Nodally 3-connected planar graphs and barycentric embeddings

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Abstract

An interesting question about planar graphs is whether they admit plane embeddings in which every bounded face is convex. Stein [10] gave as a necessary and sufficient condition that every face boundary be a simple cycle and every two bounded faces meet in a connected set, with an extra condition about the number of vertices on the outer face. Tutte [12] gave a similar characterisation, and later [13] showed that every nodally 3-connected planar graph admits a barycentric embedding. Floater [4] generalised this to convex combination mappings of triangulated graphs. White [14] showed that a chord-free triangulated graph is nodally 3-connected and showed that Tutte's result applies to all triangulated graphs.

We extend Tutte's results beyond the class of triangulated graphs.

We show that a biconnected plane-embedded graph is nodally 3-connected if and only if the intersection of any two faces, bounded or otherwise, is connected.

If a plane embedded graph admits a convex embedding, then every face boundary is a simple cycle, the intersection of every two faces is connected, and there are no inverted subgraphs (as defined in the paper). Such graphs we call *admissible.* The idea of admissible embedded graph is more useful than Stein's criterion [10] and simpler than Tutte's [12].

We show that every admissible plane embedded graph admits a barycentric embedding.

It follows immediately that a plane embedded graph has a convex embedding if and only if every barycentric map is an embedding.

Finally we show that when a plane embedded graph admits a barycentric embedding, the two embeddings are isotopic.

1 Criterion for nodally 3-connected planar graphs

We follow the usual definitions of graphs, paths, cycles, connectivity, plane embeddings, and planar graphs: [6] is a useful source on the subject. The accepted definition of graph does not allow self-loops nor multiple edges nor infinite sets of vertices, so it is a finite simple graph in Tutte's language [13], and a graph G can be specified as a pair (V, E) giving its vertices and edges. E is a set of unordered pairs of distinct vertices in V.

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Given $G = (V, E)$, when u is considered to be a vertex, $u \in G$ means $u \in V$, and when e is considered to be an edge, $e \in G$ means $e \in E$.

The degree of a vertex is the number of edges incident to it, or the number of neighbours it has. The word 'node' is reserved in [13] to denote vertices whose degree $\neq 2$.

A *path (graph)* is either a trivial graph or one in which two vertices have degree 1 and all others have degree 2. A *simple cycle (graph)* is a connected nonempty graph all of whose vertices have degree 2.

If $G_i = (V_i, E_i)$ are two graphs then we define

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2) \quad \text{and} \quad G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

If $G = (V, E)$ and $S \subseteq V$ then $G \backslash S = (V', E')$ where

$$V' = V \backslash S \quad \text{and} \quad E' = \{\{u, v\} \in E : \ u \notin S \text{ and } v \notin S\}.$$

We extend this notation loosely but with little risk of confusion: if x is a vertex then $G \backslash x = G \backslash \{x\}$, and if H is a subgraph, or a path, or a cycle, then $G \backslash H$ is the same as $G \backslash S$ where S is the set of vertices in H.

A graph G is *connected* if every two vertices are connected by a path in G; it is *biconnected* if it is connected and for every $u \in G$, $G \backslash u$ is connected. G is *triconnected* if it is biconnected and for any $u, v \in G$, $G \backslash \{u, v\}$ is connected.

This paper is concerned with *nodal 3-connectivity* (defined in 1.23), which requires biconnectivity but is weaker than triconnectivity.

We assume the usual definitions of plane embeddings which map vertices to points in $\mathbb{R}^2$ and edges to curve-segments with the usual conditions about intersection. 'G is planar' means that a plane embedding exists. Generally a plane embedding takes edges to curve-segments. If these are all straight line-segments we have a *straight-edge embedding.*

Of course a straight-edge embedding is fully determined by the placement of the vertices. We also note

(1.1) Proposition *A graph is planar if and only if it admits a straight-edge embedding [3,8,9].* ∎

(1.2) Topology in two dimensions. See [7,11]. We assume the basic notions of open and closed sets, connectedness, and path-connectedness. If $x \in \mathbb{R}^2$ and $\varepsilon > 0$ then the ε-neighbourhood of x is

$$B(x, \varepsilon) = \{y \in \mathbb{R}^2 : \ |y - x| < \varepsilon\}.$$

If S is any subset of $\mathbb{R}^2$ then its *closure*, written $\overline{S}$, is

$$\overline{S} = \{x \in \mathbb{R}^2 : \ (\forall \varepsilon > 0) B(x, \varepsilon) \cap S \neq \emptyset\},$$

and its boundary ∂S is

$$\partial S = \overline{S} \cap \overline{\mathbb{R}^2 \backslash S}.$$

If S is open then $S \cap \partial S = \emptyset$. We are not concerned with connectedness, but with the rather stronger notion of path-connectedness:[1] a set S is *path-connected* if for any $x, y \in S$ there exists a path from x to y, a continuous map $\pi : [0, 1] \to S$ such that $\pi(0) = x$ and $\pi(1) = y$.

[1]A topological space S is disconnected if $S = A \cup B$ where A and B are disjoint and nonempty and and open. Otherwise S is connected. Usually, or at least in this paper, a set is connected if and only if it is path-connected.

(1.3) Definition *Given a plane embedding of a graph G, by abuse of notation let G also denote the union of points and curve-segments constituting its image in the plane. This is a closed and bounded set of points in the plane.*

A face *of G is a path-connected component of $\mathbb{R}^2 \backslash G$.*

All faces except one are bounded. The unbounded face is called the external face *or* outer face. *Vertices on the external face are called* external*; the others are* internal.

Faces are open sets in $\mathbb{R}^2$.

One often speaks of a planar graph G with a specific plane embedding of G in mind, so it really means a plane embedded graph. A very significant difference is that a plane embedded graph has a definite external face, whereas there is no notion of external face, nor perhaps even of face, in a planar graph without a prescribed embedding. Figure 2 shows a planar graph with two quite different embeddings.

(1.4) Proposition *Let x and y be two vertices in an plane embedded graph. Then they are in the same component of G as a graph if and only if they are in the same (path-)component of G as a topological subspace of $\mathbb{R}^2$. (Proof easy.)* ▮

(1.5) Definition *Let e be an embedded edge of a plane-embedded graph, so it is a curve-segment joining two points x and y. The* relative interior *of e is the point-set*

$$\text{interior}(e) = e \backslash \{x, y\}.$$

The following proposition applies to straight-edge embeddings for the sake of simplicity. Reference to straight-edge embeddings in this paper is generally to make proofs easier, even when the proofs are omitted. The results generally hold without assuming the edges are straight. The assumption simplifies the proof of the following

(1.6) Proposition (i) *If F is a face of a straight-edge embedded graph G, then ∂F is a subgraph of G, and* (ii) $G = \bigcup_F \partial F$. *(Proof omitted.)* ▮

(1.7) Jordan curves. A *Jordan curve* is a subset of $\mathbb{R}^2$ homeomorphic to the unit circle S^1. That is, J is a Jordan curve iff there exists a continuous injective map $h : S^1 \to \mathbb{R}^2$ whose range is the subset J. Part (i) of Proposition 1.8 below states the Jordan Curve Theorem, which is a difficult result. Proofs usually involve algebraic topology [5], but less advanced methods can be used [7,11]. Actually for our purposes we need only consider polygonal Jordan curves, which makes the proofs much easier. Parts (ii) and (iii) are elementary.

(1.8) Proposition (i) *(Jordan Curve Theorem [5,7,11]) If J is a Jordan curve then $\mathbb{R}^2 \backslash J$ is the union of two open, path-connected components,* interior(J) *and* exterior(J), interior(J), *the* inside, *is bounded, and* exterior(J), *the* outside *or* exterior, *is unbounded, and* $\partial(\text{interior}(J)) = \partial(\text{exterior}(J)) = J$.

(ii) *If S is any path-connected open set such that $\partial S = J$, then $S =$* interior(J) *or $S =$* exterior(J).

(iii) *If G is a plane embedded graph and C is a nontrivial simple cycle in G, then C is a Jordan curve in $\mathbb{R}^2$.* ▮

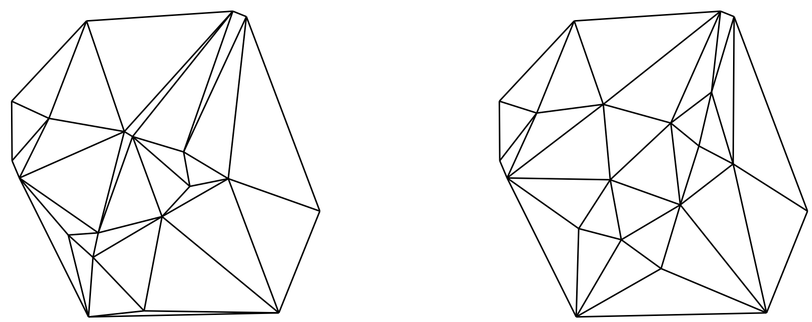

Figure 1: Delaunay triangulation of 20 points and barycentric embedding of the same graph with the same bounding polygon.

(1.9) Edges inside and outside Jordan curves. If J is a Jordan curve and $e = \{u, v\}$ an edge of an embedded graph, where e doesn't meet J except perhaps at u or v, then the relative interior of e (Definition 1.5) satisfies

$$\text{interior}(e) \subseteq \text{interior}(C) \quad \text{or} \quad \text{interior}(e) \subseteq \text{exterior}(C).$$

In this case we say e is inside or outside J as appropriate. We need a certain refinement of the Jordan curve theorem:

(1.10) Proposition (Jordan-Schönflies Theorem). *Let D^1 be the unit disc in $\mathbb{R}^2$ and $S^1 = \partial D^1$, the unit circle. Then if J is a Jordan curve (a homeomorphic image of ∂D^1), the homeomorphism of ∂D^1 extends to a homeomorphism between D^1 and* $\overline{\text{interior}(J)}$.

More generally, if J and J′ are two Jordan curves then the homeomorphism between J and J′ extends to a homeomorphism between $\mathbb{R}^2$ and itself taking interior(J) *to* interior(J') *and* exterior(J) *to* exterior(J'). *(See [7].)* ∎

(1.11) Convex sets in the plane. We note the basic definitions and results (see [1]). A set A is *convex* if for any two points $a, b \in A$, the line-segment ab is entirely contained in A. Suppose S is a finite set of points in the plane. The *convex hull* $H(S)$ is the smallest convex set containing S, that is, the intersection of all convex sets containing S. It is also the intersection of all closed half-planes containing S. Either $H(S)$ is empty, or a point, or a line-segment, or it is bounded by a convex polygon whose corners are in S. In the latter case $H(S)$ is the intersection of those closed half-planes containing S whose boundaries contain sides of S.

(1.12) Proposition *If A is convex then its closure $\overline{A}$ is convex.* ∎

(1.13) Definition *A* convex embedding *of a planar graph G is a straight-edge embedding in which all bounded faces are convex, and the outer boundary is a simple polygon.*

Let G be a plane embedded graph whose external boundary is a simple cycle C. Another map f from its vertices to points in the plane is barycentric *if the external vertices are mapped (in cyclic order) to the corners of a convex polygon, and for every internal vertex u, that is, for every vertex $u \notin C$,*

$$f(u) = \frac{1}{k} \sum_{i=1}^{k} f(v_i), \tag{1.14}$$

where $v_i, 1 \leq i \leq k$, are the vertices adjacent to u.

If a barycentric map determines a straight-edge embedding of G then it is called a barycentric embedding.

For example, Figure 1 shows a Delaunay triangulation with 20 vertices, and a barycentric embedding of the same graph.

The definition of convex embedding does not exclude the possibility that several edges on a face boundary be collinear. Tutte's definition of convex embedding [12] requires that the external boundary be a convex polygon, which would rule out most triangulated graphs. Hence we require that it be a simple polygon, though not necessarily convex.

In a barycentric map, every internal vertex is the average, centroid, or barycentre, of its neighbours. Floater [4] generalises this to a *convex combination map* where each vertex is a weighted average of its neighbours, and shows that Tutte's results generalise to convex combination maps of triangulated graphs.

(1.15) Identifying x with $f(x)$. It is natural to identify a plane embedding of a graph with the graph itself. Strictly speaking, there is a graph G and a map of its vertices and edge to points and curves in the plane, but it is generally natural to identify a vertex x with the point $f(x)$ and an edge $e = \{x, y\}$ with the curve $f(e)$ joining $f(x)$ to $f(y)$. Generally we do not distinguish the two. In Lemmas 1.16 and 1.19 the distinction is made because the maps are not necessarily injective.

(1.16) Lemma *If f is a barycentric map taking the external boundary of a connected plane embedded graph G to a convex polygon P, then all vertices and edges are mapped by f onto or inside P.*

Proof. More explicitly, let D be the closed convex set

$$D = \overline{\text{interior}(P)}.$$

We need to show that the image of every edge and vertex is in D.

Since D is convex, it is enough to show that for every vertex u, $f(u) \in D$. External vertices are mapped to corners of P, hence into D.

Suppose there is an internal vertex u such that $f(u) \notin D$. One of the half-planes defining D does not contain $f(u)$. W.l.o.g. the half-plane and D are bounded above by the x-axis. Again w.l.o.g. $f(u)$ has y-coordinate h, where h is maximal.

Since G is connected, there is a path in G from u to an external vertex w. For at least one of the vertices in this path, w.l.o.g. u itself, $f(u)$ has y-coordinate h but $f(v)$ has y-coordinate $< h$, where v is the node following u in the path. Then $f(u)$ has y-coordinate $h > 0$, so u is an internal vertex; for all its neighbours z, $f(z)$ has y-coordinate $\leq h$; and for at least one neighbour v, $f(v)$ has y-coordinate $< h$. This violates Equation 1.14, and f is not barycentric, a contradiction. **Q.E.D.**

(1.17) Lemma *If a barycentric map f is an embedding, then it is convex.*

Sketch proof. (Here we identify vertices x with points $f(x)$, etcetera, as in Paragraph 1.15.)

Otherwise there exists a bounded face F with a concave corner x, meaning that there is an open nonempty line-segment I with $I \backslash \{x\} \subseteq F$ and the endpoints y, z of I in ∂F.

If x is an external vertex then it is a corner of the polygon P: let $D = \overline{\text{interior}(P)}$. Since x is a corner of D, y and z cannot both belong to D, which contradicts Lemma 1.16, since they are on edges of G.

If x is an internal vertex, let L be the line containing I. Since $I \backslash \{x\} \subseteq F$, all neighbours v_j of x are on the same side of L, since a small open strip incident to I near x is interior to F and meets no edge incident to x. Their barycentre is also on that side of L; since $x \in L$, their barycentre differs from x, a contradiction. **Q.E.D.**

(1.18) Matrix defining a barycentric map. Given a plane embedded graph G whose external boundary is a simple cycle C, barycentric maps are easily specified using a matrix A. Suppose that G has m vertices $v_1, \ldots, v_m$, the first n of them belonging to C, the last $m - n$ being internal vertices, and the coordinates of their images are $x_i, y_i, 1 \leq i \leq m$. Any straight-edge embedding is then equivalent to a column vector of height $2m$.

Let A be the $m \times m$ matrix whose first n rows are identical with those of the identity matrix, and whose last $m - n$ rows express the barycentric mapping equations (1.14). Equivalently, for $n < i \leq m$, let

$$a_{ij} = \begin{cases} \deg(v_i) \quad \text{if } i = j, \\ -1 \quad \text{if } v_j \text{ is a neighbour of } v_i, \\ 0 \quad \text{otherwise.} \end{cases}$$

Equation 1.14 can be written in the form

$$\sum a_{ij} x_j = 0 \quad \text{and} \quad \sum a_{ij} y_j = 0.$$

For any barycentric map f, let B_x be the column vector of height m whose first n entries give the x-coordinates of the corners of P and whose other entries are zero; similarly let B_y specify the y-coordinates. Then f is equivalent to column vectors X and Y satisfying

$$AX = B_x; \quad AY = B_y.$$

(1.19) Lemma *If G is connected then the above matrix A is invertible.*

If G is a connected plane embedded graph whose external boundary is a simple cycle, and whose external vertices are mapped in cyclic order to the corners of a convex polygon, then this map extends to a unique barycentric map of G.

Proof. According to Tutte [13], the determinant of the matrix A is the number of spanning trees of a connected graph related to G [2], hence A is invertible. White [14] gives an elementary, and very elegant, argument that A is invertible. His argument may be paraphrased as follows. Suppose $X = [x_i](1 \leq i \leq m)$ and $AX = O$. We can interpret X as a map taking vertices to points on the real line $\mathbb{R}$, not of course an embedding. All external vertices map to 0. If not all x_i are zero, we can follow a path from an external vertex v_k to an internal vertex v_i where $|x_i|$ is maximal. Note $x_k = 0$. Among all the paths from v_k to v_i, choose one with as few vertices as possible.

For all neighbours v_j of v_i, $|x_j| \leq |x_i|$, and for at least one neighbour v_j — the second-last vertex on the path — $|x_j| < |x_i|$. Then

$$|x_i| > \frac{1}{\deg(v_i)} \sum |x_j|$$

where the sum is over all neighbours v_j of v_i, so

$$\deg(v_i)x_i - \sum x_j \neq 0,$$

contradicting one of the equations $\sum a_{ij}x_j = 0$.

Hence $X = O$, the nullspace of A is trivial, and A is invertible.

Since A is invertible, existence and uniqueness of the barycentric map is immediate. **Q.E.D.**

(1.20) Lemma *If G is a connected plane embedded graph bounded by a simple cycle C and f is a barycentric map taking C onto a convex polygon P, then* (i) *every internal vertex is a convex combination of corners of P.*

Moreover, (ii) *the internal vertices are all in the convex hull of those corners of P which correspond to nodes (vertices of degree $\neq 2$) belonging to C.*

Proof. (i) Since the bottom $m - n$ entries of B_x and B_y are zero, $A^{-1}B_x$ and $A^{-1}B_y$ are linear combinations of the x- and y- coordinates of corners of P.

The coefficients are the same in the x- and y-coordinates, so if one of these linear combinations is not convex then there exists an internal vertex x such that $f(x)$ is not on or inside the polygon P, contradicting Lemma 1.16.

(ii) Suppose z is a vertex on C which is not a node, so it is adjacent to no internal vertex in G. Consequently the equations 1.14 never mention z and in the matrix A the column corresponding to z, the j-th column, say, has 1 in the j-th position and 0 everywhere else. That is, the j-th column of A equals the j-th column, call it I_j, of the $m \times m$ identity matrix. Therefore $AI_j = I_j$. Therefore $A^{-1}I_j = I_j$. That is, the j-th column of A^{-1} has zeroes in the bottom $m - n$ rows, so the bottom $m-n$ entries of $A^{-1}B_x$ and $A^{-1}B_y$ do not involve the j-th elements of B_x and B_y, that is, all internal vertices are convex combinations of polygon corners not including $f(z)$. **Q.E.D.**

(1.21) Corollary *Let G be a connected plane embedded graph whose external boundary is a simple cycle C, and let f be a barycentric map mapping C onto the boundary of a convex polygon P.*

For any internal vertex x, let E be the set of paths in G beginning at x whose last vertices may belong to C but whose other vertices are internal.

Let R be the set of vertices in C which are endpoints of paths in E. Then $f(x)$ is a convex combination of the corners

$$\{f(y) : \; y \in R\}.$$

Proof. Let H be the subgraph of G containing all vertices and edges in all paths in E.

If $y \in H$ then y is the endpoint of a path in E, and by definition of H every vertex and edge on this path is in H. Therefore H is connected.

Clearly $R = H \cap C$. Since G is connected, there is a path from x to some vertex in C. The shortest initial segment reaching C is a path in E which ends in $R = H \cap C$, so $H \cap C$ is nonempty. Also C is connected. Therefore $H \cup C$ is connected. Call this graph G'. It is a connected plane embedded subgraph of G whose external boundary is C. Thus its internal vertices are those in $H \backslash C$.

Let f' be the restriction of the map f to the vertices in G'.

If y is an internal vertex in G' then it is an internal vertex in H and there is a path from x to y in E, and it does not meet C. If z is a neighbour of y in G, then this path can be extended to a path from

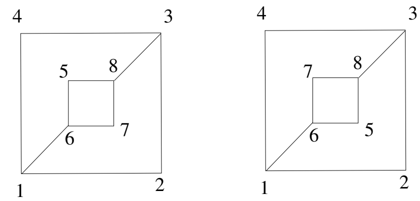

Figure 2: a graph with different plane embeddings. Also, the barycentric map is not an embedding.

x to z in E, so $z \in G'$. Therefore every neighbour of y in G is a neighbour in G'. The converse holds since G' is a subgraph, and for every internal vertex y in G',

$$f'(y) = \frac{1}{\ell} \sum f'(y_j) \tag{1.22}$$

where y_j are the ℓ neighbours of y in G'. That is, f' is a (the unique) barycentric map of G' taking C onto P.

The nodes in $C \cap G'$ are precisely the vertices in R. Therefore for every internal node y of G', $f'(y)$ is a convex combination of $\{f'(z) : z \in R\}$ (Lemma 1.20). Since f' is the restriction of f to G', $f(y)$ is a convex combination of $\{f(z) : z \in R\}$. **Q.E.D.**

(1.23) Definition *A graph G is* nodally 3-connected *if it is biconnected and for every two subgraphs H and K of G, if $G = H \cup K$ and $H \cap K$ consists of just two vertices (and no edges), then H or K is a simple path.*

(1.24) Proposition *Every triconnected graph is nodally 3-connected, and every nodally 3-connected graph with no vertices of degree 2 is triconnected. (Proof omitted.)* ▮

We depend heavily on Tutte's 1963 paper [13]. His paper is concerned directly with the graph rather than a particular embedding of the graph. Hence the notion of a peripheral polygon, which is any cycle of the graph which can occur as the boundary of a face in some plane embedding.

(1.25) Definition *A* periperhal polygon *in a connected graph G is a simple cycle C such that $G \backslash C$ is connected.*

(1.26) Proposition (Tutte [13] (1963)). *If G is a nodally 3-connected planar graph and C is a peripheral polygon, and the vertices of C are mapped (in cyclic order) onto the corners of a convex polygon P, then that map extends to a unique barycentric map which is a convex, straight-edge embedding of G.* ▮

It is easy to give a counterexample when G is not nodally 3-connected. For example, in Figure 2, any barycentric map must map the inner square face to a line-segment. The figure illustrates different plane embeddings of the same graph, which is not nodally 3-connected.

Theorem 1.33 below shows that, except regarding the external face, a *planar* graph is nodally 3-connected if and only if barycentric maps are plane embeddings.

Lemmas 1.27 and 1.29 below are fairly obvious and well-known, but still worth mentioning.

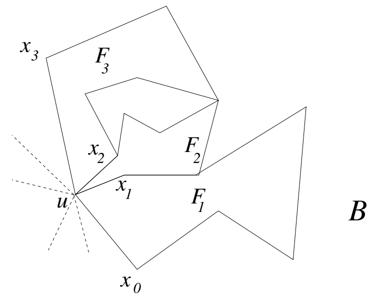

Figure 3: neighbours of u connected by paths avoiding u.

(1.27) Lemma *A plane (straight-line) embedded graph G is connected if and only if for every face F, ∂F is (path-)connected.*

Sketch proof. If G is disconnected, then there exists a Jordan curve J intersecting no edge of G and separating two subgraphs H and H'. Suppose H is inside J and H' outside. The curve J is entirely contained in a face F of G. Take any path in the plane from H to H'. It begins at a point $x \in H$, say, meets J first at a point y_0, say, leaves J for the last time at a point y_1, say (probably $y_0 = y_1$), and ends at a point $z \in H'$. Note $x \notin F$, $J \subseteq F$, and $y \notin F$. Between x and y_0 it must cross ∂F inside J, and between y_1 and z it must cross ∂F outside J. Therefore ∂F intersects the inside and outside of J but not J itself, and is not path-connected.

Conversely, if ∂F is disconnected then there is a Jordan curve separating parts of it, and this curve separates subgraphs of G. **Q.E.D.**

(1.28) Lemma *Let G be a straight-edge embedded plane graph in which all face boundaries are simple cycles, and u any vertex of G.*

Let $x_0, \ldots, x_k$ be a list of neighbours of u consecutive in anticlockwise order; possibly $x_0 = x_k$ but otherwise they are distinct. For $1 \leq j \leq k$ let F_j be the face occurring between the edges (line-segments) ux_{j-1} and ux_j in the anticlockwise sense. (The faces F_j are not necessarily distinct.)

Let B be the subgraph formed by the edges and vertices in $\bigcup_j \partial F_j$.

Then any two vertices in the list x_j are joined by a path in $B \backslash u$. See Figure 3.

Proof. $B \backslash u$ is also the subgraph consisting of all vertices and edges in $\bigcup_j (\partial F_j \backslash u)$. Since each face is a simple cycle, $\partial F_j \backslash u$ is a path joining x_{j-1} to x_j. Thus $B \backslash u$ contains paths joining all these vertices x_j. **Q.E.D.**

(1.29) Lemma *A plane straight-edge embedded graph G is biconnected if and only if the graph consists of a single vertex or a single edge or the boundary of every face is a simple cycle.*

Sketch proof. (i): If. A single vertex or edge is biconnected, so we assume that the boundary of every face is a simple cycle. G is connected (Lemma 1.27).

For any vertex x and all neighbours x_j of x there exist paths connecting these neighbours which avoid x (Lemma 1.28). Therefore all these neighbours are in the same component of $G \backslash x$, and it follows that $G \backslash x$ is connected. Hence G is biconnected.

(ii): Only if. Suppose that G is connected, not a single vertex or edge, and there exists a face F whose boundary is not a simple cycle (graph): ∂F is connected but contains a node x whose degree (in ∂F, not in G) differs from 2.

If the degree of x in F is zero, x is an isolated vertex and G is disconnected. If the degree 1 then x would have only one incident edge xy in G, and since G is not a single edge, $G \backslash y$ would be disconnected and G not biconnected. We may assume that the degree of x in F is at least 3.

Letting $x_1, \ldots, x_k$ be the neighbours of x in anticlockwise order around x, at least three of these are in ∂F, hence there exist $1 \leq i < j \leq k$ such that x_{i-1}, x_i, x_{j-1}, and x_j all belong to F (interpret x_0 as x_k and x_{k+1} as x_1). Take a point u close to x in the triangle $x_{i-1}xx_i$ and a point v close to x in the triangle x_jxx_{j+1}. Both of these triangles are part of F so there is a polygonal path connecting them inside F. This path can be made disjoint from the line-segments ux and vx, so we get a polygonal Jordan curve J meeting G at x alone and otherwise contained in F.

In a sufficiently small neighbourhood of x, the inside and outside of J are on opposite sides of the polygonal path uxv, and the line-segments $x_{i-1}x$ and x_ix meet x, and J, from opposite sides. Since J meets G only at x, it follows that x_{i-1} is inside J and x_i outside, or vice-versa. Any path in G joining these two vertices must intersect the Jordan curve J, hence must pass through x; $G \backslash x$ is disconnected and G is not biconnected. **Q.E.D.**

(1.30) Witnesses for a non-nodally 3-connected graph. Suppose G is not nodally 3-connected. We say that H, K, u, v are *witnesses* if $H \cap K$ contains just two vertices u, v and no edge, neither H nor K are path graphs, and neither H nor K equals G.

(1.31) Lemma (i) *Given witnesses H, K, u, v, if L is a path in G connecting $H \backslash K$ to $K \backslash H$, then L contains three consecutive vertices r, s, t where $\{r, s\} \in H$, and $\{s, t\} \in K$, $r \in H \backslash K$, $t \in K \backslash H$, and $s \in H \cap K$, so $s = u$ or $s = v$.*

(ii) *Any path (respectively, cycle) which avoids u and v except perhaps at its endpoints (respectively, perhaps once), is entirely in H or in K.*

Proof. (i) The first vertex in L is in $H \backslash K$, so the first edge is in H. Similarly the last edge is in K. Therefore there exist three consecutive vertices r, s, t on the path where $\{r, s\} \in H$ and $\{s, t\} \in K$. Then $s \in H \cap K$, so $s = u$ or $s = v$ and s is incident to edges from H and from K.

(ii) Now let P be a path which avoids u and v except perhaps at its endpoints. This includes the possibility of a cycle, viewed as a path which begins and ends at the same vertex w: we allow w, but no other vertex on the cycle, to equal u or v.

If the path is not entirely in H or in K, then it contains a triple r, s, t where $s = u$ or $s = v$, a contradiction. **Q.E.D.**

The proof of Theorem 1.33 is long. To lighten it somewhat, we prove

(1.32) Lemma *Let G be a plane embedded graph in which all face boundaries are simple cycles. Then*

(i) *either G is a simple cycle with two faces, or*

(ii) *for no two faces F, F' is $\partial F \cap \partial F'$ a simple cycle, and if there are 3 faces F_1, F_2, F_3 such that*

$$Q_1 = \partial F_1 \cap \partial F_2, Q_2 = \partial F_2 \cap \partial F_3, \quad \textit{and} \quad Q_3 = \partial F_3 \cap \partial F_1$$

are all nonempty and connected, therefore simple paths, and they all join the same two vertices u and v, then there are exactly three faces, and G consists of two nodes connected by three paths.

Proof. Since all face boundaries are simple cycles, G is biconnected, hence connected.

(i) Suppose $\partial F \cap \partial F' = \partial F$, that is $\partial F \cap \partial F'$ is a Jordan curve J. By Theorem 1.8 (ii), F is the inside of J and F' the outside or vice-versa, so G is a simple cycle with two faces.

(ii) W.l.o.g. F_1 and F_2 are bounded. Their intersection Q_1 is a simple path, which means that the union of their closures is simply-connected. Its boundary is $Q_2 \cup Q_3 = \partial F_3$. F_3 is either the inside or outside of ∂F_3 (Theorem 1.8), but $F_1 \cup F_2$ are inside, so it is the outside, and F_3 is the unbounded face. Thus there are three faces and G is the union of three paths $Q_1 \cup Q_2 \cup Q_3$ with two nodes in common. **Q.E.D.**

(1.33) Theorem *A plane (straight-edge) embedded graph is nodally 3-connected iff it is biconnected and the intersection of any two face boundaries is connected.*

Proof. We can assume G is biconnected, since that is required for nodal 3-connectivity. Since G is biconnected either it is empty or trivial, or a single edge, or every face is bounded by a simple cycle. In the first three cases the graph is obviously nodally 3-connected and biconnected with one face, so we need only consider the fourth case and can assume that every face is bounded by a simple cycle.

We can assume that G is straight-edge embedded. Therefore the boundary of every face is a simple polygon.

Only if: Suppose F_1 and F_2 are different faces and $\partial F_1 \cap \partial F_2$ is disconnected. R.T.P. G is not nodally 3-connected.

Let u and v be vertices in different components of $\partial F_1 \cap \partial F_2$. For $i = 1, 2$ there are two paths P_i and Q_i joining u to v in ∂F_i. These paths are polygonal.

One can also construct a path P_1' within F_1, loosely speaking by displacing P_1 slightly into F_1, and connecting its endpoints to u and v. The resulting path is in F_1 except at its endpoints. Similarly one can construct a path P_2' in F_2 except at its endpoints. These paths together form a (polygonal) Jordan curve J which meets G only at u and v. By construction, $P_1 \cup P_2$ is inside J and $Q_1 \cup Q_2$ is outside J.

Let H (respectively, K) be the subgraph consisting of all vertices and edges of G which lie inside or on J (respectively, outside or on J). The only vertices in $H \cap K$ are u and v, and $H \cap K$ contains no edge. H contains $P_1 \cup P_2$ and therefore is not a path graph, since otherwise $P_1 = P_2$ and u and v would be in the same component of $\partial F_1 \cap \partial F_2$. Similarly K is not a path graph. Therefore G is not nodally 3-connected.

If: Suppose G is biconnected but not nodally 3-connected, and H, K, u, v are witnesses. G has more than one face, so all face boundaries are simple cycles.

Claim 1. The subgraphs $H \backslash K$ and $K \backslash H$ are nonempty. If every vertex in K were also in H, then the vertices in K are in $H \cap K$, that is, u and v. Either K has no edges, in which case $H = G$, or it has the edge $\{u, v\}$ and is a path graph. Neither is possible. Therefore $H \backslash K$ and similarly $K \backslash H$ are nonempty.

Claim 2. Neither u nor v are isolated vertices in H or in K.

Otherwise suppose u is isolated in K. Let L be any path joining $H \backslash K$ to $K \backslash H$. By Lemma 1.31, every path connecting $H \backslash K$ to $K \backslash H$ contains a vertex, u or v, incident to edges from H and from K. By hypothesis, u is not; so every such path contains v. By Claim 1, at least one such path exists, so $G \backslash v$ is not connected, and G is not biconnected.

Claim 3. Both u and v have neighbours both in $H\backslash K$ and in $K\backslash H$. Suppose all neighbours of u are in H. Since u is not isolated in K, there is an edge $\{u, t\}$ in K incident to u. But t is a neighbour of u, therefore $t \in H \cap K$, so $t = v$. The only edge in K incident to u is $\{u, v\}$.

Consider a path in G joining $H\backslash K$ to $K\backslash H$. Let t be the first vertex where the path meets $K\backslash H$, and let s be the vertex before t on the path. Since $\{s, t\} \in K$ and $s \notin K\backslash H$, $s \in H \cap K$: $s = u$ or $s = v$. However, if $s = u$, then, since $t \in K$, $t = v$ and $t \notin K\backslash H$. Therefore $s = v$. This implies that every path from $H\backslash K$ to $K\backslash H$ contains v. Again by Claim 1, G is not biconnected.

This contradiction shows that not all neighbours of u are in H; neither are they in K, and the same goes for v.

Claim 4. The vertices u and v share a face in common. Otherwise let $x_1, \ldots, x_k$ be the neighbours of u. We know (Lemma 1.28) that they are all connected by paths in $B\backslash u$, where B is the union of boundaries of bounded faces incident to u. Assuming v is incident to none of these faces, these paths would also avoid v. This implies that all neighbours of u are in H or in K, contradicting Claim 3.

Claim 5. The vertices u and v have at least two faces in common. Let F_j be the faces incident to u in anticlockwise order around u. At least one of these faces is incident to u and to v. Suppose F_j is, and no other face is.

There are two cases. If u or v, w.l.o.g. u, is an internal vertex, then all faces incident to u are bounded, and by Lemma 1.28, the subgraph $\bigcup_{i \neq j}(\partial F_i \backslash u)$ would be connected and contain neither u nor v. Then all vertices in this subgraph would belong to H or to K. Since it includes all neighbours of u in G, it would contradict Claim 3.

If both u and v are external vertices, then the face common to both of them is the external face, and all bounded faces incident to u avoid v. This time we consider the subgraph $\bigcup_j(\partial F_j \backslash u)$ where F_j are the bounded faces incident to u. Again this is a connected subgraph containing all neighbours of u in G, and again it omits both u and v, so again all vertices in it are in H or in K, and again Claim 3 is contradicted.

Therefore u and v have at least two faces F and F' in common.

Claim 6. If u and v are incident to three faces F_1, F_2, and F_3, then the boundaries of at least two of these faces have disconnected intersection.

First note in general that if two faces of any plane graph have connected nonempty intersection, then the intersection is either a simple path or a cycle. The latter is possible only when the graph G is a simple cycle with just two faces.

Let

$$Q_1 = \partial F_1 \cap \partial F_2, \quad Q_2 = \partial F_2 \cap \partial F_3, \quad Q_3 = \partial F_3 \cap \partial F_1.$$

Suppose all three intersections are connected, so they are all simple paths. Any edge on Q_1 is incident to F_1 and F_2 alone; this goes also for internal vertices on Q_1. Similarly for Q_2 and Q_3. Hence the only vertices common to more than one path are u and v.

So the union of any two is a Jordan curve meeting the third only at u and v. W.l.o.g. Q_1 is inside $Q_2 \cup Q_3$.

Q_2 and Q_3 are paths joining u to v in ∂F_3. All edges on these paths are incident to F_3. Together they form a polygon contained in the polygon ∂F_3, hence $Q_2 \cup Q_3 = \partial F_3$. This implies that F_3 surrounds the other faces, so F_3 is the external face.

Put another way, $\partial F_3 \subseteq \partial F_1 \cup \partial F_2$. F_1 and F_2 are the only faces incident to F_3. Similarly for the other two faces, and we conclude that there are only three faces and $G = Q_1 \cup Q_2 \cup Q_3$. Either each Q_j is contained in the same subgraph, w.l.o.g. H, and $H = G$, or one subgraph is the union of two of the paths and the other, w.l.o.g. H, is the third path, and H is a path graph. These contradictions show that the one of the pairs $\partial F_i \cap \partial F_j$ is disconnected, as required for Claim 6.

Claim 7. If there are exactly two faces F and F' incident to u and to v, then $F \cap F'$ is disconnected.

Otherwise $F \cap F'$ is a path Q' joining a vertex u' to another vertex v' and containing a subpath Q joining u to v. Not all of u', u, v, v' need be distinct, but it is assumed that they occur in that order in Q'.

By Lemma 1.31, all vertices in Q belong to H or to K: w.l.o.g. to H. The boundary cycles ∂F and $\partial F'$ include two other paths joining u' to v' and forming a Jordan curve J.

If $u' \neq u$ then J meets $H \cap K$ at v alone, or not at all, and by Lemma 1.31, all vertices in J, plus those in $Q' \backslash Q$, belong to H or to K.

If all vertices in J belong to H, then all vertices outside J also belong to H, because for any vertex y outside J, one can choose a shortest path joining y to a vertex in J. Neither u nor v occur as internal vertices on this path, so all vertices on the path are in H or K (Lemma 1.31), H since the last vertex is in H.

We have counted all vertices in G: those outside J, those on J, and those on Q', and all are in H, so $H = G$, which is false.

On the other hand, if all vertices in J, and in $Q' \backslash Q$, belong to K, then all vertices outside J belong to K, and $H = Q$ is a path graph, which is false. This proves Claim 7 in the case $u \neq u'$, and by symmetry where $v \neq v'$.

If $u = u'$ and $v = v'$ then $Q = Q'$: let Q_1 and Q_2 be the other subpaths joining u to v in ∂F and $\partial F'$ respectively. By Lemma 1.31, each subpath Q_i is contained in H or in K. Again we have a Jordan curve J; now $J = Q_1 \cup Q_2$.

If u is an internal vertex, then F and F' are bounded faces incident to u, and since $\partial F \cap \partial F' = Q$, they are consecutive in cyclic order. By Lemma 1.28 the neighbour of u in Q_1 connected to the neighbour of u in Q_2 by a path which avoids u and v. By Lemma 1.31, both neighbours are from H or from K. Therefore all vertices in J are contained in H or all are contained in K, and, as previously, the same goes for all vertices outside J, so either $H = G$ or $H = Q$ is a path graph.

Similarly if v is an internal vertex.

This leaves the case where u and v are external vertices with exactly two faces in common, F and F', whose boundaries have connected intersection Q, a path joining u to v. In this case F or F', F', say, is the external face.

If no faces other than F and F' are incident to u, then u has degree 2 in G and also $\partial F \cap \partial F'$ contains both neighbours of u. This is inconsistent with $\partial F \cap \partial F'$ being a path joining u to v. Therefore u is incident to another bounded face; likewise v.

The definitions of Q, Q_1, Q_2 are still in force. If Q_1 (respectively, Q_2) is a single edge then all vertices in J are in Q_2 (respectively, Q_1). Otherwise u has a neighbour v_1 in Q_1 different from v and a neighbour v_2 in Q_2 different from v. The set of faces incident to u and excluding F and F' forms a connected subgraph B, and $B \backslash u$ contains a path from v_1 to v_2 which avoids u and v (Lemma 1.28). In any case, all vertices on J belong to H or to K. All vertices in Q belong to H. Notice that all vertices of G are on or inside J except for Q, which is part of the external cycle.

Suppose that all vertices on J belong to H. Again we can argue that there all vertices inside J belong to H and $H = G$, a contradiction. Otherwise they all belong to K and the only vertices in H are in Q: $H = Q$ is a path graph, another contradiction.

This finishes the proof of Claim 7. Claims 6 and 7 taken together amount to the desired result. **Q.E.D.**

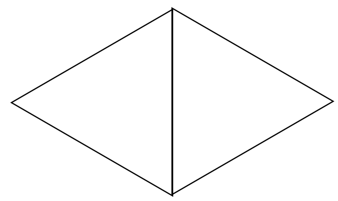

Figure 4: a nodally 3-connected but not triconnected triangulated planar graph

(1.34) Chord-free triangulated graphs. A triangulated plane embedded graph is one in which every bounded face is bounded by three edges. In a triangulated biconnected graph the external boundary is also a simple cycle. It can only fail to be nodally 3-connected if a bounded face meets the external boundary in a disconnected set. This implies that one of its edges is a chord joining two non-consecutive vertices on the external boundary, and the converse holds unless the chord joins the endpoints of two consecutive boundary edges [14].

The graph in Figure 4 is nodally 3-connected but not triconnected.

A fully triangulated planar graph is a triangulated planar graph in which there are three external edges. In other words, the external face also is bounded by a 3-cycle. Therefore the external cycle has no chords, so every fully triangulated planar graph is nodally 3-connected.

Also let G be a fully triangulated planar graph containing a vertex v of degree 2. Let u and w be the neighbours of v. There are only two faces incident to v and they are both incident to $u, v,$ and w. One of them must be the external face. Thus $u, v,$ and w are the three external vertices. They also bound the only bounded face. G is a 3-cycle, and therefore triconnected.

On the other hand, if G is fully triangulated then it is nodally 3-connected, so if it contains no vertex of degree 2 then it is triconnected (Proposition 1.24). Therefore

(1.35) Corollary *Every fully triangulated planar graph is triconnected.* ∎

2 Conditions for a barycentric embedding

This section investigates the general conditions for a barycentric embedding to exist. The notion of inverted subgraph given below is not discussed by Stein [10] but it does figure, in a different form, in Tutte's paper on convex embeddings [12, Theorem I].

(2.1) Definition *Let G be a connected plane (straight-edge) embedded graph whose external boundary C is a simple cycle. Suppose there is an edge $\{x, y\}$ on C and a bounded face F such that x and y are incident to F but the edge (line-segment) xy is not. There is a Jordan curve J containing the line-segment xy such that the other path joining x to y in J is in the face F except at x and y. Then the subgraph consisting of x and y and all edges and vertices of G inside J is called an* inverted subgraph *of G. See Figure 5.*

(2.2) Definition *A plane-embedded graph G is* semi-admissible *if* (i) *it has at least 3 vertices,* (ii) *every face boundary is a simple cycle, and* (iii) *every two face boundaries have connected intersection.*[2]

It is admissible *if it is semi-admissible and* (iv) *there are no inverted subgraphs.*

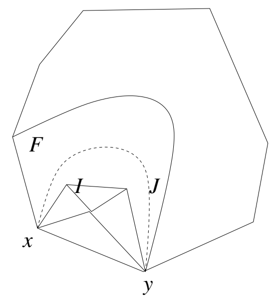

Figure 5: inverted subgraph I.

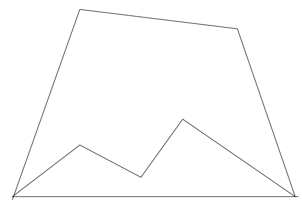

Figure 6: the topologically unique nodally 3-connected plane embedded graph without a barycentric embedding.

(2.3) Lemma *Let G be a convex plane-embedded graph. Then G is admissible.*

Proof. (i) The external boundary is a simple polygon with at least three vertices.

(ii) The boundary of every bounded face is a convex polygon, hence a simple polygon and a simple cycle. The external boundary is a simple polygon and a simple cycle.

(iii) Let $F_1 \neq F_2$ be bounded faces. Then $\partial F_1 \cap \partial F_2 = \overline{F_1} \cap \overline{F_2}$ is convex, hence connected.

(iv) For an inverted subgraph to exist, there would have to be a bounded face F meeting the external face in two vertices x and y, where xy is an edge of the bounding polygon, but is not incident to F. The closure $\overline{F}$ is closed and convex, and x and y are in $\overline{F}$, so $xy \subseteq \overline{F}$, a contradiction. **Q.E.D.**

The barycentric embedding result in [13] is derived in the following way. It is proved directly for any plane embedding (choice of peripheral polygon, Definition 1.25) of G, if G has at least three nodes (vertices of degree $\neq 2$).

(2.4) Other nodally 3-connected planar graphs. Nodally 3-connected graphs with fewer than 3 nodes can be considered separately. A θ-graph is a graph with two nodes connected by three paths (it resembles the letter θ). The only nodally 3-connected planar graphs with fewer than 3 nodes are single vertices, single edges, simple cycles, and θ-graphs. They all admit barycentric embeddings. However, if one of the paths joining the nodes in a θ-graph is a single edge, then that edge cannot be on the outer boundary since otherwise there would be an inverted subgraph. This is the only example of a nodally 3-connected graph for which a barycentric map is not an embedding (Figure 6).

(2.5) Lemma *If G is a semi-admissible straight-edge embedded plane graph (2.2), then there is another straight-edge embedding which is admissible.*

[2]Equivalently, if F and F' are bounded faces and $\overline{F} \cap \overline{F'} \neq \emptyset$, then $\overline{F \cup F'}$ is a closed disc.

Sketch proof. Suppose that a bounded face F meets two ends of an external edge e without being incident to the edge. Take the subgraph enclosed between e and F and move it to the other side of e. This alters the external boundary and removes the inverted subgraph due to F. Nor can it introduce a new external edge e' forming an inverted subgraph with a bounded face F', for otherwise before the alteration the boundary of the face corresponding to F' would have met ∂F in a disconnected set, violating condition (iii). The procedure can be repeated to remove all inverted subgraphs. **Q.E.D.**

(2.6) Lemma *Let G' be an admissible plane embedded graph (Definition 2.2). Let C be its outer cycle, and $e = \{a, b\}$ be a chord, that is, an edge not on C where a and b are external vertices. The embedded graph $C \cup \{e\}$ is a kind of θ-graph. The vertices a and b are connected by two subpaths P_1, P_2, of C. Let $C_i = P_i \cup \{e\}, i = 1, 2$. Then C_1 and C_2 are simple cycles, $C_1 \cup C_2 = C \cup \{e\}$, and $C_1 \cap C_2$ consists of the single edge e.*

For $i = 1, 2$, let G_i be the subgraph consisting of

- *All vertices of G' on and inside C_i*
- *All edges of G' on and inside C_i (see Paragraph 1.9)*

Then $G' = G_1 \cup G_2$, $G_1 \cap G_2 = \{\{a, b\}, \{e\}\}$, G_i are connected with external boundary C_i, the internal vertices of G' are those of G_1 and of G_2, and the bounded faces of G' are those of G_1 (inside C_1) and G_2 (inside C_2).

Finally, G_1 and G_2 are admissible.

Sketch Proof. Most of what has been said is obvious but tricky to prove, so we shall discuss properties 2.2 (i–iv) for G_1. The same observations apply to G_2.

(i) C_1 has at least 3 vertices: so has G_1.

(ii) The bounded faces of G_1 are those of G' inside C_1, so their boundaries are simple cycles. The boundary of the unbounded face of G_1 is C_1, also a simple cycle.

(iii) If F' and F'' are two bounded faces of G_1, then they are bounded faces of G', so $\partial F' \cap \partial F''$ is connected.

(iv) Suppose G_1 has an inverted subgraph: G_1 has a bounded face F', and an edge e' on C_1, so $\partial F' \cap C_1$ contains both ends of e' but not e' itself.

If $e' \neq e$ then $e' \in C$ and $\partial F' \cap C$ contains both ends of e' but not e' itself — so G' would have an inverted subgraph, a contradiction.

Hence $e' = e$. The edge e is incident to two bounded faces F_1, F_2 of G', where F_2 is inside C_2. Since

$$\overline{F'} \subseteq \overline{\text{interior}(C_1)} \quad \text{and} \quad \overline{F_2} \subseteq \overline{\text{interior}(C_2)},$$

$$\partial F' \cap \partial F_2 \subseteq C_1 \cap C_2 = e.$$

But $\partial F'$ intersects e only at a and b, so $\partial F' \cap \partial F_2$ is disconnected, contradicting condition (iii). **Q.E.D.**

(2.7) Preview of Theorem 2.9. The theorem below, which gives the most general form possible of Tutte's barycentric embedding theorem, proceeds by induction where the base case is Tutte's theorem [13]. Inductively, given a plane embedded graph G with external boundary C, a face F is chosen such that $\partial F \cap C$ is disconnected, with vertices a and b in different components, and dividing G into two graphs separated by the edge $\{a, b\}$. There are complications if $\{a, b\}$ is not already an edge of G. The first is that it must be represented by a plane curve. This is easily done on the assumption that the embedding of G is piecewise linear; then it is possible to join a to b by a polygonal path within F. We can no longer assume that G is embedded with straight edges, but we can assume that the embedding is piecewise linear (i.e., the edges are embedded as polygonal paths), and this is just as convenient.

Another complication is that the divided graphs can violate condition (iii). The facts can be stated in a lemma.

(2.8) Lemma *Let G be an admissible (Definition 2.2) piecewise-linear plane embedded graph. Let C be the external boundary of G, and suppose a and b are vertices in $\partial F \cap C$ where F is a bounded face and $\{a, b\}$ is not an edge on C. If $e = \{a, b\}$ is not already an edge of G then add it, and embed it as a polygonal path within F (thereby dividing F into two faces, F_1 and F_2.) This results in a new graph G''.*

It is then possible that G'' have two bounded faces whose intersection is disconnected, but the only way this can happen is that one of the two subpaths of ∂F joining a to b has inner vertices of degree 2 only. If Q is this path then G'' can be modified by removing the edges and inner vertices of Q. Let G' be the new graph in this case; if Q does not exist, let $G' = G''$, and if $\{a, b\}$ was an edge of G let $G' = G$. Then if $G' \neq G$ it differs from G in that it has a new edge, or the path Q is replaced by the single edge e.

Then G' is admissible.

Proof. If $G' = G$ there is nothing to prove.

Otherwise $G'' \neq G$. In this case, consider how G'' can violate any of conditions (i–iv) of Definition 2.2.

(i) G'' has at least 3 vertices. (ii) Its new bounded faces are F_1 and F_2 whose boundaries are simple cycles, and its external boundary is C, a simple cycle.

(iv) If G'' has inverted subgraphs then a bounded face F' meets both ends of an edge e' on C, (so $e' \neq e$). Suppose e' is not incident to F'. In this case F' cannot be a face of G. W.l.o.g. $F' = F_1$.

The edge e separates interior(C) into two open simply-connected sets whose closures may be denoted D_1 and D_2, where w.l.o.g. D_1 is the region containing F_1. Let $Q_i = \partial F \cap D_i$, $i = 1, 2$. Then

$$\partial F = Q_1 \cup Q_2,$$

the union of two paths disjoint except at a and b.

$$\partial F_1 = Q_1 \cup e,$$

so all vertices on ∂F_1 are on Q_1 and both ends of e' are incident to Q_1, therefore to ∂F, and e' is incident to F. Since e' is not on Q_1, it must be on Q_2, and $Q_2 = ab = e' = e$, a contradiction. Therefore G'' has no inverted subgraphs.

(iii) There may exist bounded faces F' and F'' of G'' such that $\partial F' \cap \partial F''$ is disconnected. They cannot both be faces of G, and $\partial F_1 \cap \partial F_2 = e$, so w.l.o.g. F' is a face of G and $F'' = F_2$.

$$\partial F_1 \cup \partial F_2 = Q_1 \cup e \cup Q_2 = \partial F \cup e.$$

Also, since e is not incident to F',

$$\partial F' \cap \partial F_i = \partial F' \cap Q_i, \qquad (i = 1, 2),$$

and

$$\partial F' \cap \partial F = (\partial F' \cap Q_1) \cup (\partial F' \cap Q_2).$$

Since the left-hand side is connected and $\partial F' \cap Q_2$ is disconnected, $\partial F' \cap Q_1$ must be connected. But $\{a, b\} \subseteq \partial F' \cap Q_1$, so $\partial F' \cap Q_1 = Q_1$. Since Q_1 meets the interior of D_1, so does F': $F' \subseteq D_1$. Since the relative interior of Q_2 is disjoint from D_1, it is disjoint from $\partial F'$, and $\partial F' \cap \partial F = Q_1$. There can be at most one such face F'.

Alternatively, if we had assumed that $F' = F_1$ and F'' is a face of G, we should have concluded that F'' is the unique face such that $\partial F'' \cap F = Q_2$. These two alternatives are incompatible: there cannot be faces F' and F'' of G which are incident to F along Q_1 and Q_2, since then $\partial F' \cap \partial F'' = \{a, b\}$ would be disconnected.

Thus either G'' satisfies conditions (i–iv), or there exists a path Q as predicted ($Q = Q_1$ or $Q = Q_2$.)

In the latter case, G' is obtained from G'' by removing the path, w.l.o.g. Q_1. This merges F' and F_1 into a single face. Note that G was admissible, so the edge e cannot have been an edge of G.

It remains to show that G' is admissible. First consider condition (iv): whether G' has an inverted subgraph.

Write $\partial F' = Q_3 \cup Q_1$, and let $F_3 = F' \cup \text{interior}(Q_1) \cup F_1$, the merged face.

$$\partial F_3 = Q_3 \cup e, \quad \text{so} \quad C \cap \partial F_3 = C \cap \partial F'.$$

The face F_3 cannot give rise to an inverted subgraph in G', since if both ends of an edge $e' \in C$ are incident to ∂F_3, then they are incident to $\partial F'$, so e' is incident to F' and to F_3. All other faces of G' are faces of G'', so G' has no inverted subgraphs.

G and G' have the same vertices, so admissibility condition (i) is obvious. Admissibility conditions (ii,iii) are topologically invariant, so it is enough to show that the embeddings of G and G' are topologically equivalent. Let

$$D = \overline{F' \cup Q \cup F} \supseteq F' \cup Q \cup F_1 \cup e \cup F_2.$$

Since G is admissible, D is a closed disc (conditions ii,iii).

It follows from the Schönflies theorem (1.10) that there is a homeomorphism of $\mathbb{R}^2$ to itself which fixes $\partial D \cup \text{exterior}(\partial D)$ and carries F' onto F_3, F onto F_2, and Q onto e. This homeomorphism carries all vertices, edges, and faces of G onto corresponding vertices, edges, and faces of G', so G' satisfies admissibility conditions (i–iv). **Q.E.D.**

(2.9) Theorem *Let G be an admissible piecewise-linear plane embedded graph. Let f be any mapping of its external cycle C to the corners of a convex polygon (in cyclic order), and let g be the unique barycentric map extending f to G.*

Then g is an embedding.

Proof. The condition on face boundaries implies that G is biconnected. Define the *excess* of G to be the quantity

$$\sum_F (|\text{components of } F \cap C| - 1),$$

where the sum is over all bounded faces F meeting C.

If the excess is 0 then G is nodally 3-connected. By Paragraph 2.4, since G has no inverted subgraphs, the barycentric extension of f is an embedding.

For the inductive step, suppose the result is true for graphs of excess $< s$ and G has excess $s \geq 1$. Choose a bounded face F which meets the bounding cycle C in two or more components, one containing a vertex a and the other containing a vertex b. Construct the graph G' as in Lemma 2.8. The edge $e = ab$ divides C into cycles C_1, C_2 and Lemma 2.6 holds. Let

$$D = \overline{\text{interior}(C)} \quad \text{and} \quad D_i = \overline{\text{interior}(C_i)}, \quad i = 1, 2.$$

The graphs G' and G may be different, so let us write g' for the unique barycentric extension of f to G'.

Consider the subgraphs G_1, G_2, of G', as in Lemma 2.6.

Claim: the excess of G_1 (and of G_2) is less than that of G.

Bounded faces of G_1 are either bounded faces of G (possibly including the face F), or a sub-face F_1 of F bounded by e, or a face $F_3 = F \cup \text{interior}(Q) \cup F_1$.

If $F' \subseteq D_1$ is a bounded face of G different from F, then $C \cap \partial F' \subseteq C \cap C_1$, a path joining a to b, and

$$C \cap \partial F' = C_1 \cap \partial F',$$

so these sets have the same number of components.

If e is an edge of G then $F \subseteq D_1$ is a bounded face of G_1, and

$$C_1 \cap \partial F = (C \cap \partial F) \cup e.$$

But e meets two components of $C \cap \partial F$, so the excess of G_1 is less than that of G in this case.

In this case the excess of G_2 is also less than that of G, since every face of G_2 is a face of G, and F is not a face of G_2.

If e is not an edge of G then F is split into two faces F_1 and F_2, and every component of $C_1 \cap \partial F_1$ disjoint from e is a component of $C \cap \partial F$. Again, two components of $C \cap \partial F$ meet the component containing e in $C_1 \cap \partial F_1$, so this has fewer components that $C \cap \partial G$, and in case also the excess of G_1 is less than that of G. By symmetry, the same holds for G_2 in this case, proving the claim.

A vertex of G, G_1, or G_2, is an inner vertex if and only if it is in $\text{interior}(C), \text{interior}(C_1)$, or $\text{interior}(C_2)$.

Let v be an inner vertex of G_1. Since G_1 is a subgraph of G, every edge of G_1 incident to v is an edge of G. Let e' be an edge of G incident to v. Then

$$\text{interior}(e') \subseteq \text{interior}(C), \quad \text{interior}(e') \cap e = \emptyset, \quad \text{and } \; e' \cap \text{interior}(C_1) \neq \emptyset,$$

so $\text{interior}(e') \subseteq \text{interior}(C_1)$ and $e' \in G_1$.

Thus the neighbours of v in G_1 are its neighbours in G. Let f_i, $i = 1, 2$, be the restrictions of f to the vertices in C_i.

Since G_1 is connected with external boundary C_1 (Lemma 2.6), there is a unique barycentric extension g_1 of f_1 to G_1. Since the set of neighbours of inner vertices in G_1 is the same in G_1 as in G, g_1 and g satisfy the same equations for vertices in G_1, so g_1 is the restriction of g' to G_1.

By induction, g_1 is an embedding of G_1 into D_1. Thus the restriction of g' to G_1 is an embedding. Similarly the restriction of g' to G_2 is an embedding. If g' were not an embedding then there would be an edge of G_1 which intersects the relative interior of an edge of G_2, or vice-versa. Such an intersection can only occur in $D_1 \cap D_2$. But $D_1 \cap D_2 = e$, so no such intersection can occur.

If $G' \neq G$ then e is not an edge of G. Either G' consists of G with one more edge e, joining two external vertices, so g' and g are defined for the same vertices and g is an embedding of G, or G' consists of G with a path Q replaced by a single edge e. In this case g would embed Q in a straight line from a to b, so its image as a point-set coincides with $g'(e)$, and again g is an embedding. **Q.E.D.**

(2.10) Corollary *An embedded graph admits a convex embedding iff it admits a barycentric embedding.* ∎

3 Ambient isotopy

Stein [10] mentions that different embeddings of a semi-admissible[3] plane embedded graph are ambient isotopic, but does not prove it. Semi-admissibility and admissibility differ in the matter of inverted subgraphs (see also [12]). Stein's paper is concerned with when a planar embedding exists in which the bounded faces are convex (with polygonal boundary). It does not stipulate a straight-edge embedding. He disposes of the difficulty of inverted subgraphs, multiple edges, etcetera, by allowing edges to be subdivided with new vertices.

Thus Stein's result is about piecewise-linear convex embeddings of graphs. In fact, to insist on condition (iv) of Definition 2.2 would hamper the isotopy proof, so in this section we discuss *semi-admissible* graphs.

(3.1) Definition *Given topological spaces X and Y, an* isotopy *is a continuous map $h : [0,1] \times X \to Y$ such that for each t, $0 \le t \le 1$, the map $h_t : X \to Y;\ \ x \mapsto h(t,x)$ is a homeomorphism.*

This section gives an outline proof of the following isotopy theorem (Corollary 3.6). Let G^1 and G^2 be two semi-admissible plane embeddings of the same graph G, and such that their external boundaries are images of the same cycle C of G, with the same orientation. Then there exists an isotopy: $\mathbb{R}^2 \to \mathbb{R}^2$ taking the vertices, edges, and faces of G^1 to those of G^2.

(3.2) Proposition *Suppose G is a semi-admissible plane embedded graph (Definition 2.2,* (i–iii)*). Then either G has one bounded face or there exist two bounded faces F' and F'' such that $\partial F' \cap \partial F'' = Q$ is nonempty (and connected), and if $F = F' \cup \text{interior}(Q) \cup F''$, then for every other face A of G, $\partial A \cap \partial F$ is connected.*

Furthermore, if G' is the embedded graph obtained by removing the edges and inner vertices on Q, hence merging F' and F'' into a single face F, then G' is semi-admissible with the same external boundary as G. (The first part was proved in [10], and the rest follow immediately.) ∎

(3.3) Definition *Let G^1 and G^2 be two plane embedded graphs. The embeddings are* ambient homeomorphic *(respectively,* ambient isotopic*) if there is a homeomorphism (respectively, an isotopy) from $\mathbb{R}^2$ to itself taking the vertices, edges, and faces of G^1 bijectively onto those of G^2.*

(3.4) Lemma *If G^1 and G^2 are plane embeddings of a θ-graph G (Paragraph 2.4), then they are ambient homeomorphic. (Follows from the Schönflies theorem 1.10: proof omitted.)* ∎

[3] The term 'semi-admissible' is not used in [10].

(3.5) Corollary *If G^1 and G^2 be two semi-admissible embeddings of the same graph G with the same boundary cycle, then they are ambient homeomorphic.*

Proof. This is a simple application of Stein's result (Lemma 3.2), and is by induction on the number of bounded faces. If G is a simple cycle then this is just the Schönflies Theorem (Proposition 1.10).

For the inductive step, choose faces F' and F'' of G^1 separated by a path Q such that $F = F' \cup \text{interior}(Q) \cup F''$ has the properties stated in Lemma 3.2. Let H be the subgraph of G obtained by removing the edges and inner vertices of Q, and let H^1 be the modified embedding where F' and F'' are merged into F. Then H^1 is a semi-admissible embedding of H. Similarly a modified embedding H^2 is obtained from G^2. By induction, H^1 and H^2 are ambient homeomorphic through a homeomorphism h'. Let D^1 and D^2 be the images of $\overline{F}$ under the respective embeddings. $D^2 = h'(D^1)$. They contain images Q^1 and Q^2 of the path Q.

By Lemma 3.4, there exists a homeomorphism $h : D^1 \to D^2$ which agrees with h' on ∂D^1 and takes $(F')^1$ to $(F')^2$, $(F'')^1$ to $(F'')^2$, and Q^1 to Q^2, and also takes the vertices and edges in Q^1 to those in Q^2. Extend h to $\mathbb{R}^2$ by making it coincide with h' outside $(\partial F)^1$. Then h is an ambient homeomorphism between G^1 and G^2. **Q.E.D.**

(3.6) Corollary *If G^1 and G^2 are semi-admissible embeddings of the same graph with the same external boundary in the same anticlockwise order C^1 and C^2, then the embeddings are connected by an isotopy.*

Sketch proof. There is an ambient homeomorphism h connecting them (Corollary 3.5). According to [11], h is isotopic to the identity or to reflection in the x-axis. Furthermore, if h preserves the orientation of any Jordan curve, as it does in this case, it is isotopic to the identity. This yields an isotopy carrying G^2 to G^1. ∎

Specialising G^1 to G itself, an admissible embedding, and G^2 to a barycentric embedding, we get

(3.7) Corollary *Let G be an admissible embedding of a planar graph, and suppose f is a map taking its external vertices to the corners of a convex polygon, in the same anticlockwise order. Then G is isotopic to the barycentric extension of f.* ∎

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5 References

1. Arne Brøndstred (1983). *An introduction to convex polytopes.* Springer Graduate Texts in Mathematics 90.

2. R.L. Brooks, C.A.B. Smith, A.H. Stone, and W.T. Tutte (1940). The dissection of rectangles into squares. *Duke Mathematical Journal* **7**, 312–340.

3. H. De Fraysseix, J. Pach, and R. Pollack (1990). How to draw a planar graph on a grid. *Combinatorica* **10:1**, 41–51.

4. Michael Floater (2003). One-to-one piecewise linear mappings over triangulations. *Math. Comp.* **72**, 685-696.

5. Marvin Greenberg (1967). *Lectures on Algebraic Topology.* Benjamin, New York.

6. Goos Kant (1993). Algorithms for Drawing Planar Graphs. Ph.D. Dissertation, Computer Science Department, University of Utrecht.

7. E.E. Moise (1977). *Geometric topology in dimensions 2 and 3.* Springer Graduate Texts in Mathematics 47.

8. Colm Ó Dúnlaing (1994). A simple linear-time planar layout algorithm with a LEDA implementation. Report ALCOM-II-429, also published as report TCDMATH 98–06.

9. Ronald C. Read (1987). A new method for drawing a planar graph given the cyclic order of the edges at each vertex. *Congressus Numerantii* **56**, 31–44.

10. S.K. Stein (1951). Convex Maps. *Proc. American Math. Soc.* **2**, 464–466.

11. John Stillwell (1980). *Classical topology and combinatorial group theory.* Springer Graduate Texts in Mathematics 72.

12. W.T. Tutte (1960). Convex representations of graphs. *Proc. London Math. Soc. (3)* **10**, 304–320.

13. W.T. Tutte (1963). How to draw a graph. *Proc. London Math. Soc. (3)* **13**, 743–768.

14. Geoffrey White (2004). Mesh parametrization for texture mapping. Undergraduate computer science project, Oxford University.