathcal{O}_n^\alpha = \int_{e_\alpha} I_n \left(\frac{\phi+\psi+F_A}{2\pi i} \right)$.

Examples. $I_1(\phi) = \mathrm{Tr}\phi^2$, $d_1 = 2$, $I_2(\phi) = \mathrm{Tr}\phi^3, I_3 = \mathrm{Tr}\phi^4, I_4 = \left(\mathrm{Tr}\phi^2\right)^2$, $d_2 = 3, d_3 = d_4 = 4$.

Denote $\mathcal{M} = \amalg_\ell \mathcal{M}_\ell$, $\mathcal{E} = \amalg \mathcal{E}_\ell$. There is a a characteristic class $c_I(\mathcal{E})$ associated to each invariant $I \in \mathrm{Fun}(\mathbf{g})^{\mathbf{G}}$.

Let Ω_n^α be the slant product $\int_{e_\alpha} c_{I_n}(\mathcal{E}) \in \mathrm{H}^{2d_n - \mathrm{dim} e_\alpha}(\mathcal{M})$.

Definition. The following integral over $\mathcal{M}$ is the attempt to define the intersection theory of Ω_n^α

$$\left\langle \Omega_{n_1}^{\alpha_1} \dots \Omega_{n_k}^{\alpha_k} \right\rangle = \sum_{\ell} \int_{\mathcal{M}_\ell} \mathcal{O}_{n_1}^{\alpha_1} \wedge \dots \wedge \mathcal{O}_{n_k}^{\alpha_k}$$

Definition. The prepotential of the refined Donaldson-Witten theory is the generating function:

$$\mathcal{Z}_A(T) = \left\langle \exp\left(T_\alpha^k \Omega_k^\alpha\right) \right\rangle \equiv$$

$$\sum \frac{1}{k!} T_{\alpha_1}^{n_1} \dots T_{\alpha_k}^{n_k} \left\langle \Omega_{n_1}^{\alpha_1} \dots \Omega_{n_k}^{\alpha_k} \right\rangle$$

Physical Picture

<u>*The fields*</u>: twisted $\mathcal{N} = 2$ vector multiplet

Bosons: gauge field $A = A_\mu dx^\mu$, the complex scalar ϕ and its conjugate $\bar{\phi}$, self-dual two form H

Fermions: the one-form ψ, the scalar η and the self-dual two-form χ.

All fields take values in the adjoint representation.

<u>*Nilpotent Symmetry*</u>:

$$\delta\phi = 0, \quad \delta\bar{\phi} = \eta, \quad \delta\eta = [\phi, \bar{\phi}]$$

$$\delta\chi = H, \quad \delta H = [\phi, \chi]$$

$$\delta A = \psi, \quad \delta\psi = D_A\phi$$

$\delta^2 = \mathcal{L}_\phi =$ infinitesimal gauge transformation generated by $\phi \Rightarrow$ nilpotent on the gauge invariant functionals of the fields (equivariant cohomology).

Definition. Observables - gauge invariant functionals of the fields, annihilated by δ.

The correlation functions of observables do not change under a small variation of metric on the four-manifold X.

Observables: Invariant polynomial $\mathcal{P} = \sum_k t^k I_k$ on the algebra $\mathbf{g}$, $C^k, k = 0, \dots 4$ – closed k-cycles on X. Their homology cycles are denoted as $[C^k] \in \mathrm{H}_k(X; \mathbf{C})$. The observables form the descend sequence:

$$\mathcal{O}^{(0)} = \mathcal{P}(\phi), \quad \delta\mathcal{O}^{(0)} = 0$$

$$d\mathcal{O}^{(0)} = -\delta\mathcal{O}^{(1)} \quad (\mathcal{O}^{(1)}, [C^1]) \equiv \int_{C^{(1)}} \mathcal{O}^{(1)} \equiv \int_{C^1} \frac{\partial\mathcal{P}}{\partial\phi^a}\psi^a$$

$$d\mathcal{O}^{(1)} = -\delta\mathcal{O}^{(2)} \quad (\mathcal{O}^{(2)}, [C^2]) = \int_{C^2} \mathcal{O}^{(2)} =$$

$$\int_{C^2} \frac{\partial\mathcal{P}}{\partial\phi^a}F^a + \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial\phi^a\partial\phi^b}\psi^a \wedge \psi^b$$

$$\dots$$

top degree observable: $\mathcal{O}^{(4)}_{\mathcal{P}} = \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial\phi^a\partial\phi^b}F^a F^b +$

$$+\frac{1}{3!}\frac{\partial^3\mathcal{P}}{\partial\phi^a\partial\phi^b\partial\phi^c}F^a\psi^b\psi^c + \frac{1}{4!}\frac{\partial^4\mathcal{P}}{\partial\phi^a\partial\phi^b\partial\phi^c\partial\phi^d}\psi^a\psi^b\psi^c\psi^d$$

Action S equals the sum of the 4-observable, constructed out of the *prepotential* $\mathcal{F}$ and the δ-exact term:

$$S = \mathcal{O}^{(4)}_{\mathcal{F}} + \delta R$$

The standard choice: $\mathcal{F} = \left(\frac{i\theta}{8\pi^2} + \frac{1}{e^2}\right) \mathrm{Tr}\phi^2$,

$$R = \frac{1}{e^2} \mathrm{Tr} \left(\chi F^+ - \chi H + D_A \bar{\phi} \star \psi + \eta \star [\phi, \bar{\phi}] \right),$$

Tr denotes the Killing form.

The bosonic part of the action S is then:

$$S = \int_X \tau \mathrm{Tr} F \wedge F +$$

$$+ \frac{1}{e^2} \left(\mathrm{Tr} F \wedge \star F + \mathrm{Tr} D_A \phi \wedge \star D_A \bar{\phi} + \mathrm{Tr}[\phi, \bar{\phi}]^2 \right)$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

The e^2-dependence – only via $\delta\,(\ldots)$ terms:

$$\mathcal{S} = \frac{\theta}{2\pi} \int_X F \wedge F + \frac{1}{e^2} \delta(...)$$

$\Rightarrow$ can take $e^2 \to 0$ limit for correlators of observables: the path integral measure gets localized near solutions to $F^+ = 0$, $D_A \phi = 0$

Moral. **The correlation functions of observables reduce to the integrals over $\mathcal{M}_\ell$.**

• Donaldson theory ($G = SU(2)$ or $G = SO(3)$): aim is to compute:

$$\langle \exp((\mathcal{O}_u^{(2)}, w) + \lambda \mathcal{O}_u^{(0)}) \rangle,$$

for $w \in H^2(X, \mathbf{R})$, $\mathcal{O}_u^{(0)} = u \equiv \mathrm{Tr}\phi^2$,

$$(\mathcal{O}_u^{(2)}, w) = -\frac{1}{4\pi^2} \int_X \mathrm{Tr}(\phi F + \frac{1}{2}\psi\psi) \wedge w$$

• Refinement: generating function of all correlators of all observables:

$$\mathcal{Z}_A(T^k) = \langle e^{T^{k,\alpha}(\mathcal{O}_{I_k}^{(4-d_\alpha)}, e_\alpha)} \rangle$$

$$T^k = T^{k,\alpha} e_\alpha \in \mathcal{V} = \oplus_{p=0}^{4} \mathrm{H}^p(X, \mathbf{C})$$

This is a physical definition of the four dimensional type A theory

Very important tool of computing infinite-dimensional path integral over all fields entering in the definition of correlators $\Rightarrow$ **Abelianization.**

Problem. $\mathcal{M}_\ell$ is non-compact. Need to compactify it in order to have a nice intersection theory.

• Donaldson compactification: add point-like instantons as above (for high enough instanton charges get a manifold, perhaps with orbifold singularities)

• For Kähler X a refinement of the compactification above: Gieseker compactification:

Idea: On Kähler X with Kähler form ω :

$$F^+ = 0 \Leftrightarrow \bar{\partial}_A^2 = 0, \quad F \wedge \omega = 0$$

$\bar{\partial}_A$ defines a holomorphic bundle $\mathcal{E}$ over X: its local sections are annihilated by $\bar{\partial}_A$. Then $F \wedge \omega = 0$ is a stability condition.

Replace $\mathcal{E}$ by its (holomorphic) sheaf of sections. Consider the moduli space $\overline{\mathcal{M}}_\ell^G$ of sheaves which are *torsion free* as $\mathcal{O}_X$-modules. The latter has sheaves which are not *locally free*, i.e. which are not holomorphic bundles. However, for each such sheaf $\mathcal{E}'$ there is a zero-dimensional subscheme $Z \subset X$, such that on $X \backslash Z$ $\mathcal{E}'$ **is** a holomorphic bundle and has a connection.

Problem. Find an analogue of Kontsevich compactification.

Problem. Find a physical realization of all these compactifications.

Partial answer to the last problem: On $X = \mathbf{CP}^2$ the compactification by sheaves corresponds to the *gauge theory on a non-commutative space.*

Intersection theory in four dimensions

Take $X = \mathbf{CP}^2$, $G = U(r)$, w - Kähler form.

$p \in \mathrm{H}^2(X, \mathbf{Z}), k \in \mathrm{H}^4(X, \mathbf{Z})$.

• Monad construction of the torsion free sheaves on X: Let V_0, V_1, V_2 be the complex vector spaces of dimensions $v_{0,1,2}$ respectively. Consider the complex of bundles over X:

$$0 \to V_0 \otimes \mathcal{O}(-1) \xrightarrow{a} V_1 \otimes \mathcal{O} \xrightarrow{b} V_2 \otimes \mathcal{O}(1) \to 0$$

In down-to-earth terms this sequence has the following meaning. The maps a, b in the homogeneous coordinates $(z^0 : z^1 : z^2)$ are the matrix-valued linear functions: $a(z) = z^\alpha a_\alpha, b(z) = z^\alpha b_\alpha$. The words "complex" mean that

$$b(z) \cdot a(z) = z^\alpha z^\beta b_\alpha a_\beta = 0 \Leftrightarrow$$

$$b_\alpha a_\alpha = 0,\ \alpha = 0, 1, 2, \quad b_\alpha a_\beta + b_\beta a_\alpha = 0,\ \alpha \neq \beta$$

For the pair (b, a) of the maps between the sheaves obeying this condition we can define a sheaf $\mathcal{F}$ over X, whose space of sections over an open set U is

$$\Gamma\left(\mathcal{F}|_U\right) = \mathrm{Ker} b(z) / \mathrm{Im} a(z), \quad \text{for} \quad (z^0 : z^1 : z^2) \in U$$

$$\beta^{ij}(z)\Psi^j(z) = 0, \quad \text{modulo} \quad \Psi^j(z) = a^{jk}(z)\tilde{\Psi}^k(z)$$

Definition: The space of monads is the space M_{mon} of triples of matrices $a_\beta \in \mathrm{Hom}(V_0, V_1)$, $b_\alpha \in \mathrm{Hom}(V_1, V_2)$ obeying $b(z)a(z) = 0$. This space is acted on by the group

$$G^c_{\mathrm{mon}} = \left(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)\right) / \mathbf{C}^\star$$

$$(b,a) \mapsto g \cdot (b,a) = (g_2 b g_1^{-1}, g_1 a g_0^{-1}), \text{ for } (g_0, g_1, g_2) \in G^c_{\text{mon}}$$

The sheaves defined by the pairs (b,a) and $g \cdot (b,a)$ are isomorphic. The maximal compact subgroup of G^c_{mon}

$$G_{\text{mon}} \approx \left(U(V_0) \times U(V_1) \times U(V_2)\right)/U(1)$$

acts in M_{mon} preserving its natural symplectic structure

$$\Omega = \frac{1}{2i} \sum_{\beta} \mathrm{Tr} \delta a_{\beta} \wedge \delta a^{\dagger}_{\beta} + \frac{1}{2i} \sum_{\alpha} \mathrm{Tr} \delta b^{\dagger}_{\alpha} \wedge \delta b_{\alpha}$$

Fix the real numbers r_0, r_1, r_2, such that $\sum_{\alpha} v_{\alpha} r_{\alpha} = 0$, $r_0, r_2 > 0$. Write the moment maps:

$$\mu_1 = -r_0 \mathbf{1}_{v_0} + \sum_{\beta} a^{\dagger}_{\beta} a_{\beta}$$

$$\mu_2 = -r_1 \mathbf{1}_{v_1} + \sum_{\alpha} b^{\dagger}_{\alpha} b_{\alpha} - \sum_{\beta} a_{\beta} a^{\dagger}_{\beta}$$

$$\mu_3 = -r_2 \mathbf{1}_{v_2} + \sum_{\alpha} b_{\alpha} b^{\dagger}_{\alpha}$$

Then the moduli space of the semistable sheaves is

$$\overline{\mathcal{M}}_{c_*} = \left(\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)\right)/G_{\text{mon}}$$

This is typical example of hyperkähler quotient (Integration over Higgs Branches - MNS'97). The compactness of the space is obvious: if we first perform a reduction with respect to the groups $U(V_0) \times U(V_2)$ then the resulting space is the product of two Grassmanians: $\mathrm{Gr}(v_0, 3v_1) \times \mathrm{Gr}(v_2, 3v_1)$ which is already compact. The subsequent reduction does not spoil this.

The Chern classes, $c_* = \{r, c_1, c_2\}$, of the sheaf $\mathcal{F}$ determined by the pair (b, a) are:

$$r = v_1 - v_0 - v_2,\; c_1 = (v_0 - v_2)\,,\; c_2 = \frac{1}{2}\left((v_2 - v_0)^2 + v_0 + v_2\right)$$

Let $(i\psi, i\phi, i\chi)$ denote the elements of the Lie algebra of $G_{\rm mon}$, i.e. $i\psi \in {\rm u}(V_0), i\phi \in {\rm u}(V_1), i\chi \in {\rm u}(V_2)$ and $(\psi, \phi, \chi) \sim (\psi + \mathbf{1}_{v_0}, \phi + \mathbf{1}_{v_1}, \chi + \mathbf{1}_{v_2})$. We are interested in computing certain integrals over $\overline{\mathcal{M}}_{c_*}$. This can be accomplished by computing an integral over $M_{\rm mon}$ with the insertion of the delta function in μ_i and dividing by the volume of $G_{\rm mon}$ provided that the expression we integrate is $G_{\rm mon}$-invariant:

$$\int_{\overline{\mathcal{M}}_{c_*}} (\ldots) =$$

$$\frac{1}{{\rm Vol}(G_{\rm mon})} \int_{{\rm Lie}G_{\rm mon}} d\psi d\phi d\chi e^{i{\rm Tr}\psi\mu_1 + i{\rm Tr}\phi\mu_2 + i{\rm Tr}\chi\mu_3} (\ldots)$$

The useful fact is that the observables of the gauge theory we are interested in are the gauge-invariant functions on (ψ, ϕ, χ) only. More specifically, there is a *universal sheaf* $\mathcal{U}$ over $\overline{\mathcal{M}}_{c_*} \times X$, defined again as ${\rm Ker}b(z)/{\rm Im}a(z)$ but now the space of parameters contains (b, a) in addition to z. Its Chern character is given by:

$$Ch(\mathcal{U}) = {\rm Tr}e^{\phi} - {\rm Tr}e^{\psi - \omega} - {\rm Tr}e^{\chi + \omega}$$

In particular:

$$\mathcal{O}^{(0)}_{u_1} = \frac{1}{2}\left({\rm Tr}\chi^2 + {\rm Tr}\psi^2 - {\rm Tr}\phi^2\right); \int_X \omega \wedge \mathcal{O}^{(2)}_{u_1} = {\rm Tr}\chi - {\rm Tr}\psi$$

Since the observables are expressed through ψ, ϕ, χ only we can integrate out a_β, b_α to obtain:

$$\langle \exp t_1 \mathcal{O}^{(0)}_{u_1} + T_1 \int_S \omega \wedge \mathcal{O}^{(2)}_{u_1} \rangle^{\text{torsion free}} = \oint \prod_{i,j,k} d\psi_i d\chi_j d\psi_k$$

$$\frac{\prod_{i'<i''} (\psi_{i'} - \psi_{i''})^2 \prod_{j'<j''} (\phi_{j'} - \phi_{j''})^2}{\prod_{i,j} (\phi_j - \psi_i + i0)^3}$$

$$\frac{\prod_{k'<k''} (\chi_{k'} - \chi_{k''})^2 \prod_{i,k} (\chi_k - \psi_i)^6}{\prod_{j,k} (\chi_k - \phi_j + i0)^3}$$

$$\times e^{t_1 \frac{1}{2} \left(\sum_k {\chi_k}^2 + \sum_i \psi_i^2 - \sum_j \phi_j^2\right) + T_1 \left(\sum_k \chi_k - \sum_i \psi_i\right)} \times$$

$$e^{ir_1 \sum_i \psi_i + ir_2 \sum_j \phi_j + ir_3 \sum_k \chi_k}$$

Abelianization - Theory B, Physical Picture

• Integrate out non-abelian components of all fields (quadratic, Gaussian, integral). Result - some abelelain theory, defined on Cartan subgroup of Gauge group with abelian fields: $\phi^i, \bar{\phi}^i, \eta^i, A^i, \psi^i, \chi^i$

• Again, on the space of fields δ-operators acts (original topological, δ, symmetry is preserved - not broken): $\delta^2 = 0$. Define observables for abelian theory as in original, non-abelian theory: $\mathcal{O}^i$.

• Find for every observable in non-abelian theory corresponding observable after abelianization.

• Write the action in abelian theory as 4-observable descedning from some function $\mathcal{F}(u)$, where $u_1, ..., u_N$ are invariant polynimials of ϕ, functions of ϕ^i.

•From general principles the abelian action must have the form:

$$S_0 = \mathcal{O}_{\mathcal{F}}^{(4)} + \delta R$$

and deformed action is:

$$S = S_0 + t_i \mathcal{O}^i$$

The generating function for correlators is given by partition function on **B** side by:

$$Z_B(t) =< e^{-S(t)} >= \int D\phi^i D\bar{\phi}^i D\eta^i DA^i D\psi^i DH^i e^{-S_0 - t^i \mathcal{O}^i}$$

and finally:

$$Z_A(T) = Z_B(t(T))$$

This shows that one needs:

1. Explicit expression for δ in terms of abelian fields,

2. Explicit form of $\mathcal{F}(u)(u = Tr\phi^2$ for $SU(2))$

3. Explicit relation between observables $\mathcal{O}^i$ between non-abelian and abelian theories

4. Explicit relation between parameters T^i in non-abelian theory and t^i in abelian theory - $t^i(T)$.

1. & 2. δ and formula for prepotential $\mathcal{F}$ was found by Seiberg & Witten in 1994 (for $SU(2)$). Other groups - various authors after SW found prepotential $\mathcal{F}$ for all groups and all generalizations of 4d $N = 2$ SYM with matter.

3. & 4. Solution to these was found by Moore & WItten and by Losev, Nekrasov & S. Sh. in 1997 ("universal formula for contact terms" etc.).

Integral over abelian fields in theory **B** is reduced to finite-dimensional integral via localization technique and is related to nice and simple symplectic geometry problem.

Few words on prepotential $\mathcal{F}$:

In abelianized theory $\phi = diag(a_1, .., a_r)$. Let (a_i, a_D^i) coordinates in C^{2r} with complex symplectic worm $\omega = da_i \wedge da_D^i$.

$\mathcal{F}$ - generating function of Lagrangian submanifold $\Theta = a_D^i da_i = d\mathcal{F}$ invariant under certain discrete subgroup Γ of $SP(2r, \mathbf{Z})$.

Turning on couplings T corresponds to deformations of this Lagrangian submanifold - flows described explicitly in LNS.