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A zero-mode quantization of the Skyrmion

Conor Houghton and Shane Magee,
School of Mathematics, Trinity College Dublin, Dublin 2, Ireland
Email: houghton@maths.tcd.ie, magees@maths.tcd.ie

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Abstract

In the semi-classical approach to the Skyrme model, nuclei are approximated by quantum mechanical states on a finite-dimensional space of field configurations; in zero-mode quantization this space is generated by rotations and isorotations. Here, simulated annealing is used to find the axially symmetric Skyrme configuration which extremizes the zero-mode quantized energy for the nucleon.

1 Introduction

The Skyrme model is an effective theory of pions and nucleons. It is a non-linear field theory in which nuclei correspond, classically, to topological soliton solutions, called Skyrmions. The model is non-renormalizable and so the approach usually taken is to reduce to a finite-dimensional space of Skyrme configurations before quantizing. In this approach, quantum-mechanical states on a space of topological charge B Skyrme configurations model baryon number B nuclei at low energies.

The Skyrme Lagrangian is acted on invariantly by rotations and isorotations; this generates from a single Skyrme configuration a space of energy-degenerate configurations. This space is known as the space of zero modes and quantization on this space is called zero-mode quantization. This approach began with the seminal paper [1] where the nucleon and delta are constructed as quantum mechanical states on the zero-mode space generated from the classical minimum energy configuration of unit baryon number.

In this paper, the spin-half, isospin-half quantum Hamiltonian is calculated on the zero-mode space generated from a general axially symmetric configuration and simulated annealing is used to find the Skyrme configuration that minimizes the energy of the lowest state.

This approach has previously been considered for the zero-mode space of a general spherical symmetric configuration ([2],[3]). In this space the quantum Hamiltonian reduces to a scalar, here it is a matrix. Aspects of our approach are also shared with a very recent paper [4] which appeared when our paper was in preparation. The quantum Hamiltonian used in that paper is a scalar ansatz motivated by the spherical Hamiltonian used in [1] and differs from the Hamiltonian we derive here. In fact, we will see that this will not make a significant difference, the numerical results in both papers do not differ much from each other or from what would be calculated using the classical minimum. However, the approach here is more direct than the approach described in [4] and can be generalized to higher charge nuclei.

2 Quantization procedure

Written in terms of the vector currents $R_\mu = \partial_\mu U U^\dagger$ of an SU(2) field $U(\mathbf{x})$, the Skyrme model has the Lagrangian

$$L = \int \mathrm{d}^3\mathbf{x} \left[-\frac{1}{2} \,\mathrm{Tr}\,(R_\mu R^\mu) + \frac{1}{16} \,\mathrm{Tr}\,([R_\mu, R_\nu][R^\mu, R^\nu]) + \left(\frac{2m_\pi}{F_\pi e}\right)^2 \,\mathrm{Tr}\,(U - 1)\right], \quad (1)$$

where m_π, F_π and e are parameters that are adjusted to fit experimental data. The Skyrmion mass for a static field $U_s(\mathbf{x})$ can be derived from this Lagrangian and is

$$M = \int \mathrm{d}^3\mathbf{x} \left[-\frac{1}{2} \,\mathrm{Tr}\,(R_i R_i) - \frac{1}{16} ([R_i, R_j][R_i, R_j]) - \left(\frac{2m_\pi}{F_\pi e}\right)^2 \,\mathrm{Tr}\,(U - 1)\right]. \quad (2)$$

We wish to quantize the rotational and isorotational degrees of freedom. Rather than acting on a specific Skyrme configuration, we want to consider the zero-mode space of fields generated from a general static configuration $U_s(\mathbf{x})$ by isorotation A and rotation B:

$$U(\mathbf{x}) = A U_s\left(\mathbf{x}^B\right) A^\dagger, \quad (3)$$

where A is in the 2×2 representation,

$$x_i^B = B_{ij} x_j, \quad (4)$$

and B_{ij} is a three-dimensional matrix representation of B. Since rotation and isorotation are symmetries of the original Lagrangian, these configurations are all energy-degenerate. The effective Lagrangian on this restricted space of configurations can be calculated by allowing A and B to depend on time, giving $L = -M + L_{rot}$, where L_{rot} is the kinetic Lagrangian

$$L_{rot} = \frac{1}{2}\Omega_i U_{ij} \Omega_j + \frac{1}{2}\omega_i V_{ij} \omega_j - \omega_i W_{ij} \Omega_j, \quad (5)$$

with the rotational and isorotational angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ given by

$$\begin{aligned} \Omega_i &= -i\mathrm{Tr}\left(\sigma_\mathrm{i} \mathrm{A}^\dagger \dot{\mathrm{A}}\right), \\ \omega_i &= -i\mathrm{Tr}\left(\sigma_\mathrm{i} \mathrm{B}^\dagger \dot{\mathrm{B}}\right), \end{aligned} \quad (6)$$

and the moment of inertia tensors U_{ij}, V_{ij} and W_{ij} given by the following integrals

$$U_{ij} = -\int \mathrm{d}^3\mathbf{x} \left[\mathrm{Tr}\,(T_i T_j) + \frac{1}{4} \,\mathrm{Tr}\,([R_k, T_i][R_k, T_j])\right],$$

$$V_{ij} = -\epsilon_{ilm}\epsilon_{jpq}\int \mathrm{d}^3\mathbf{x}\; x_l x_p \left[\mathrm{Tr}\left(R_m R_q\right) + \frac{1}{4}\,\mathrm{Tr}\left([R_k, R_m][R_k, R_q]\right)\right],$$

$$W_{ij} = \epsilon_{jlm}\int \mathrm{d}^3\mathbf{x}\; x_l \left[\mathrm{Tr}\left(T_i R_m\right) + \frac{1}{4}\,\mathrm{Tr}\left([R_k, T_i][R_k, R_m]\right)\right]. \quad (7)$$

and

$$T_i = i\left[\frac{\sigma_i}{2}, U\right] U^\dagger, \quad (8)$$

where the σ_i are the usual Pauli matrices.

In this paper, we will restrict our discussion to axially symmetric Skyrmion solutions. Numerical simulations indicate that there is also a reflection symmetry in the xy-plane and so the principal axes of inertia can be taken as the standard orthogonal axes, with $U_{ij} = V_{ij} = W_{ij} = 0$ where $i \neq j$, and we can set $U_{ii} = U_i$ and so forth. By inspection, cylindrical symmetry in the xy-plane will also mean

$$U_1 = U_2, \quad V_1 = V_2, \quad W_1 = W_2. \quad (9)$$

We can use axial symmetry to establish an additional identification between the normal moments of inertia (see Appendix):

$$U_3 = V_3 = W_3 \quad (10)$$

Applying these restrictions to L_{rot} (5) we get

$$L_{rot} = \frac{1}{2}\left(\Omega_1^2 + \Omega_2^2\right) U_2 + \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) V_2 + \frac{1}{2}\left(\Omega_3 - \omega_3\right)^2 U_3 - \left(\Omega_1\omega_1 + \Omega_2\omega_2\right) W_2 \quad (11)$$

or, written as a sum of complete squares,

$$L_{rot} = \frac{1}{2}\left(V_2 - \frac{W_2^2}{U_2}\right)\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}\left(\Omega_3 - \omega_3\right)^2 U_3 + \frac{1}{2}\left[\left(\Omega_1 - \frac{W_2}{U_2}\omega_1\right)^2 + \left(\Omega_2 - \frac{W_2}{U_2}\omega_2\right)^2\right] U_2. \quad (12)$$

The rotation and isorotation angular momentum vectors $\mathbf{L}$ and $\mathbf{K}$ canonically conjugate to $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are

$$\mathbf{L} = \frac{\partial L_{rot}}{\partial \boldsymbol{\omega}} = \left(V_2\omega_1 - W_2\Omega_1, V_2\omega_2 - W_2\Omega_2, U_3\left(\omega_3 - \Omega_3\right)\right),$$

$$\mathbf{K} = \frac{\partial L_{rot}}{\partial \boldsymbol{\Omega}} = \left(U_2\Omega_1 - W_2\omega_1, U_2\Omega_2 - W_2\omega_2, -U_3\left(\omega_3 - \Omega_3\right)\right). \quad (13)$$

Note that $L_3{=}-K_3$: upon quantization, this is the axially symmetric condition expressed in operator form. Our approach will be to find the minimum energy Skyrmion such that its energy eigenstate is also a null eigenstate of L_3+K_3. From (12) and (13), the Hamiltonian for the rotational and isorotational degrees of freedom can now be calculated:

$$\begin{aligned} H &= \mathbf{L}.\boldsymbol{\omega}+\mathbf{K}.\boldsymbol{\Omega}-L_{rot} \\ &= \frac{1}{2}\left[\frac{\left(L_1+\frac{W_2}{U_2}K_1\right)^2}{V_2-\frac{W_2^2}{U_2}}+\frac{\left(L_2+\frac{W_2}{U_2}K_2\right)^2}{V_2-\frac{W_2^2}{U_2}}+\frac{K_1^2}{U_1}+\frac{K_2^2}{U_2}+\frac{L_3^2}{U_3}\right]. \end{aligned} \tag{14}$$

For a spin-n,isospin-n particle, the angular momentum operators $\mathbf{L}$ can be written as $(2n+1)\times(2n+1)$ dimensional matrix representation $\Sigma_1^L,\Sigma_2^L,\Sigma_3^L$ of SU(2), as can the isospin operators $\Sigma_1^K,\Sigma_2^K,\Sigma_3^K$ of $\mathbf{K}$ for an isospin-n particle. If we define our quantum state using the $|l,l_3\rangle\otimes|k,k_3\rangle$ basis, l, l_3 and k, k_3 being the quantum numbers for $\mathbf{L}$ and $\mathbf{K}$ respectively, we can embed $\mathbf{L}$ and $\mathbf{K}$ into the resulting $SO(3)^L\times SO(3)^K$ direct product space:

$$\begin{aligned} \mathbf{L} &\mapsto \hbar\boldsymbol{\Sigma}^L\otimes I_{2n+1} \\ \mathbf{K} &\mapsto I_{2n+1}\otimes\hbar\boldsymbol{\Sigma}^K \end{aligned} \tag{15}$$

where I_{2n+1} is the $(2n+1)$-dimensional identity matrix.

We can now find the lowest energy nucleon state; first we insert the spin-half, isospin-half matrix representation of $\mathbf{L}$ and $\mathbf{K}$ into the Hamiltonian (14) to get:

$$H=\frac{\hbar}{4}\begin{pmatrix}\kappa_1&0&0&0\\0&\kappa_1&\kappa_2&0\\0&\kappa_2&\kappa_1&0\\0&0&0&\kappa_1\end{pmatrix} \tag{16}$$

where

$$\kappa_1=\frac{1+\left(\frac{W_2}{U_2}\right)^2}{V_2-\frac{W_2^2}{U_2}}+\frac{1}{U_2}+\frac{1}{2U_3} \tag{17}$$

and

$$\kappa_2=\frac{2\frac{W_2}{U_2}}{V_2-\frac{W_2^2}{U_2}}. \tag{18}$$

There are two eigenvectors of H which are also eigenvectors of L_3+K_3 with eigenvalue zero:

$$|0,0\rangle\equiv\frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle\otimes|\frac{1}{2},\frac{1}{2}\rangle-|\frac{1}{2},\frac{1}{2}\rangle\otimes|\frac{1}{2},-\frac{1}{2}\rangle\right),\quad E_{0,0}=\frac{\hbar}{4}(\kappa_1-\kappa_2) \tag{19}$$

and

$$|1,0\rangle\equiv\frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle\otimes|\frac{1}{2},\frac{1}{2}\rangle+|\frac{1}{2},\frac{1}{2}\rangle\otimes|\frac{1}{2},-\frac{1}{2}\rangle\right),\quad E_{1,0}=\frac{\hbar}{4}(\kappa_1+\kappa_2) \tag{20}$$

The first eigenvector is a spherically symmetric state and also has the lower energy since κ_1 and κ_2 are always positive; hence

$$E_N = \frac{\hbar}{4}\left[\frac{\left(1-\frac{W_2}{U_2}\right)^2}{V_2-\frac{W_2^2}{U_2}}+\frac{1}{U_2}+\frac{1}{2U_3}\right] \tag{21}$$

and is therefore the energy of an axially symmetric nucleon. We see that in the spherically symmetric case $U_2 = U_3 = W_2 = V_2$, and E_N reduces to the rotational energy formula obtained in [1]:

$$E_N^{sym} = \frac{\hbar}{2V_3} l\,(l+1) = \frac{3}{4}\frac{\hbar}{2\Lambda} \tag{22}$$

where $\Lambda = \frac{1}{3}(U_1+U_2+U_3) = U_3$. The energies of all other eigenstates go to infinity in the spherically symmetric limit.

The energy, $M+E_N$, of the quantum state would be difficult to extremize using gradient-based methods; instead, simulated annealing is used to find the Skyrmion configuration that minimizes this energy. Since the configuration is assumed to be axially symmetric, the cartoon method is used. The configuration is annealed on a quarter-disk two-dimensional lattice with a radius of 250 lattice points and a lattice spacing of 0.06. A variant of the Adaptive Simulated Annealing probability distribution [5] is used for the field perturbations; this seems to improve the speed of convergence and allows an exponential cooling schedule. The algorithm needs an initial configuration to perturb; any configuration of unit baryon number will suffice, and the ansatz given in [6] is probably the easiest to implement.

In Fig.1 and Fig.2, the results of our simulations are shown in comparison with results obtained using the rigid body approximation used in [7]. The pion mass m_π was set to its experimental value of 138 MeV in Fig.1, whereas in Fig.2 it was set to the larger value of 345MeV suggested in [4]; we see that the nucleon deformation only becomes noticeable at high values of the pion mass.

3 Discussion

Although it has appealing mathematical and physical properties, the Skyrme model has only been modestly successful in modeling nuclear dynamics. Perhaps one of the biggest challenges is that the zero-mode quantization of the classical minimum fails to give even the correct lowest energy state for many values of the baryon number ([8],[9]). It is possible that this is because inappropriate parameter values are used to calculate the classical minima: for example, in [10] it is suggested that using a non-zero, or even an unphysically large, pion mass will significantly affect the structure of higher charge Skyrmions. Another possibility is that the classical minimum is inappropriate and that the energy minimum of the effective quantum Hamiltonian should be used for zero-mode quantization. In this paper this has been done in the simple case of a axial symmetric nucleon; the higher charge cases will be more challenging computationally but should follow in a similar way.

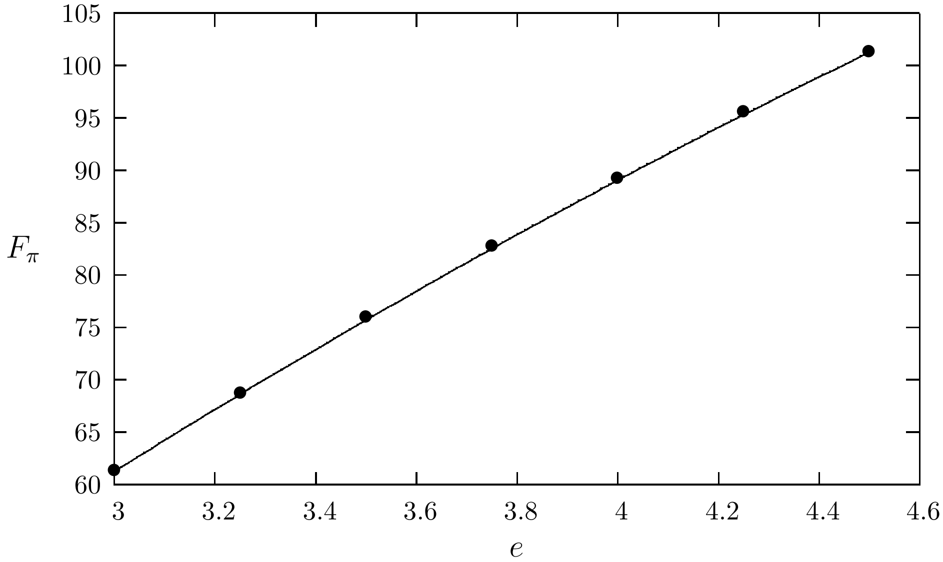

Figure 1: Plot of F_π against e with the pion mass parameter m_π set to its experimental value of 138 MeV. Our results (bold circles) are compared with those obtained using the rigid body approach taken in [7] (solid line).

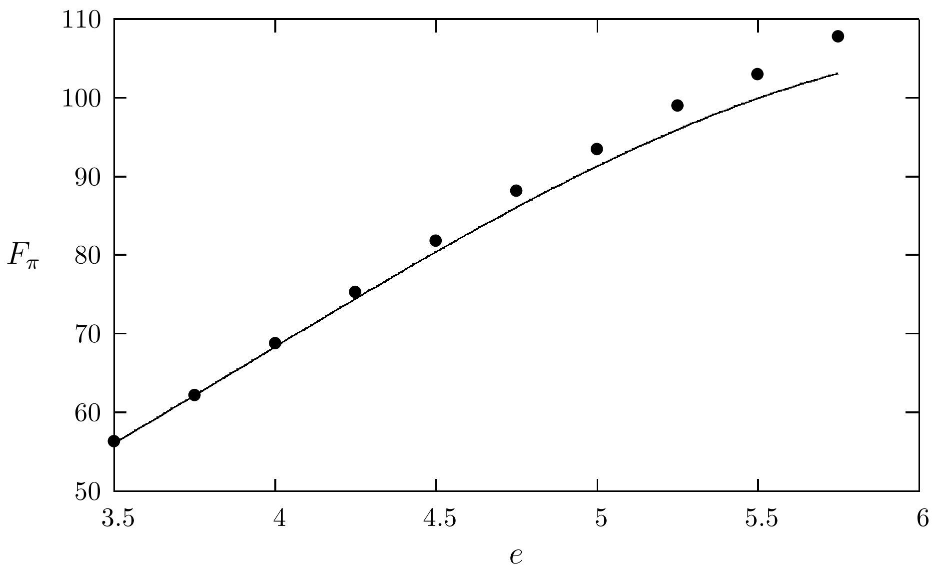

Figure 2: Plot of F_π against e with the pion mass parameter m_π set to the value of 345 MeV suggested in [4]. Our results (bold circles) are compared with those obtained using the rigid body approach taken in [7] (solid line).

We have not considered the delta here; in fact, the delta is not physically stable and it is not certain that it is useful to investigate the delta in an axially symmetric ansatz.

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Appendix: Identification of normal moments of inertia

Following [11], we establish a relation between U_3, V_3 and W_3 for axially symmetric fields. We first express the quantity $\epsilon_{3jk}x_jR_k$ in polar coordinates:

$$\begin{aligned}\epsilon_{3jk}x_jR_k &= (\mathbf{x}\times\nabla)_3\, UU^\dagger \\ &= \frac{\partial U}{\partial\phi}U^\dagger. \end{aligned} \tag{23}$$

We can then identify $\epsilon_{ij3}x_jR_i$ and $-\frac{i}{2}\left[\sigma_3, U\right]U^\dagger$ by looking at the general form of an axially symmetric field with unit baryon number:

$$U = \mathrm{e}^{\frac{-i\sigma_3\phi}{2}}\mathrm{e}^{if(r,z)n_i\sigma_i}\mathrm{e}^{\frac{i\sigma_3\phi}{2}} \tag{24}$$

and taking its derivative with respect to ϕ:

$$\begin{aligned}\frac{\partial U}{\partial\phi} &= -\frac{i\sigma_3}{2}\left(\mathrm{e}^{-\frac{i\sigma_3\phi}{2}}\mathrm{e}^{if(r,z)}\mathrm{e}^{\frac{i\sigma_3\phi}{2}}\right) + \left(\mathrm{e}^{\frac{-i\sigma_3\phi}{2}}\mathrm{e}^{if(r,z)}\mathrm{e}^{\frac{i\sigma_3\phi}{2}}\right)\frac{i\sigma_3}{2} \\ &= -\frac{i}{2}\left[\sigma_3, U\right]. \end{aligned} \tag{25}$$

We see the expressions for the inertias U_3, V_3 and W_3 in the moment of inertia integrals (7) differ only in the terms $\epsilon_{3jk}x_jR_k$ and $-\frac{i}{2}\left[\sigma_3, U\right]U^\dagger$, and so

$$U_3 = V_3 = W_3. \tag{26}$$

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