GEOMETRY AND PHYSICS
of
INSTANTONS

Simons Lectures

by

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These are the lecture notes of Simons Lectures delivered by author at SUNY Stony Brook Mathematics Department in 1999. Lectures cover the material contained in the papers written by author in collaboration with A. Losev and N. Nekrasov during period of 1996 - 1999; these lectures also contain some material which has not been published previously.

LECTURE 1:

Mirror formula

Type A sigma model on V = Type B sigma model on $\tilde{V}$

Manifolds V and $\tilde{V}$ are called mirrors.

For Kähler manifolds:

$$h^{p,q}(V) = h^{-p,q}(\tilde{V})$$

The concept of mirror symmetry extends to

V symplectic and $\tilde{V}$ complex.

Mirror exchanges *kähler* (A) and *complex* (B) deformations.

$$\sum_{n;\{k_1,\ldots,k_n\}} \frac{T^{k_1}\ldots T^{k_n}}{n!} \left\langle \mathcal{O}_a^{(0)}\mathcal{O}_b^{(0)}\mathcal{O}_c^{(0)} \int_\Sigma \mathcal{O}_{k_1}^{(2)} \ldots \int_\Sigma \mathcal{O}_{k_n}^{(2)} \right\rangle_A$$

$$= \frac{\partial^3 \mathcal{F}_B\left(T\right)}{\partial T^a \partial T^b \partial T^c}$$

Type A sigma models: Gromov-Witten theory.

Two dimensional sigma model - maps

$$\Phi : \Sigma \to V$$

Σ - two dimensional manifold, V - some Riemannian manifold.

Let V be complex manifold. Mathematical reformulation of what physicists call the computation of the path integral in the topological type A sigma model:

Given a set of submanifolds $C_1, \dots, C_k$, $C_i \subset V$, *compute the number* $N_{C_1,\dots,C_k;\beta}$ *of rigid genus* g *holomorphic curves* $\Sigma \subset V$, $[\Sigma] = \beta \in \mathrm{H}_2(V; \mathbf{Z})$ *passing through them*

The cycles in $\mathrm{H}_*(V)$ represented by $C_1, \dots, C_k$ are Poincare dual to some cohomology classes $\omega_1, \dots, \omega_k \in \mathrm{H}^*(V)$.

Physical picture

(Supersymmetric) Sigma model - defined through classical action and path integral.

Φ - a map, Σ - Riemann surface and V - Riemannian manifold of metric g.

Pick local coordinates: on Σ - $z, \bar{z}$, on V - Φ^I. Map - locally described by $\Phi^I(z, \bar{z})$.

K ($\overline{K}$) - the canonical (anti-canonical) line bundles of Σ (the bundle of one forms of types $(1,0)$ ($(0,1)$))

TV - complexified tangent bundle of V.

to get supersymmetry $\Rightarrow$ add Grassmann variables:

ψ_+^I - a section of $K^{1/2} \otimes \Phi^*(TV)$

ψ_-^I - a section of $\overline{K}^{1/2} \otimes \Phi^*(TV)$.

Physical Sigma Model action - the functional on the space of maps Φ and sections ψ :

$$\mathcal{L} = \frac{1}{f^2}\int_\Sigma \left(\frac{1}{2}g_{IJ}(\Phi)\partial_z\phi^I\partial_{\bar{z}}\phi^J + \frac{i}{2}g_{IJ}\psi_-^I D_z\psi_-^J\right) +$$

$$+\left(\frac{i}{2}g_{IJ}\psi_+^I D_{\bar{z}}\psi_+^J + \frac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L\right)$$

f^2 - **coupling constant**, R_{IJKL} - Riemann tensor of V.

$D_{\bar{z}}$ - $\bar{\partial}$ operator on $K^{1/2}\otimes\Phi^*(TV)$ constructed using the pullback of the Levi-Civita connection on TV.

Now suppose V is Kähler $\Rightarrow$ sigma model has extended susy ($\mathcal{N}=2$).

Local coordinates: $\phi^i, \phi^{\bar{i}} = \overline{\phi^i}$.

Decompose: $TV = T^{1,0}V \oplus T^{0,1}V$.

ψ_+^i ($\psi_+^{\bar{i}}$) - the projection of ψ_+ in:

$$K^{1/2} \otimes \Phi^*(T^{1,0}V) \qquad (K^{1/2} \otimes \Phi^*(T^{0,1}V))$$

ψ_-^i ($\psi_-^{\bar{i}}$) - the projections of ψ_- in:

$$\bar{K}^{1/2} \otimes \Phi^*(T^{1,0}V) \qquad (\bar{K}^{1/2} \otimes \Phi^*(T^{0,1}V))$$

Action has more parameters:

$$\mathcal{L} = i\theta \int_\Sigma \frac{1}{2} g_{i\bar{j}} \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} \right) + \frac{1}{f^2} \int_\Sigma \frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J +$$

$$+ i\psi_-^{\bar{i}} D_z \psi_-^i g_{\bar{i}i} + i\psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i g_{\bar{i}i} + R_{i\,\bar{i}j\,\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}}$$

θ-another parameter, **theta-angle**.

Twist:

$+$: ${\psi_+}^i$ and ${\psi_+}^{\bar{i}}$ - sections of $\Phi^*(T^{1,0}X)$ and $K \otimes \Phi^*(T^{0,1}X)$.

$-$: ψ_+^i and $\psi_+^{\bar{i}}$ - sections of $K \otimes \Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$.

A Model: $+$ twist of ψ_+ and a $-$ twist of ψ_-.

B Model: $-$ twists of both ψ_+ and ψ_-

Locally the twisting does nothing at all, since locally K and $\overline{K}$ are trivial.

χ - section of $\Phi^*(TX)$ ($\chi^i = \psi^i_+$, and $\chi^{\bar{i}} = \psi^{\bar{i}}_-$);

$\psi^{\bar{i}}_+$ - in the A model a $(1,0)$ form on Σ with values in $\Phi^*(T^{0,1}X)$; $\psi^{\bar{i}}_+ = \psi^{\bar{i}}_z$.

ψ^i_- is $(0,1)$ form with values in $\Phi^*(T^{1,0}X)$; $\psi^i_- = \psi^i_{\bar{z}}$.

Topological transformation laws:

$$\delta\Phi^I = i\alpha\chi^I$$

$$\delta\chi^I = 0$$

$$\delta\psi^{\bar{i}}_z = -\alpha\partial_z\phi^{\bar{i}} - i\alpha\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_z$$

$$\delta\psi^i_{\bar{z}} = -\alpha\partial_{\bar{z}}\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi^m_{\bar{z}}.$$

$\delta^2 = 0$ - on the space of solutions of equations of motion (minimizing the action). Can be made "off-shell" by introducing auxiliary fields.

Let $t = \theta + \frac{i}{f^2}$.

Action:

$$\mathcal{S} = \frac{1}{f^2}\int_\Sigma d^2z\ \delta R + t\int_\Sigma \Phi^*(\omega)$$

$$R = g_{i\bar{j}}\left(\psi^{\bar{i}}_z\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{i}}\psi^j_{\bar{z}}\right),$$

$$\int_\Sigma \Phi^*(\omega) = i\int_\Sigma d^2z\ \left(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}g_{i\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}g_{i\bar{j}}\right)$$

– the integral of the pullback of the Kähler form $\omega = -ig_{i\bar{j}}dz^i dz^{\bar{j}}$.

$\int\Phi^*(\omega)$ - depends only on the cohomology class of ω and the homology class $\beta \in \mathrm{H}_2(V)$ of the image of the map Φ.

In physics one computes correlation functions of some operators (observables) in given theory.

Definition. Observable $\{\mathcal{O}_i\}$ – a functional of the fields, s.t. $\delta \mathcal{O}_i = 0$.

Definition. Physical observable = a δ - cohomology class, $\mathcal{O}_i \sim \mathcal{O}_i + \delta \Psi_i$.

Definition. Correlator - path integral:

$$\langle \prod_a \mathcal{O}_a \rangle_\beta = e^{-2\pi t \int_\beta \omega} \int_{\mathcal{B}_\beta} D\phi \; D\chi \; D\psi \;\; e^{-\frac{1}{f^2}\delta \int R} \cdot \prod_a \mathcal{O}_a .$$

$\mathcal{B}_\beta$ - the component of the field space for maps of degree $\beta = [\Phi(\Sigma)] \in \mathrm{H}_2(V, \mathbf{Z})$, and $\langle \;\; \rangle_\beta$ - degree β contribution to the expectation value.

Correlators of the observables depend only on their δ-cohomology class, in particular — independent of the complex structure of Σ and V, and depend only on the cohomology class of the Kähler form ω.

Standard argument: δ $\sim$ exterior derivative on the field space $\mathcal{B} \rightarrow \langle \delta\Psi \rangle_\beta = 0$ for any reasonable Ψ. Thus, the $\mathcal{O}_i$ should be considered as representatives of the δ-cohomology classes.

Correlator is independent of f^2. If $f^2 \rightarrow \infty$ - Gaussian model.

Bosonic part of the *Action*

$$it \int \Phi^*(\omega) + \frac{1}{f^2} \int_\Sigma g_{i\bar{j}}(\phi)\partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i$$

for given β is minimized by holomorphic map:

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = 0.$$

The entire path integral, for maps of degree β, reduces to an integral over the space of degree β holomorphic maps $\mathcal{M}_\beta$.

Pick an n-form $W = W_{I_1 I_2 \dots I_n}(\phi) d\phi^{I_1} \wedge d\phi^{I_2} \wedge \dots \wedge d\phi^{I_n}$ on $V \Rightarrow$ a local functional

$$\mathcal{O}_W(P) = W_{I_1 I_2 \dots I_n}(\Phi(P)) \chi^{I_1} \dots \chi^{I_n}(P).$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{dW},$$

d the exterior derivative on V.

$\Rightarrow W \mapsto \mathcal{O}_W$ - natural map from the de Rham cohomology of V to the space of physical observables of quantum field theory $A(V)$. For local operators - isomorphism.

Let d - be the DeRham differential on Σ. We have **descend equations**:

$\mathrm{d}\mathcal{O}_W = \delta \mathcal{O}_W^{(1)}$, $\oint_C \mathcal{O}_W^{(1)}$ - 1-observable. The physical observable depends on the homology class of C in $\mathrm{H}_1(\Sigma)$.

$\mathrm{d}\mathcal{O}_W^{(1)} = \delta \mathcal{O}_W^{(2)}$, $\int_\Sigma \mathcal{O}_W^{(2)}$ - 2-observable.

Deformations of the theory: change the action as follows:

$$\mathcal{S}_T = \mathcal{S} + T^a \int_\Sigma \mathcal{O}_{W_a}$$

T^a are the formal parameters (nilpotent). The path integral with the action $\mathcal{S}_T$ computes the *generating function* $\mathcal{F}_A(T)$ of the correlation functions of the two-observables:

$$\mathcal{F}_A(T) = \langle e^{-\int_\Sigma \mathcal{S}(T)} \rangle$$

$$\mathcal{S}(0) = \mathcal{S}, \qquad \frac{\partial \mathcal{S}}{\partial T^a}|_{T=0} = \int_\Sigma \mathcal{O}_{W_a}$$

Reduction to the enumerative problem

C - submanifold of V (only its homology class matters).

The "Poincaré dual" to C - cohomology class that counts intersections with C. Represent by a differential form $W(C)$ that has delta function support on C:

$$W(C) = \delta_C$$

<u>*Conclude*</u>:

Correlators of topological observables $\mathcal{O}_{W(C_1)} \dots \mathcal{O}_{W(C_k)}$ are integrals over $\mathcal{M}_\beta$ of the products of delta functions which pick out the holomorphic maps whose image intersects the submanifolds $C_1, \dots, C_n$:

Let $C_1, \dots, C_k \subset V$ - complex submanifolds, $\dim C_l = d_l$.

$\omega_m = W(C_m) \in H^*(V)$ - their Poincare duals.

Let $z_1, \dots, z_m \in \Sigma$, $m \leq k$ be the marked points.

For a complex submanifold $C \subset V$ and for $1 \leq l \leq m$ define the following submanifolds $\mathcal{M}^0_{C,l} \subset \mathcal{M}$, $\mathcal{M}^2_C \subset \mathcal{M}$:

Definition. $\mathcal{M}^0_{C,l} = \{\Phi : \Sigma \to V | \, \Phi \in \mathcal{M}, \, \Phi(z_l) \in C\}$

Definition. $\mathcal{M}^2_C = \{\Phi : \Sigma \to V | \, \Phi(\Sigma) \cap C \neq \emptyset\}$

The correlation functions in the type A sigma model are simply the intersection numbers:

$$\langle \mathcal{O}^{(0)}_{C_1}(z_1)\ldots\mathcal{O}^{(0)}_{C_m}(z_m)\int_\Sigma \mathcal{O}^{(2)}_{C_{m+1}}\cdots\int_\Sigma \mathcal{O}^{(2)}_{C_k}\rangle \; =$$

$$\#\mathcal{M}^0_{C_1,1}\cap\ldots\mathcal{M}^0_{C_m,m}\cap\mathcal{M}^2_{C_{m+1}}\cap\ldots\cap\mathcal{M}^2_{C_k}$$

$$\sum \mathrm{dim}\mathcal{M}^0_{C_i,i}+\sum \mathrm{dim}\mathcal{M}^2_{C_i}=\mathrm{dim}\mathcal{M}_\beta$$

otherwise $\langle\ldots\rangle$ vanishes,

$$\mathrm{dim}\mathcal{M}_\beta=\int_\beta c_1(V)+(1-g)dimV$$

Problem: $\mathcal{M}_\beta$ is non-compact. Need to compactify it in order to get a nice intersection theory.

Compactification is not unique.

Option I. Kontsevich stable maps.

Option II. Freckled instantons – in case where V is a symplectic quotient of a G-equivariant submanifold of a vector (affine) symplectic space A: $V \subset A//G$.

Compactification of $\mathcal{M}$ - Regularization

Non-compactness of $\mathcal{M}$ comes from ultraviolet non-compactness of the fields space $\mathcal{B}$. (UV $= \|d\Phi\|^2 \to \infty$)

Physical picture

Option I = coupling to topological gravity $\approx$ averaging over conformal structures on Σ.

Option II = gauged linear sigma model with target A and gauge group G (and perhaps superpotential).

Option I. Intersection theory of stable maps

For simplicity $g = 0$ - counting rational curves.

Definition.

$$\left\langle \mathcal{O}_1^{(0)} \mathcal{O}_2^{(0)} \mathcal{O}_3^{(0)} \int_\Sigma \mathcal{O}_4^{(2)} \dots \int_\Sigma \mathcal{O}_k^{(2)} \right\rangle_{A;\beta} =$$

$$N_{C_1,\dots,C_k;\beta}$$

The curve embedded into V has a parameterization.

$g = 0$ - the space of all parameterizations is acted on by the group $\mathrm{PGL}_2(\mathbf{C})$ of automorphismes of $\mathbf{P}^1$. This freedom can be partially fixed - the points 0, 1, ∞ on $\mathbf{P}^1$ are mapped to C_1, C_2, C_3.

The positions $z_4, \dots, z_k$ - preimages of $\Sigma \cap C_4, \dots \Sigma \cap C_k$, are not fixed, can be arbitrary.

Consider the k-punctured curves - the number $N_{C_1,\dots,C_k}$ can be expressed as the integral over the moduli space $\overline{\mathcal{M}}_{0,k}$ of such curves. This space has complex dimension $k - 3$ and the positions of $z_4, \dots, z_k$ are integrated over, hence the asymmetry in the notations in the definition.

It follows from the connectivity of $\overline{\mathcal{M}}_{0,k}$ that the result is independent on the ordering of $C_1, \dots, C_k$.

Defintion. A stable map is the structure: $(\Sigma, x_1, \ldots, x_k; \phi)$, consisting of

A connected reduced curve Σ with $k \geq 0$ pairwise distinct marked non-singular points $x_1, \ldots, x_k \in \Sigma$ and at most ordinary double singular points;

A map $\phi : \Sigma \to V$ having no non-trivial first order infinitesimal automorphismes, identical on V and $\{x_1, \ldots x_k\}$ - every component of Σ of genus 0 (resp. 1) which is mapped to a point by ϕ must have at least 3 (resp. 1) marked or singular points on its normalization.

Reduced curve The compact algebraic curve is a zero locus of an appropriate number of homogeneous polynomials $f_1, \ldots, f_k$ in a projective space $\mathbf{P}^{k+1}$. The curve is reduced if none of the linear combinations of polynomials f_i is a square of another polynomial.

Normalization. For a curve C with only simple double singular points (i.e. locally given by the equation $xy = 0$ in $\mathbf{C}^2$) the normalization is a (perhaps disconnected) curve $\tilde{C}$ and the holomorphic map $\pi : \tilde{C} \to C$ such that π is isomorphism over the set of smooth points in C and the preimage of each singular point consists of two points.

Lemma. The number $N_{C_1,\ldots,C_k;\beta}$ can also be represented as:

$$\int_{\overline{\mathcal{M}}_{n+3,\beta}} \Omega_1^{(0)} \wedge \Omega_2^{(0)} \wedge \Omega_3^{(0)} \wedge \Omega_4^{(2)} \wedge \ldots \wedge \Omega_k^{(2)},$$

$\overline{\mathcal{M}}_{k,\beta}$ - the moduli space of stable holomorphic maps of the k-punctured worldsheet $\Sigma \approx \mathbf{P}^1$ to V,

$\beta \in \mathrm{H}_2(V)$ - the homology class $[\phi(\Sigma)]$,

$\Omega_m^{(i)}$ - the cohomology classes of $\overline{\mathcal{M}}_{k,\beta}$, defined as follows.

For each $m = 1, \ldots, k$ there is evaluation map:

$$e_m : \overline{\mathcal{M}}_{k,\beta} \to V$$

which sends a stable map $(\Sigma, x_1, \ldots, x_k; \phi)$ to the image $\phi(x_m) \in V$ of the m'th puncture: $e_m = \phi(x_m)$. Then

$$\Omega_m^{(0)} = e_m^* \omega_m, \quad \Omega_m^{(2)} = (p_m)_* \, e_m^* \omega_m = \int_{\Sigma, x_m} \Omega^{(0)}$$

where $p_m : \overline{\mathcal{M}}_{k,\beta} \to \overline{\mathcal{M}}_{k-1,\beta}$ is the projection forgetting m'th puncture (and contracting all unwanted components of Σ which may occur).

Option II. Freckled Instantons

At first sight one does not need complicated objects such as the stable maps.

Let $V = \mathbf{CP}^N$ (one may easily generalize to the case of submanifold in the generaic symplectic quotient), $\Sigma = \mathbf{CP}^1$.

Homogeneous coordinates in $V:\ \left(Q^0 : \ldots : Q^N\right)$, Homogeneous coordinates on Σ: (ξ_0, ξ_1).

Statement. Holomorphic degree d genus 0 map $\Phi : \Sigma = \mathbf{CP}^1 \to V$ is the same thing as the collection of $N+1$ homogeneous polynomials:

$$Q^i(\xi_0, \xi_1) = \sum_{m=0}^{d} Q^i_m \xi_0^m \xi_1^{d-m}, i = 0, \ldots, N$$

which obey the following requirement:

<u>*for any*</u> $(\xi_0 : \xi_1) \in \Sigma$ *there exists* i*, s.t.* $Q^i(\xi_0, \xi_1) \neq 0 \quad (\star)$

The map is defined as follows:

$$\Phi : \xi = (\xi_0 : \xi_1) \in \Sigma \mapsto \left(Q^0(\xi) : \ldots : Q^N(\xi)\right)$$

Note. Multiplication of all Q^i_m by the same number $\lambda \in \mathbf{C}^*$ does not change the map $\Rightarrow$ the space $\mathcal{M}_d$ of holomorphic maps of degree d is a subspace in the projective space $\mathbf{P}^{(N+1)(d+1)-1}$.

Let us relax the condition $(\star)$ to the following:

<u>*there exists*</u> $(\xi_0 : \xi_1) \in \Sigma$ *and* i, *s.t.* $Q^i(\xi_0, \xi_1) \neq 0 \quad (\star\star)$

In this way we obtain a compactification (originally due to Drinfeld) $\overline{\mathcal{M}}_d = \mathbf{P}^{(N+1)(d+1)-1}$ of the space of parameterized holomorphic maps. What does this space parameterize?
A point $Q \in \overline{\mathcal{M}}_d$ determines a collection of polynomials which may have a common factor:

$$Q^i(\xi) = P(\xi)\tilde{Q}^i(\xi)$$

where $\tilde{Q}^i$ do not have common factors. Let $k = \deg P$ We have:

$$d = \deg Q^i = \deg P + \deg \tilde{Q}$$

Hence $\tilde{Q}$ defines a degree $d - k$ map from $\mathbf{P}^1$ to V. The polynomial P plays no role in this map. It plays crucial role in keeping the total degree conserved.

Definition. The zeroes of the polynomial P (there are k of them) are called **freckles**. The structure (a degree $d - k$ holomorphic map $\Sigma \to V$, a set of k (perhaps coincident) points on $\mathbf{P}^1$) is called a **degree** d **freckled instanton.**

Stratification:

$$\overline{\mathcal{M}}_d = \mathcal{M}_d \cup \mathcal{M}_{d-1} \times \Sigma \cup \ldots \cup \mathcal{M}_{d-p} \times \mathrm{Sym}^p\Sigma \cup \ldots$$

The importance of the freckled instantons is that the path integral motivated integral over the non-compact space $\mathcal{M}_d$ can be replaced by the intersection theory on the compact space $\overline{\mathcal{M}}_d$.

Intersection theory with freckles

For $V = \mathbf{CP}^N$ or in more general case described above we can compactify $\mathcal{M}_\beta$ by considering the space $\overline{\mathcal{M}}_\beta$ of freckled instantons.

In this way we get *a priori* another definition of the correlation functions:

$$\langle \mathcal{O}^{(0)}_{C_1}(z_1)\dots\mathcal{O}^{(0)}_{C_m}(z_m)\int_\Sigma \mathcal{O}^{(2)}_{C_{m+1}}\dots\int_\Sigma \mathcal{O}^{(2)}_{C_k}\rangle' =$$

$$\#\overline{\mathcal{M}}^0_{C_1,1}\cap\dots\overline{\mathcal{M}}^0_{C_m,m}\cap\overline{\mathcal{M}}^2_{C_{m+1}}\cap\dots\cap\overline{\mathcal{M}}^2_{C_k}$$

The computation of $\langle\dots\rangle'$ is a simple problem due to the compactness of all submanifolds involved.

The difficulty of computing $\langle\dots\rangle$ — extracting of the boundary contribution:

$$\Delta = \#\overline{\mathcal{M}}^0_{C_1,1}\cap\dots\overline{\mathcal{M}}^0_{C_m,m}\cap\overline{\mathcal{M}}^2_{C_{m+1}}\cap\dots\cap\overline{\mathcal{M}}^2_{C_k}\cap(\overline{\mathcal{M}}\backslash\mathcal{M})$$

Example. $V = \mathbf{P}^2$, $\Sigma = \mathbf{P}^1$, C_1, C_2, C_3 are lines in V, C_4, C_5 - points. $z_1 = 0, z_2 = 1, z_3 = \infty \in \Sigma$.

• The elementary geometry tells us that $\langle \ldots \rangle = 1$ in this case.

$\overline{\mathcal{M}} = \mathbf{P}^5$, $\mathcal{M}^0_{C_l,l}$ = a hyperplane in $\mathbf{P}^5$, $\mathcal{M}^2_{C_l}$, $l = 4, 5$ are quadric hypersurfaces. Hence the Besout theorem gives:

$$\langle \ldots \rangle' = 2 \times 2 = 4$$

• The discrepancy 3 is due to the contribution of the boundary: the freckles hitting the points $0, 1$ or ∞ contribute 1 to the intersection number.
This example will be studied in more detail in the last lecture.

The moral. The generating function

$$\partial^3_{T^X T^Y T^Z} \mathcal{F}_A(T) = \langle \mathcal{O}^{(0)}_X(0) \mathcal{O}^{(0)}_Y(1) \mathcal{O}^{(0)}_Z(\infty) \exp \sum T^k \int_\Sigma \mathcal{O}^{(2)}_{C_k} \rangle$$

differs from

$$\partial^3_{t^X t^Y t^Z} \mathcal{F}'_A(t) = \langle \mathcal{O}^{(0)}_X(0) \mathcal{O}^{(0)}_Y(1) \mathcal{O}^{(0)}_Z(\infty) \exp \sum t^k \int_\Sigma \mathcal{O}^{(2)}_{C_k} \rangle'$$

by a (triangular in the case of V with $c_1(V)$ positive) change of variables:

$$T^k = T^k(t^k, t^{k-1}, \ldots, t^{k-p}, \ldots).$$

(physically - contact terms)

One can compute $\mathcal{F}'_A$ for $V = \mathbf{CP}^N$ rather easily. The submanifolds C_k are the planes $\mathbf{CP}^k \subset V$, $k = 0, \ldots, N$.

$$\mathcal{F}'_A(t) = \oint \frac{d\sigma}{\sigma^N - \exp\left(\sum_r r t_r \sigma^{r-1}\right)}$$

Type B sigma models: Kodaira-Spencer theory.

Consider the space S of generalized (in the sense of Kontsevich-Witten) deformations of complex structures of variety $\tilde{V}$ ($\tilde{V}$ - mirror to V).

The tangent space to S at some point s represented by a variety V_s' is given by:

$$T_s S = \bigoplus_{p,q} \mathrm{H}^p \left(\tilde{V}_s, \Lambda^q \mathcal{T}_{V_s} \right) \equiv \bigoplus_{p,q} \mathrm{H}^{-q,p}(\tilde{V}_s)$$

Let T denote special coordinates on this space.

The right-hand side of the mirror formula - essentially a partition function in type B sigma model expressed in terms of special coordinates, whose choice is *absolutely necessary* for the formulation of mirror symmetry.

Physical Picture

$\psi^{\bar{i}}_{\pm}$ - sections of $\Phi^*(T^{0,1}\tilde{V})$

ψ^{i}_{+} - section of $K \otimes \Phi^*(T^{1,0}\tilde{V})$

ψ^{i}_{-} - section of $\overline{K} \otimes \Phi^*(T^{1,0}\tilde{V})$.

ρ - one form with values in $\Phi^*(T^{1,0}\tilde{V})$; $\rho^i_z = \psi^i_+$, $\rho^i_{\bar{z}} = \psi^i_-$.

all fields above are valued in Grassmann algebra

Denote:

$$\eta^{\bar{i}} = \psi^{\bar{i}}_+ + \psi^{\bar{i}}_-$$
$$\theta_i = g_{i\bar{i}}\left(\psi^{\bar{i}}_+ - \psi^{\bar{i}}_-\right).$$

Transformations:

$$\delta\phi^i = 0$$
$$\delta\phi^{\bar{i}} = i\alpha\eta^{\bar{i}}$$
$$\delta\eta^{\bar{i}} = \delta\theta_i = 0$$
$$\delta\rho^i = -\alpha\ d\phi^i.$$

nilpotent symmetry: $\delta^2 = 0$ modulo the equations of motion.

Action:

$$\mathcal{S} = \frac{1}{f^2}\int_\Sigma d^2z\left(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{i}}(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_z)g_{i\bar{i}}\right.$$
$$\left. + i\theta_i(D_{\bar{z}}{\rho_z}^i - D_z{\rho_{\bar{z}}}^i) + R_{i\bar{i}j\bar{j}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{i}}\theta_k g^{k\bar{j}}\right).$$

Again one can rewrite the action using δ:

$$\mathcal{S} = \frac{1}{f^2} \int \delta U + \mathcal{S}_0$$

$$U = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right)$$

$$\mathcal{S}_0 = \int_\Sigma \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right).$$

B theory is independent of the complex structure of Σ and the Kähler metric of $\tilde{V}$. Change of complex structure of Σ or Kähler metric of $\tilde{V}$ - Action changes by irrelevant terms of the form $\delta(\ldots)$.

The theory depends on the complex structure of $\tilde{V}$, which enters δ

B model is independent of f^2; take limit $f^2 \to \infty$; In this limit, one expands around minima of the bosonic part of the Action = constant maps $\Phi : \Sigma \to \tilde{V}$:

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0$$

The space of such constant maps is a copy of $\tilde{V}$; the path integral reduces to an integral over $\tilde{V}$.

Observables:

Consider $(0,p)$ forms on $\tilde{V}$ with values in $\wedge^q T^{1,0}\tilde{V}$, the q^{th} exterior power of the holomorphic tangent bundle of $\tilde{V}$.

$$W = d\bar{z}^{i_1} d\bar{z}^{i_2} \ldots d\bar{z}^{i_p} W_{\bar{i}_1\,\bar{i}_2 \ldots \bar{i}_p}{}^{j_1 j_2 \ldots j_q} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_q}}$$

W is antisymmetric in the j's as well as in the $\bar{i}$'s.

Form local operator:

$$\mathcal{O}_W = \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} W_{\bar{i}_1 \ldots \bar{i}_p}{}^{j_1 \ldots j_q} \psi_{j_1} \ldots \psi_{j_q}.$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{\bar{\partial} W},$$

$\mathcal{O}_W$ is δ-invariant iff $\bar{\partial} W = 0$ and δ-exact if $W = \bar{\partial} S$ for some S.

$W \mapsto \mathcal{O}_W$ - natural map from $\oplus_{p,q} H^p(V, \wedge^q T^{1,0} V)$ to the δ-cohomology of the B model. It is isomorphism for local operators.

The story of *Correlators in B model*, *Descend Equations*, *Deformation of the action by 2-observables*, *Generating function* $\mathcal{F}_B(T)$ is completely paralell.

• Interesting examples of the deformations:

$W = \mu_{\bar{i}}^{j} \frac{\partial}{\partial z^j} d\bar{z}^{\bar{j}}$ - deformation of the complex structure of $\tilde{V}$

$W = W(z)$ - holomorphic function (for non-compact $\tilde{V}$)- singularity (Landau-Ginzburg in physical terminology) theory

$W = \frac{1}{2}\pi^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$ - non-commutative deformation

Example. For variation of complex structure of a Calabi-Yau manifold the (projective) special coordinates are given by periods of a holomorphic top form.

$\tilde{V}_s$ – family of d complex dimensional projective varieties with $c_1(\tilde{V}_s) = 0$.
Unique up to a multiplicative constant holomorphic $(d, 0)$ form Ω.

$\mathcal{M}$ - moduli of cmplx structures $\tilde{V}_{s_0}$

$$\mathcal{T}_{s_0}\mathcal{M} \approx \mathrm{H}^{d-1,1}(\tilde{V}_{s_0})$$

The universal cover $\widetilde{\mathcal{M}}$ has special coordinates T^i, $i = 0, \ldots, h^{d-1,1}(\tilde{V}$

Let $\alpha_I(s), \beta^I(s)$, $I = 0, \ldots, h^{d-1,1}(Y)$ be a symplectic basis in $\mathrm{H}^d(\tilde{V}_s, \mathbf{Z})$:

$$\alpha_I \cap \alpha_J = \beta^I \cap \beta^J = 0, \quad \alpha_I \cap \beta^J = \delta_I^J$$

On the $\widetilde{\mathcal{M}}$ this basis is defined uniquely once it is chosen at some marked point $p_0 \in \widetilde{\mathcal{M}}$.
Let

$$A^I(s) = \int_{\alpha_I(s)} \Omega, \quad A_{D,I}(s) = \int_{\beta^I(s)} \Omega$$

Ω - defined uniquely up to a constant. Let us fix this freedom by choosing a distinguished cycle α_0 and demanding $A^0 = 1$.
Then

$$T^i = A^i, \quad i = 1, \ldots, \mathrm{dim}\mathcal{M}$$

There exists a function $\mathcal{F}_B$ on $\widetilde{\mathcal{M}}$ such that

$$d\mathcal{F}_B = \sum_i A_{D,i} dA^i$$

Locally $\mathcal{F}_B$ can be viewed as a function of T^i and it is in this sense that it appears in the right-hand-side of the **2d mirror formula**.

Physical motivation: For $d = 3$:

$$\frac{\partial^3 \mathcal{F}}{\partial T^i \partial T^j \partial T^k} = \int_{\tilde{V}_s} \Omega \wedge \iota_{\mu_i \wedge \mu_j \wedge \mu_k} \Omega$$

-the three point function on a sphere. μ_i- Beltrami differentials:

$$\iota_{\mu_i} \Omega = \left(\frac{\partial \Omega}{\partial T^i} \right)^{2,1}$$

Mirror symmetry: A=B

not only for CY, but more general

Special case of CY threefolds: physical intuition

As $\mathcal{N} = 2$ SCFT's the theories A and B don't differ (internal authomorphism of the $\mathcal{N} = 2$ algebra maps A to B and vice versa)

SCFT has different large volume limits - the same theory looks as different sigma models with different target spaces V and $\tilde{V}$ in different limits.

T-duality - the simplest example.

LECTURE 2

FOUR DIMENSIONAL THEORY A

REFINED

DONALDSON-WITTEN THEORY

- X – compact smooth Riemannian manifold;
- $b_i = b_i(X)$ – Betti numbers.
- On $\mathrm{H}^*(X)$: intersection form $(,)$; metric $\langle , \rangle$:

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2, \quad \langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \star\omega_2$$

$\star$ - the Hodge star operation.

$b_2^{\pm}$ – dim's of the positive and negative subspaces of $\mathrm{H}^2(X)$.

$\omega \in \mathrm{H}^2(X)$: $\omega^{\pm}$ – orthogonal projections to the spaces of self- and antiselfdual classes: $\mathrm{H}^{2,\pm}(X)$ – $(\omega^{\pm}, \cdot) = \pm\langle \omega^{\pm}, \cdot \rangle$, $\omega = \omega^+ + \omega^-$.

$\chi = \sum_{i=0}^4 (-1)^i b_i$, – the Euler characteristics of X

$\sigma = b_2^+ - b_2^-$ the signature of X

- e_α is a basis in $\mathrm{H}_*(X, \mathbf{C})$,
- e^α the dual basis in $\mathrm{H}^*(X, \mathbf{C})$:

$$(e^\alpha, \omega) = \int_{e_\alpha} \omega$$

for any $\omega \in \mathrm{H}^*(X)$.

$\mathbf{G}' = SU(r+1)$, $\mathbf{G} = \mathbf{G}'/Z$, $Z \approx \mathbf{Z}_{r+1}$, $\mathbf{g} = \mathrm{Lie}\mathbf{G}$.

$\mathbf{T} = U(1)^r$ – maximal torus of $\mathbf{G}$, $W = \mathcal{S}_{r+1}$ the Weyl group,

$\mathbf{g} = \mathrm{Lie}(\mathbf{G}), \mathbf{t} = \mathrm{Lie}(\mathbf{T})$.

$h = r + 1$ – dual Coxeter number.

$\ell = (w_2; k)$, $k \in \mathbf{Z}$, $w_2 \in \mathrm{H}^2(X, Z)$ – generalized Stiefel-Whitney class.

$\mathcal{P}_\ell$ - a principal $\mathbf{G}$ bundle over X and E_ℓ the associated vector bundle with $w_2(E_\ell) = w_2$,

$c_2(E_\ell) + \frac{1}{2} w_2 \cdot w_2 = k$.

$\mathcal{A}_\ell$ - the space of connections in $\mathcal{P}_\ell$.

$\mathcal{G}_\ell$ - the group of gauge transformations of $\mathcal{P}_\ell$.

The Lie algebra of $\mathcal{G}_\ell$ - the algebra of sections of the associated adjoint bundle $\mathbf{g}_\ell = \mathcal{P}_\ell \times_{\mathbf{Ad}} \mathbf{g}$. ϕ - an element of Lie$\mathcal{G}_\ell$.

For the connection A (= the gauge field) let F_A denote its curvature (it is a section of $\Lambda^2 T^*_X \otimes \mathbf{g}_\ell$).

Definition. $\mathbf{G}$-instanton is the solution to the equation $F_A^+ = 0$ where $+$ acts on the $\Lambda^2 T^*_X$ part of F_A.

Definition. a $\mathbf{G}$-instanton A is called irreducible if there are no infinitesimal gauge transformations, preserving A. This condition is equivalent to the absence of the solutions to the equation

$$d_A \phi = 0, \quad 0 \neq \phi \in \Gamma(\mathbf{g}_\ell)$$

where d_A is the connection on $\mathbf{g}_\ell$ associated with A.

Definition. a $\mathbf{G}$-instanton is called unobstructed if there are no solutions to the equation $(d_A^+)^* \chi = 0$, $0 \neq \chi \in \Gamma\left(\Lambda^{2,+} T^*_X \otimes \mathbf{g}_\ell\right)$.

Definition. The moduli space $\mathcal{M}_\ell$ of $\mathbf{G}$-instantons is the space of all irreducible unobstructed $\mathbf{G}$-instantons modulo action of $\mathcal{G}_\ell$. For the instanton A let $[A]$ denote its gauge equivalence class - a point in $\mathcal{M}_\ell$.

The tangent space to $\mathcal{M}_\ell$ at A is the middle cohomology group of the Atiyah-Hitchin-Singer (AHS) complex of bundles over X:

$$0 \to \Lambda^0 T_X^* \otimes \mathbf{g}_\ell \to \Lambda^1 T_X^* \otimes \mathbf{g}_\ell \to \Lambda^{2,+} T_X^* \otimes \mathbf{g}_\ell \to 0$$

the first arrow is d_A, the second is $d_A^+ = \mathrm{P}_+ d_A$.

P_+ - the projection $\Lambda^2 T_X^* \otimes \mathbf{g}_\ell \to \Lambda^{2,+} T_X^* \otimes \mathbf{g}_\ell$.
$d_A^+ \circ d_A = F_A^+ = 0 \to$ the sequence is the complex.

$H^0(AHS) = 0$ for irred. instantons. $H^2(AHS) = 0$ - obstruction space; absent for unobstructed instantons.

Lemma. The dimension of the moduli space $\mathcal{M}_\ell$:

$$\mathrm{dim}\mathcal{M}_\ell = 4hk - \mathrm{dim}\mathbf{G}\frac{\chi + \sigma}{2}$$

Proof: index theorem applied to the AHS complex.

Remark. $\mathcal{M}_\ell$ is non-compact. Sometimes it can be compactified (Donaldson-Uhlenbeck) by adding the point-like instantons:

$$\overline{\mathcal{M}}_\ell = \mathcal{M}_\ell \cup \mathcal{M}_{\ell-(0;1)} \times X \cup \ldots \cup \mathcal{M}_{\ell-(0;k)} \times S^k X$$

For A from class $[A] \in \mathcal{M}_\ell$ the space $T_{[A]}\mathcal{M}_\ell$ can be identified with the space of solutions α:

$$d_A^+ \alpha = 0, \quad d_A^* \alpha = 0$$

$\alpha \in \Gamma\left(\Lambda^1 T^* X \otimes \mathbf{g}_\ell\right)$.

Consider the product $\mathcal{M}_\ell \times X$ and form the *universal bundle* $\mathcal{E}_\ell$ - the bundle whose restriction onto $[A] \times X \subset \mathcal{M}_\ell \times X$ coincides with E_ℓ.

$\mathbf{d}$ be the differential in the DeRham complex on $\mathcal{M}_\ell \times X$ and d_m, d be its components along $\mathcal{M}_\ell$, X respectively.

Definition. The *universal* connection is the $\mathbf{G}$-connection $\mathbf{a}$ in $\mathcal{E}_\ell$ with the following properties:

1. $\mathbf{a}|_{[A]\times X} \in [A]$
2. $\mathbf{a}|_{\mathcal{M}_\ell \times \{x\}} = \frac{1}{\Delta_A} d_A^* d_m A$ with $\Delta_A = d_A^* d_A$

Lemma. The curvature of the universal connection can be expanded as:

$$\mathcal{F}_{\mathbf{a}} = F_A + \psi + \phi$$

ψ is the fundamental solution to the equations:

$$d_A^+ \psi = 0, \quad d_A^* \psi = 0$$

ϕ is given by:

$$\phi = \frac{1}{\Delta_A} [\psi, \star\psi]$$

Comments. We view ψ as the mixed $(\mathcal{M}_\ell, X)$ component of the curvature of $\mathbf{a}$. It means that locally we view ψ as one-form on $\mathcal{M}_\ell$ with values in $\mathbf{g}$. Using metric on X and the induced metric on $\mathcal{M}_\ell$ we identify $T_{[A]}\mathcal{M}_\ell$ with $T_{[A]}^*\mathcal{M}_\ell$.

Similarly ϕ is the $(\mathcal{M}_\ell, \mathcal{M}_\ell)$ component of the curvature of $\mathbf{a}$.

$\{I_k\}$ - additive basis in the space of invariants: $\mathrm{Fun}(\mathbf{g})^{\mathbf{G}} \approx \mathrm{Fun}(\mathbf{t})^W$.

d_k - the degree of I_k.

$\mathcal{O}_n^\alpha = \int_{e_\alpha} I_n \left(\frac{\phi+\psi+F_A}{2\pi i} \right)$.

Examples. $I_1(\phi) = \mathrm{Tr}\phi^2,\ d_1 = 2,\ I_2(\phi) = \mathrm{Tr}\phi^3, I_3 = \mathrm{Tr}\phi^4, I_4 = \left(\mathrm{Tr}\phi^2\right)^2,\ d_2 = 3, d_3 = d_4 = 4$.

Denote $\mathcal{M} = \amalg_\ell \mathcal{M}_\ell,\ \mathcal{E} = \amalg \mathcal{E}_\ell$. There is a a characteristic class $c_I(\mathcal{E})$ associated to each invariant $I \in \mathrm{Fun}(\mathbf{g})^{\mathbf{G}}$.

Let Ω_n^α be the slant product $\int_{e_\alpha} c_{I_n}(\mathcal{E}) \in \mathrm{H}^{2d_n - \mathrm{dim} e_\alpha}(\mathcal{M})$.

Definition. The following integral over $\mathcal{M}$ is the attempt to define the intersection theory of Ω_n^α

$$\left\langle \Omega_{n_1}^{\alpha_1} \dots \Omega_{n_k}^{\alpha_k} \right\rangle = \sum_\ell \int_{\mathcal{M}_\ell} \mathcal{O}_{n_1}^{\alpha_1} \wedge \dots \wedge \mathcal{O}_{n_k}^{\alpha_k}$$

• the problem is with the choice of representatives of a cohomology classes on a non-compact manifolds, see Donaldson's papers for $r = 1, n = 1$ case

Definition. The prepotential of the refined Donaldson-Witten theory is the generating function:

$$\mathcal{Z}_A(T) = \left\langle \exp\left(T_\alpha^k \Omega_k^\alpha\right) \right\rangle \equiv$$

$$\sum \frac{1}{k!} T_{\alpha_1}^{n_1} \dots T_{\alpha_k}^{n_k} \left\langle \Omega_{n_1}^{\alpha_1} \dots \Omega_{n_k}^{\alpha_k} \right\rangle$$

Physical Picture

<u>*The fields*</u>: twisted $\mathcal{N} = 2$ vector multiplet

Bosons: gauge field $A = A_\mu dx^\mu$, the complex scalar ϕ and its conjugate $\bar{\phi}$, self-dual two form H

Fermions: the one-form ψ, the scalar η and the self-dual two-form χ.

All fields take values in the adjoint representation.

<u>*Nilpotent Symmetry*</u>:

$$\delta\phi = 0, \quad \delta\bar{\phi} = \eta, \quad \delta\eta = [\phi, \bar{\phi}]$$

$$\delta\chi = H, \quad \delta H = [\phi, \chi]$$

$$\delta A = \psi, \quad \delta\psi = D_A\phi$$

$\delta^2 =$ infinitesimal gauge transformation generated by $\phi \Rightarrow$ nilpotent on the gauge invariant functionals of the fields (equivariant cohomology).

Definition. Observables - gauge invariant functionals of the fields, annihilated by δ.

The correlation functions of observables do not change under a small variation of metric on the four-manifold X.

Observables: Invariant polynomial $\mathcal{P} = \sum_k t^k I_k$ on the algebra $\mathbf{g}$, $C^k, k = 0, \ldots 4$ – closed k-cycles on X. Their homology cycles are denoted as $[C^k] \in \mathrm{H}_k(X; \mathbf{C})$. The observables form the descend sequence:

$$\mathcal{O}^{(0)} = \mathcal{P}(\phi), \quad \delta\mathcal{O}^{(0)} = 0$$

$$d\mathcal{O}^{(0)} = -\delta\mathcal{O}^{(1)} \quad (\mathcal{O}^{(1)}, [C^1]) \equiv \int_{C^{(1)}} \mathcal{O}^{(1)} \equiv \int_{C^1} \frac{\partial\mathcal{P}}{\partial\phi^a}\psi^a$$

$$d\mathcal{O}^{(1)} = -\delta\mathcal{O}^{(2)} \quad (\mathcal{O}^{(2)}, [C^2]) = \int_{C^2} \mathcal{O}^{(2)} =$$

$$\int_{C^2} \frac{\partial\mathcal{P}}{\partial\phi^a}F^a + \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial\phi^a\partial\phi^b}\psi^a \wedge \psi^b$$

$$\ldots$$

top degree observable: $\mathcal{O}^{(4)}_{\mathcal{P}} = \frac{1}{2}\frac{\partial^2\mathcal{P}}{\partial\phi^a\partial\phi^b}F^aF^b +$

$$+\frac{1}{3!}\frac{\partial^3\mathcal{P}}{\partial\phi^a\partial\phi^b\partial\phi^c}F^a\psi^b\psi^c + \frac{1}{4!}\frac{\partial^4\mathcal{P}}{\partial\phi^a\partial\phi^b\partial\phi^c\partial\phi^d}\psi^a\psi^b\psi^c\psi^d$$

Action S equals the sum of the 4-observable, constructed out of the *prepotential* $\mathcal{F}$ and the δ-exact term:

$$S = \mathcal{O}^{(4)}_{\mathcal{F}} + \delta R$$

The standard choice: $\mathcal{F} = \left(\frac{i\theta}{8\pi^2} + \frac{1}{e^2}\right) \mathrm{Tr}\phi^2$,

$$R = \frac{1}{e^2}\mathrm{Tr}\left(\chi F^+ - \chi H + D_A\bar{\phi} \star \psi + \eta \star [\phi, \bar{\phi}]\right),$$

Tr denotes the Killing form.

The bosonic part of the action S is then:

$$S = \int_X \tau \mathrm{Tr} F \wedge F +$$

$$+\frac{1}{e^2}\left(\mathrm{Tr}F \wedge \star F + \mathrm{Tr}D_A\phi \wedge \star D_A\bar{\phi} + \mathrm{Tr}[\phi, \bar{\phi}]^2\right)$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

The e^2-dependence – only via $\delta\left(\ldots\right)$ terms $\Rightarrow$ can take $e^2 \to 0$ limit for correlators of observables: the path integral measure gets localized near solutions to $F^+ = 0$, $D_A\phi = 0$

Moral. **The correlation functions of observables reduce to the integrals over $\mathcal{M}_\ell$.**

• Donaldson theory ($G = SU(2)$ or $G = SO(3)$): aim is to compute:

$$\langle \exp((\mathcal{O}_u^{(2)}, w) + \lambda \mathcal{O}_u^{(0)}) \rangle,$$

for $w \in H^2(X, \mathbf{R})$, $\mathcal{O}_u^{(0)} = u \equiv \mathrm{Tr}\phi^2$,

$$(\mathcal{O}_u^{(2)}, w) = -\frac{1}{4\pi^2} \int_X \mathrm{Tr}(\phi F + \frac{1}{2}\psi\psi) \wedge w$$

• Refinement: generating function of all correlators of all observables:

$$\mathcal{Z}_A(T^k) = \langle e^{T^{k,\alpha}(\mathcal{O}_{I_k}^{(4-d_\alpha)}, e_\alpha)} \rangle$$

$$T^k = T^{k,\alpha} e_\alpha \in \mathcal{V} = \oplus_{p=0}^{4} \mathrm{H}^p(X, \mathbf{C})$$

This is a physical definition of the four dimensional type A theory

Problem. $\mathcal{M}_\ell$ is non-compact. Need to compactify it in order to have a nice intersection theory.

• Donaldson compactification: add point-like instantons as above (for high enough instanton charges get a manifold, perhaps with orbifold singularities)

• For Kähler X a refinement of the compactification above: Gieseker compactification:

Idea: On Kähler X with Kähler form ω :

$$F^+ = 0 \Leftrightarrow \bar{\partial}_A^2 = 0, \quad F \wedge \omega = 0$$

$\bar{\partial}_A$ defines a holomorphic bundle $\mathcal{E}$ over X: its local sections are annihilated by $\bar{\partial}_A$. Then $F \wedge \omega = 0$ is a stability condition.

Replace $\mathcal{E}$ by its (holomorphic) sheaf of sections. Consider the moduli space $\overline{\mathcal{M}}_\ell^G$ of sheaves which are *torsion free* as $\mathcal{O}_X$-modules. The latter has sheaves which are not *locally free*, i.e. which are not holomorphic bundles. However, for each such sheaf $\mathcal{E}'$ there is a zero-dimensional subscheme $Z \subset X$, such that on $X \backslash Z$ $\mathcal{E}'$ **is** a holomorphic bundle and has a connection.

Problem. Find an analogue of Kontsevich compactification.

Problem. Find a physical realization of all these compactifications.

Partial answer to the last problem: On $X = \mathbf{CP}^2$ the compactification by sheaves corresponds to the *gauge theory on a non-commutative space*.

Intersection theory with freckles in four dimensions

Take $X = \mathbf{CP}^2$, $G = U(r)$, w - Kähler form.

$p \in \mathrm{H}^2(X, \mathbf{Z}), k \in \mathrm{H}^4(X, \mathbf{Z})$.

• Monad construction of the torsion free sheaves on X: Let V_0, V_1, V_2 be the complex vector spaces of dimensions $v_{0,1,2}$ respectively. Consider the complex of bundles over X:

$$0 \to V_0 \otimes \mathcal{O}(-1) \xrightarrow{a} V_1 \otimes \mathcal{O} \xrightarrow{b} V_2 \otimes \mathcal{O}(1) \to 0$$

In down-to-earth terms this sequence has the following meaning. The maps a, b in the homogeneous coordinates $(z^0 : z^1 : z^2)$ are the matrix-valued linear functions: $a(z) = z^\alpha a_\alpha, b(z) = z^\alpha b_\alpha$. The words "complex" mean that

$$b(z) \cdot a(z) = z^\alpha z^\beta b_\alpha a_\beta = 0 \Leftrightarrow$$

$$b_\alpha a_\alpha = 0,\ \alpha = 0, 1, 2, \quad b_\alpha a_\beta + b_\beta a_\alpha = 0,\ \alpha \neq \beta$$

For the pair (b, a) of the maps between the sheaves obeying this condition we can define a sheaf $\mathcal{F}$ over X, whose space of sections over an open set U is

$$\Gamma\left(\mathcal{F}|_U\right) = \mathrm{Ker} b(z) / \mathrm{Im} a(z), \quad \text{for} \quad (z^0 : z^1 : z^2) \in U$$

$$\beta^{ij}(z)\Psi^j(z) = 0, \quad \text{modulo} \quad \Psi^j(z) = a^{jk}(z)\tilde{\Psi}^k(z)$$

Definition: The space of monads is the space M_{mon} of triples of matrices $a_\beta \in \mathrm{Hom}(V_0, V_1)$, $b_\alpha \in \mathrm{Hom}(V_1, V_2)$ obeying $b(z)a(z) = 0$. This space is acted on by the group

$$G^c_{\mathrm{mon}} = \left(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)\right) / \mathbf{C}^\star$$

$$(b,a) \mapsto g\cdot(b,a) = (g_2 b g_1^{-1}, g_1 a g_0^{-1}), \text{ for } (g_0,g_1,g_2)\in G^c_{\mathrm{mon}}$$

The sheaves defined by the pairs (b,a) and $g\cdot(b,a)$ are isomorphic. The maximal compact subgroup of G^c_{mon}

$$G_{\mathrm{mon}} \approx \left(U(V_0)\times U(V_1)\times U(V_2)\right)/U(1)$$

acts in M_{mon} preserving its natural symplectic structure

$$\Omega = \frac{1}{2i}\sum_\beta \mathrm{Tr}\delta a_\beta \wedge \delta a^\dagger_\beta + \frac{1}{2i}\sum_\alpha \mathrm{Tr}\delta b^\dagger_\alpha \wedge \delta b_\alpha$$

Fix the real numbers r_0, r_1, r_2, such that $\sum_\alpha v_\alpha r_\alpha = 0$, $r_0, r_2 > 0$. Write the moment maps:

$$\mu_1 = -r_0 \mathbf{1}_{v_0} + \sum_\beta a^\dagger_\beta a_\beta$$

$$\mu_2 = -r_1 \mathbf{1}_{v_1} + \sum_\alpha b^\dagger_\alpha b_\alpha - \sum_\beta a_\beta a^\dagger_\beta$$

$$\mu_3 = -r_2 \mathbf{1}_{v_2} + \sum_\alpha b_\alpha b^\dagger_\alpha$$

Then the moduli space of the semistable sheaves is

$$\overline{\mathcal{M}}_{c_*} = \left(\mu_1^{-1}(0)\cap\mu_2^{-1}(0)\cap\mu_3^{-1}(0)\right)/G_{\mathrm{mon}}$$

The compactness of the space is obvious: if we first perform a reduction with respect to the groups $U(V_0)\times U(V_2)$ then the resulting space is the product of two Grassmanians:

$\mathrm{Gr}(v_0, 3v_1) \times \mathrm{Gr}(v_2, 3v_1)$ which is already compact. The subsequent reduction does not spoil this.

The Chern classes, $c_* = \{r, c_1, c_2\}$, of the sheaf $\mathcal{F}$ determined by the pair (b, a) are:

$$r = v_1 - v_0 - v_2, \; c_1 = (v_0 - v_2)\,\omega, \; c_2 = \frac{1}{2}\left((v_2 - v_0)^2 + v_0 + v_2\right)$$

Let $(i\psi, i\phi, i\chi)$ denote the elements of the Lie algebra of G_{mon}, i.e. $i\psi \in \mathrm{u}(V_0), i\phi \in \mathrm{u}(V_1), i\chi \in \mathrm{u}(V_2)$ and $(\psi, \phi, \chi) \sim (\psi + \mathbf{1}_{v_0}, \phi + \mathbf{1}_{v_1}, \chi + \mathbf{1}_{v_2})$. We are interested in computing certain integrals over $\overline{\mathcal{M}}_{c_*}$. This can be accomplished by computing an integral over M_{mon} with the insertion of the delta function in μ_i and dividing by the volume of G_{mon} provided that the expression we integrate is G_{mon}-invariant:

$$\int_{\overline{\mathcal{M}}_{c_*}} (\ldots) =$$

$$\frac{1}{\mathrm{Vol}(G_{\mathrm{mon}})} \int_{\mathrm{Lie} G_{\mathrm{mon}}} d\psi d\phi d\chi e^{i\mathrm{Tr}\psi\mu_1 + i\mathrm{Tr}\phi\mu_2 + i\mathrm{Tr}\chi\mu_3} (\ldots)$$

The useful fact is that the observables of the gauge theory we are interested in are the gauge-invariant functions on (ψ, ϕ, χ) only. More specifically, there is a *universal sheaf* $\mathcal{U}$ over $\overline{\mathcal{M}}_{c_*} \times X$, defined again as $\mathrm{Ker} b(z)/\mathrm{Im} a(z)$ but now the space of parameters contains (b, a) in addition to z. Its Chern character is given by:

$$Ch(\mathcal{U}) = \mathrm{Tr} e^{\phi} - \mathrm{Tr} e^{\psi - \omega} - \mathrm{Tr} e^{\chi + \omega}$$

In particular:

$$\mathcal{O}^{(0)}_{u_1} = \frac{1}{2}\left(\mathrm{Tr}\chi^2 + \mathrm{Tr}\psi^2 - \mathrm{Tr}\phi^2\right),$$

$$\int_X \omega \wedge \mathcal{O}^{(2)}_{u_1} = \mathrm{Tr}\chi - \mathrm{Tr}\psi$$

Since the observables are expressed through ψ, ϕ, χ only we can integrate out a_β, b_α to obtain:

$$\langle \exp t_1 \mathcal{O}^{(0)}_{u_1} + T_1 \int_S \omega \wedge \mathcal{O}^{(2)}_{u_1} \rangle^{\text{torsion free}} = \oint \prod_{i,j,k} d\psi_i d\chi_j d\psi_k$$

$$\frac{\prod_{i'<i''} \left(\psi_{i'} - \psi_{i''}\right)^2 \prod_{j'<j''} \left(\phi_{j'} - \phi_{j''}\right)^2}{\prod_{i,j} \left(\phi_j - \psi_i + i0\right)^3}$$

$$\frac{\prod_{k'<k''} \left(\chi_{k'} - \chi_{k''}\right)^2 \prod_{i,k} \left(\chi_k - \psi_i\right)^6}{\prod_{j,k} \left(\chi_k - \phi_j + i0\right)^3}$$

$$\times e^{t_1 \frac{1}{2}\left(\sum_k {\chi_k}^2 + \sum_i \psi_i^2 - \sum_j \phi_j^2\right) + T_1\left(\sum_k \chi_k - \sum_i \psi_i\right)} \times$$

$$e^{ir_1 \sum_i \psi_i + ir_2 \sum_j \phi_j + ir_3 \sum_k \chi_k}$$

this integral formula is the four dimensional analogue of the integral formulae of two dimensional sigma models with freckles.

LECTURE 3

FOUR DIMENSIONAL THEORY B

DEFORMATIONS OF COMPLEX LAGRANGIAN SUBMANIFOLDS

General setup. We study holomorphic symplectic manifolds, i.e. complex varieties M^{2r} of complex dimensiuon $2r$ with holomorphic $(2,0)$-form ω such that ω^r is nowhere zero.

"Symplectic " - means "holomorphic symplectic".

"Lagrangian submanifold" = complex subvariety $L^r \subset M^{2r}$ of complex dimension r s. t. $\omega|_L$ vanishes.

Definition. Algebraically integrable system is the quadruple $(\mathcal{V}^{2r}, \omega, B^r, \pi)$ where

• $\mathcal{V}^{2r}$ is an algebraic variety over $\mathbf{C}$ of dimension $2r$;

• ω is a symplectic form on $\mathcal{V}^{2r}$;

• B^r is an algebraic variety of dimension r;

• $\pi : \mathcal{V} \to B$ is the projection, whose fibers are Lagrangian with respect to ω (i.e. $\omega|_{\pi^{-1}(u)} = 0$ for any $u \in B$) and are in addition polarized abelian varieties (this means that every fiber has a distinguished $(1,1)$ cohomology class t which is also integral).

$$\text{For } u \in B \text{ let } J_u = \pi^{-1}(u).$$

BASIC EXAMPLE

(S, ω_S) - a symplectic surface (e.g. $S = T^*\Sigma$, where Σ is an algebraic curve, or S can be a K3 surface);

$\beta \in \mathrm{H}_2^{BM}(S, \mathbf{Z})$ (Borel-Moore homology) - a two-cycle represented by a algebraic curve.

$\mathcal{M}_{S,\beta}$ - space of pairs (C, L);

C - a smooth curve in S whose homology class equals β;

L - a degree h line bundle on C.

h - the genus of C, which depends only on β (for example $h = 1 + \beta \cdot \beta$ for compact S).

$B_{S,\beta}$ - the space of smooth compact curves $C \subset S$ whose homology class equals β.

$\pi : \mathcal{M}_{S,\beta} \to B_{S,\beta}$ - the projection forgetting the line bundles.

Lemma. The space $\mathcal{M}_{S,\beta}$ has a natural symplectic form ω. The quadruple $(\mathcal{M}_{S,\beta}; \omega; B_{S,\beta}; \pi)$ is algebraically integrable system.

Proof. Fix the curve C. Let $i : C \to S$ be the embedding. Notice that it is Lagrangian with respect to ω_S.

Normal bundle NC to the curve C in S – canonically $\approx T^*C$.

Follows from the exact sequence of holomorphic bundles:

$$0 \to TC \to TS|_C \to T^*C \to 0$$

second arrow: i_* - differential of the map i;

third arrow: $v \mapsto i^* \iota_v \omega_S \in T^*C$.

Tangent space $T = T_{(C,L)}\mathcal{M}_{S,\beta}$ at (C, L) fits into the exact sequence:

$$0 \to V^* \to T \to V \to 0$$

$V = \mathrm{H}^0(C, NC) \approx$ tangent space to $B_{S,\beta}$,

$V^* \approx$ tangent space to the Jacobian of C:
$\mathrm{H}^1(C, \mathcal{O}_C) \approx \mathrm{H}^0(C, K_C)^*$ (Serre duality), $K_C = T^*C$.

Canonical pairing $V \times V^* \to \mathbf{C}$ induces symplectic form ω on T. Restriction of ω on the fiber of π is zero. By construction the fiber (Jacobian of C) is a polarized abelian variety.

Moreover, ω is closed. Darboux coordinates: choose a set of A-cycles $\sigma_i \in \mathrm{H}_1(C, \mathbf{Z}), i = 1, \ldots, h$, they define a set of h closed one-forms on $B_{S,\beta}$:

$$da^i = \oint_{\sigma_i} \omega_S$$

The same set of A-cycles define a set of h closed one-forms on the Jacobian $\mathrm{Jac}(C)$ of C: let $\varpi_i \in \mathrm{H}^0(C, K_C)$ be the basis in the space of holomorphic differentials on C which are normalized as:

$$\oint_{\sigma_j} \varpi_i = \delta_{ij};$$

define $d\varphi_i \in T^*\mathrm{Jac}(C)$ as follows: for $\xi \in \mathrm{H}^0(C, K_C)^*$

$$d\varphi_i(\xi) = \varpi_i(\xi)$$

It is easy to check that

$$\omega = \sum_{i=1}^{h} da^i \wedge d\varphi_i$$

The lemma is proved.

SECONDARY INTEGRABLE SYSTEM

Consider an algebraic integrable system. Suppose that the generic fiber $J_u = \pi^{-1}(u), u \in B$ is compact.

Let $\Sigma \subset B$ be the setof $u \in B$, s.t. J_u is singular or non-compact. $\mathcal{L}$ - the universal cover of $B - \Sigma$, and $\tilde{\pi} : \mathcal{L} \to B$ - the projection.

Choose a basepoint $p_0 \in \mathcal{L}$. Let $u_0 = \tilde{\pi}(p_0) \in B - \Sigma$, $W_{\mathbf{Z}} = \mathrm{H}^1(\pi^{-1}(u_0), \mathbf{Z})$, $W_{\mathbf{C}} = W_{\mathbf{Z}} \otimes \mathbf{C}$.

Lemma. $W_{\mathbf{C}}$ is a symplectic vector space.

Proof. Consider the class $\left[t^{r-1}\right]$ of the fiber $\pi^{-1}(u_0)$. By Poincare duality it determines a class $t_* \in \mathrm{H}_2(\pi^{-1}(u_0), \mathbf{C})$. Define the symplectic form Ω on $W_{\mathbf{C}}$ as follows: for $\alpha, \beta \in W_{\mathbf{C}}$

$$\Omega(\alpha, \beta) = \int_{t_*} \alpha \wedge \beta$$

It is obviously non-degenerate.

Let Γ be the image of $\pi_1(B - \Sigma, u_0)$ in the symplectic group $Sp(W_{\mathbf{Z}})$ under the monodromy map.

Theorem. There exists a canonical embedding $\rho : \mathcal{L} \to W_{\mathbf{C}}$, whose image $\mathbf{L} = \rho(\mathcal{L})$ is

a) Lagrangian with respect to Ω;

b) Γ-invariant.

Proof. Consider a flat vector bundle W over $\mathcal{L}$, whose fiber over $p \in \mathcal{L}$ is $\mathrm{H}^1(\pi^{-1}(\tilde{\pi}(p)), \mathbf{Z}) \otimes \mathbf{C}$.
- $\mathcal{L}$ is simply-connected $\Rightarrow$ the bundle W is trivial.
- The choice of p_0 identifies W with $\mathcal{L} \times W_{\mathbf{C}}$.
- Let $W'_{\mathbf{Z}} = \mathrm{H}_1(\pi^{-1}(u_0), \mathbf{Z})$.
- For $p \in \mathcal{L}$ we identify $\mathrm{H}_1(\pi^{-1}(\tilde{\pi}(p)), \mathbf{Z})$ with $W'_{\mathbf{Z}}$.

Define ρ: $\rho(p)$ is the element of $W_{\mathbf{C}}$ whose value on the element $\sigma \in W'_{\mathbf{Z}}$ is equal to:

$$\rho(p)[\sigma] = \int_{\gamma_{p_0}^{p} \times \sigma} \omega$$

where $\gamma_{p_0}^{p}$ is any path connecting p_0 and p. The property a) of ρ follows from symmetricity of the period matrix of abelian variety, the property b) follows from the definition of Γ.

Let α_i, β^j, $i = 1, \ldots, r$ be a canonical (up to the action $Sp(W_{\mathbf{Z}})$ basis in $W_{\mathbf{Z}}$ (with respect to the intersection form $\int_{t_*} \alpha \wedge \beta$). It determines distinguished (again up to $Sp(W_{\mathbf{Z}})$) Darboux coordinates $a^i, a_{D,i}$, $1 = 1, \ldots, r$ on $W_{\mathbf{C}}$:

$$da^i = \oint_{\alpha_i} \omega, \quad da_{D,i} = \oint_{\beta^i} \omega$$

Let $\theta = a_{D,i} da^i$ be one-form on $W_{\mathbf{C}}$ such that $d\theta = \Omega$.

• This form is not invariant under the action of $Sp(W_{\mathbf{Z}})$, but the form: $\tilde{\theta} = \theta - \frac{1}{2} d \sum_{i=1}^{r} \left(a^i a_{D,i} \right)$ is.

Definition. On $\mathbf{L}$ there is a well-defined *Generating function* $\mathcal{F}_0$, such that $d\mathcal{F}_0 = \sum_i a_{D,i} da^i|_{\mathbf{L}}$, $\mathcal{F}_0(\rho(p_0)) = 0$. Locally $\mathcal{F}_0$ can be viewed as a function on a^i.

Consider the space $\mathcal{S}$ of formal Γ-invariant deformations of $\mathbf{L}$ leaving it Lagrangian.

THE SECONDARY SYSTEM, associated to the original algebraic integrable system governs the formal deformations of $\mathbf{L}$ in the class of Γ-invariant Lagrangian submanifolds and the special coordinates on the space $\mathcal{S}$.

Theorem. The tangent space to the space $\mathcal{S}$ of such deformations is the space $\mathbf{T}$ of Γ-invariant exact one-forms on $\mathbf{L}$.

Proof. The tangent space to the space of all deformations is the space of the holomorphic sections v of the normal bundle $N\mathcal{L}$ to $\mathcal{L}$. The latter is the quotient of the restriction $T\mathbf{C}^{2r}|_{\mathcal{L}}$ of the tangent bundle $T\mathbf{C}^{2r}$ to $\mathcal{L}$ by the tangent bundle of $\mathcal{L}$.

Claim: $N\mathcal{L} \approx T^*\mathcal{L}$. Indeed, the following sequence is exact:

$$0 \to T\mathcal{L} \to T\mathbf{C}^{2r}|_{\mathcal{L}} \to T^*\mathcal{L} \to 0$$

the second arrow is the natural embedding,
the third arrow is the map which sends $v \in \Gamma\left(T\mathbf{C}^{2r}|_{\mathcal{L}}\right)$ $\iota_v\omega \in \Gamma\left(T^*\mathcal{L}\right)$.

The sequence is exact $\Leftrightarrow$ $\mathcal{L}$ – Lagrangian.

- v determines a Lagrangian deformation of $\mathcal{L} \Rightarrow d\iota_v\omega = 0$.
  For simply-connected $\mathcal{L} \Rightarrow \iota_v\omega = df_v$.
- Deformed $\mathcal{L}$ – Γ-invariant $\Rightarrow$ df_v is Γ-invariant.
- In particular, Γ-invariant **functions** u on $\mathbf{L}$

  determine infinitesimal deformations of $\mathbf{L}$.

Physical Picture

The physical arena for the constructions above is the four dimensional $\mathcal{N}=2$ supersymmetry.

Fields: r abelian twisted $\mathcal{N}=2$ vector multiplets:

bosons: a^i - complex scalar, H_i - self-dual two-form, $A^i = A^i_\mu dx^\mu$ $U(1)$-gauge field;

fermions: ψ^i-one-form, χ_i - self-dual two-form, $\eta^{\bar{i}}$ - scalar

Nilpotent symmetry: $\delta A^i = \psi^i, \quad \delta\psi^i = da^i \quad \delta a^i = 0,$

$$\delta\bar{a}^{\bar{i}} = \eta^{\bar{i}}, \quad \delta\eta^{\bar{i}} = 0, \quad \delta\chi_i = H_i, \quad \delta H_i = 0$$

Just like in two dimensions

Observables: are identified with the deformations of the theory. 0-observables: local functionals of the fields, annhilated by δ. Higher observables are the functionals of the fields, annihilated by δ, taking values in forms on X. The deformation of the action is achieved by means of 4-observables.

Action:

$$S = \int_X \frac{i}{4}\mathcal{O}^{(4)} + \delta R_0$$

again a sum of the 4-observable, constructed out of the holomorphic function $\mathcal{F}(a)$:

$$\mathcal{O}^{(4)}_{\mathcal{F}} = \frac{1}{2}\tau F\wedge F + \frac{1}{2}\frac{\partial\tau}{\partial a}F\psi^2 + \frac{1}{24}\frac{\partial^2\tau}{\partial a^2}\psi^4 + FF_D$$

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j},$$

we write $F_D = dA_D$ in order to stress the fact that F_D may be closed, but not exact form with integral periods,

and a δ-exact term δR_0, which would enforce electric-magnetic duality, discussed below:

$$R_0 = \tau_2 \left(\chi(F^+ - H) + d\bar{a} \star \psi\right) + \frac{1}{2}\frac{d\tau_2}{da}\psi^2\chi + \frac{1}{6}\frac{d\tau_2}{d\bar{a}}\chi^3$$

Expanding $\delta(\ldots)$ out we get:

$$\begin{aligned} L = &\frac{i}{8}\tau F^2 + FF_D + \tau_2\left(H(F^+ - H) + da \star d\bar{a}\right) + \\ &+ \tau_2\left(\chi(d\psi)^+ + \eta d^*\psi\right) + \\ &+ \frac{i}{8}\frac{d\tau}{da}F\psi^2 + \frac{d\tau}{da}\chi(da \wedge \psi) + H\left(\frac{d\tau_2}{d\bar{a}}(\frac{1}{2}\chi^2 + \chi\eta) + \frac{1}{2}\frac{d\tau_2}{da}\psi^2\right) \\ &+ \frac{i}{96}\frac{d^2\tau}{da^2}\psi^4 - \frac{1}{2}\frac{d\log\tau_2}{d\bar{a}}\frac{d\tau}{da}\chi\eta\psi^2 - \frac{1}{12}\frac{d^2(\tau_2^{-2})}{d\bar{a}^2}\eta(\tau_2\chi)^3 \end{aligned}$$

Gaussian integration over H gives:

$$H = \frac{1}{2}F^+ + \frac{1}{\tau_2}\left(\frac{d\tau_2}{d\bar{a}}(\frac{1}{2}(\chi^2)^+ + \chi\eta) + \frac{d\tau_2}{da}(\psi^2)^+\right)$$

and

$$\begin{aligned} -i\mathcal{L} = &\frac{1}{2}(\tau(F^-)^2 - \bar{\tau}(F^+)^2) + \tau_2(\chi(d\psi)^+ + \eta d^*\psi + da \star d\bar{a}) \\ &+ \frac{1}{2}\frac{d\tau}{da}F(\psi^2)^- + \frac{d\tau}{da}\chi(da \wedge \psi) + FF_D + \\ &+ F^+\frac{d\tau_2}{d\bar{a}}(\frac{1}{2}(\chi^2)^+ + \chi\eta) + \ldots \end{aligned}$$

where $\ldots$ denote the quartic fermionic terms.

Electric-magnetic duality

The rôle of the discrete group Γ is very important. It reflects the electric-magnetic duality of the gauge fields in four dimensions.

Maxwell equations. A-gauge field, $F = dA$ - curvature.

$$dF = 0, \quad d \star F = 0$$

The equations are invariant under the following symmetry:

$$F \leftrightarrow \star F$$

Literally does not quite make sense – F must be integral $\in \mathrm{H}^2(X, 2\pi i \mathbf{Z})$, while $\star F$ needs not. Nevertheless, look at the *canonical approach.*

Classical story

• Space-time $X = M^3 \times \mathbf{R}^1$, M^3 - Riemannian three-dimensional manifold.

• Vector space $\mathbf{t} \approx \mathbf{R}^r$, lattice $\Lambda \subset \mathbf{t}$, $\Lambda \approx \mathbf{Z}^r$, torus $\mathbf{T} = \mathbf{t}/\Lambda$.
Let $e_1, \ldots, e_r \in \mathbf{t}$ be the basis in Λ and in $\mathbf{t} = \Lambda \otimes \mathbf{R}$.

Notation: $\Omega^i(M^3, \mathbf{t})$ - $\mathbf{t}$-valued i-forms on M^3
$\Omega^i_\Lambda(M^3, \mathbf{t})$ - $\mathbf{t}$-valued i-forms on M^3 whose periods belong to Λ.

• Phase space $\mathcal{X}$ = set of pairs: (F, E),
$F = F^i e_i \in \Omega^2_\Lambda(M^3, \mathbf{t})$, $E \in \Omega^2(M^3, \mathbf{t}^*)$

• $\mathcal{X}$ = cotangent bundle to the space of connections A in all $\mathbf{T}$-bundles over M^3.

• Choose a metric g_{ij} and a symmetric pairing θ_{ij} on $\mathbf{t}$ - *couplings*.

• Symplectic form on $\mathcal{X}$: $\Omega = \int_{M^3} \delta A^i \wedge \delta E_i + \theta_{ij} \delta A^i \wedge d\delta A^j$ where $d\delta A = \delta F$, and we use the canonical pairing between E and A.

• Hamiltonian: $\mathbf{H} = \int_{M^3} \frac{1}{2} g_{ij} dA^i \wedge \star dA^j + \frac{1}{2} g^{ij} E_i \wedge \star E_j$ where we used the metric g^{ij} on $\mathbf{t}^*$ induced from $(,)$.

• Gauge group $\mathcal{G} \approx \Omega^1_\Lambda(M^3, \mathbf{t})$ acts on $\mathcal{X}$ symplectically:

$$E \mapsto E, \quad A \mapsto A + \ell, \quad \ell \in \Omega^1_\Lambda(M^3, \mathbf{t})$$

• Exact sequence: $\mathcal{G}_p \to \mathcal{G} \to \mathrm{H}^1(M^3, \Lambda)$, with $\mathcal{G}_p \approx \mathrm{Maps}(M^3, \mathbf{T})$: where the first arrow is the map $\varphi \mapsto d\varphi$ and the second arrow is $\ell \mapsto \mathbf{l} = [\ell] \in \mathrm{H}^1(M^3, \Lambda)$.

• The moment map takes values in $\mathrm{Lie}^*\mathcal{G}_p$: $\mu = dE$

• The reduced phase space $\mathcal{P} = \mu^{-1}(0)/\mathcal{G}$.

Quantization of the Maxwell Theory

♠ Quantize $\mathcal{P}$ = Quantize $\mathcal{X}$ and then impose the gauge invariance.

♢ Quantized $\mathcal{X}$ = the space $\mathcal{H}_{M^3}$ of functionals Ψ on $\Omega^2_\Lambda(M^3, \mathbf{t})$.

Exact sequence:

$$\Omega^1(M^3, \mathbf{t}) \to \Omega^2_\Lambda(M^3, \mathbf{t}) \to \mathrm{H}^2(M^3, \Lambda)$$

the first arrow: $A \mapsto dA$, the second: $F \mapsto [F] \in \mathrm{H}^2(M^3, \Lambda)$.
♠ Hence the functional Ψ on $\Omega^2_\Lambda(M^3, \mathbf{t})$ = a collection of the functionals:

$$\Psi(F) = \{\Psi_{\mathbf{m}}(A)\}, \quad A \in \Omega^1(M^3, \mathbf{t}), \quad \mathbf{m} \in \mathrm{H}^2(M^3, \Lambda)$$

♣ The $\mathcal{G}$ invariance of Ψ:

$$\Psi_{\mathbf{m}}(A + \ell) = \exp 2\pi i \theta_{ij} (\mathbf{l}^i, \mathbf{m}^j)\, \Psi_{\mathbf{m}}(A)$$

where $(,)$ denotes the intersection pairing in $H^*(M^3, \mathbf{R})$.
♢ The function E on $\mathcal{X}$ becomes an operator in $\mathcal{H}_{M^3}$:

$$E_i \mapsto \hat{E}_i = -i \frac{\delta}{\delta A^i} + \theta_{ij} F^j$$

♠ In the sector $\mathbf{m}$: $A = A_0 + \alpha$, where

• A_0 is a $\mathbf{T}$-connection whose curvature $F_0 = dA_0$ is harmonic: $d \star F_0 = 0$; $[F_0] = \mathbf{m} \in \mathrm{H}^2(M^3, \mathbf{t})$, $\alpha \in \Omega^1(M^3, \mathbf{t})$,

$$\int_{M^3} \alpha^i \wedge \star H_i = 0$$

for any $H_i \in \Omega^2(M^3, \mathbf{t}^*)$, $\quad dH_i = d \star H_i = 0$. Two choices of A_0 differ by an element of $\mathrm{H}^1(M^3, \mathbf{t})$.

• Under the action of $\mathcal{G}$ A_0 is transformed by the shifts by $\mathbf{l} \in \mathrm{H}^1(M^3, \Lambda)$, while $\alpha \mapsto \alpha + d\varphi$, $\varphi \in \mathcal{G}_p$.

• $\Psi_{\mathbf{m}}(A) = \psi_{\mathbf{m}}(A_0)\Psi(\alpha)$:

$$\psi_{\mathbf{m}}(A_0 + \mathbf{l}) = \exp 2\pi i \theta_{ij} (\mathbf{l}^i, \mathbf{m}^j) \psi_{\mathbf{m}}(A_0)$$

$$\Psi(\alpha + d\varphi) = \Psi(\alpha)$$

The Hilbert space $\mathcal{H}_{M^3}$ splits as an infinite direct sum:

$$\mathcal{H}_{M^3} = \bigoplus_{\mathbf{m} \in \mathrm{H}^2(M^3, \Lambda), \mathbf{m}^* \in \mathrm{H}^2(M^3, \Lambda^*)} \mathcal{H}_{M^3}[\mathbf{m}, \mathbf{m}^*]$$

where $\mathcal{H}_{M^3}[\mathbf{m}, \mathbf{m}^*] = \bigotimes$

$\diamondsuit$ of the one-dimensional space of the sections of a trivial $U(1)$-line bundle over the torus

$$\mathrm{H}^1(M^3, \mathbf{t})/\mathrm{H}^1(M^3, \Lambda)$$

of the form: $\exp 2\pi i \left(-\mathbf{m}^*_i + \theta_{ij}\mathbf{m}^j, A^i_0\right)$

$\diamondsuit$ and the space $\mathcal{F}$ of functionals $\psi([\alpha])$ on $\Omega^1(M^3, \mathbf{t})/d\Omega^0(M^3, \mathbf{t})$.

The Hamiltonian $\mathbf{H}$ acts in $\mathcal{H}$ preserving the spaces $\mathcal{H}_{M^3}[\mathbf{m}, \mathbf{m}^*]$:

$$\mathbf{H}|_{\mathcal{H}_{M^3}[\mathbf{m},\mathbf{m}^*]} =$$

$$= \frac{1}{2} g_{ij} \langle \mathbf{m}^i, \mathbf{m}^j \rangle + \frac{1}{2} g^{ij} \langle \mathbf{m}^*_i - \theta_{ik}\mathbf{m}^k, \mathbf{m}^*_j - \theta_{jl}\mathbf{m}^l \rangle +$$

$$+\tilde{H}|_{\mathcal{F}}$$

where

$$\tilde{H}|_{\mathcal{F}} =: \left(-i\frac{\delta}{\delta\alpha^i} + \theta_{ij} d\alpha^j\right)^2 + g_{ij} d\alpha^i \wedge \star d\alpha^j \; :$$

and we denoted by $\langle \cdot, \cdot \rangle = (\cdot, \star \cdot)$ the pairing in cohomology induced from the metric on M^3.

Duality, at last

♠ The space $\mathcal{T} = \mathbf{t} \oplus \mathbf{t}^*$ is a symplectic vector space.

• The group $\mathbf{\Gamma}$ =Sp$(2r, \mathbf{Z})$ acts there
preserving the lattice $\bigwedge = \Lambda \oplus \Lambda^*$.

♣ This action can be extended to the action of $\mathbf{\Gamma}$ in $\mathcal{H}$. The obvious action on $[\mathbf{m}, \mathbf{m}^*]$ is supplemented by the non-trivial *Bogolyubov transform* on $\mathcal{F}$.

♢ The latter is obtained by quantizing the infinite-dimensional space $\tilde{\mathcal{X}} = \Omega^1(M^3, \mathcal{T})/d\Omega^0(M^3, \bigwedge)$ on which $\mathbf{\Gamma}$ acts preserving its symplectic form.

• The $\mathbf{\Gamma}$ action on $\mathcal{H}$ transforms the **couplings**: introduce the matrix $\tau_i j = \theta_{ij} + i g_{ij}$ of an operator $\tau : \mathbf{t} \to \mathbf{t}^*$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau = (C\tau + D)^{-1} (A\tau + B)$$

Supersymmetry

Relates the scalars a^i to the gauge field A^i. Also the couplings g_{ij}, θ_{ij} are not constant but rather depend on a in a peculiar way:

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}$$

where $\mathcal{F}$ is holomorphic.

♢ The electric-magnetic duality acting on the gauge fields extends to the action of (a subgroup of, in general) $\mathrm{Sp}(2r, \mathbf{Z})$ on the scalars a^i.

♠ This action transforms the couplings τ_{ij} as before and therefore transforms $\mathcal{F}$. It turns out that the geometric meaning of these transformations is:

Claim. $\mathcal{F}$ is a generating function of a Lagrangian submanifold $\mathcal{L}$ in $\mathbf{C}^{2r}$ invariant under a subgroup Γ of $\mathrm{Sp}(2r, \mathbf{Z})$. The four dimensional fields are the (super)maps of ΠTX into $\Pi T\mathcal{L}$.

♣ The gauge fields arise as particular components of these supermaps. Other components are the fermions, auxilliary fields and so on.

• Just like in two dimensions, the correlators of the observables reduce to the integrals over the target space $\mathcal{L}/\Gamma$.

♢ For $r = 1$ the typical subgroups $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ are $\Gamma(2)$ and $\Gamma^0(4)$.

Periodic Toda system.

The theory B connected with the theory A which we described earlier revolves around the following algebraically integrable system:

♠ Base B is the space $\mathbf{C}^r$ of hyperelliptic curves $\mathcal{C}_u$ of the form:

$$z + \frac{1}{z} = P_u(x) \equiv x^{r+1} + u_1 x^{r-1} + \dots u_r$$

♣ Fiber J_u over a point $u = (u_1, \dots, u_r)$ is the Jacobian of $\mathcal{C}_u$.

• Let $\Delta(u)$ be the discriminant of the polynomial $P_u^2(x) - 4$. Let $\Sigma = \Delta^{-1}(0) \subset B$.

Theorem. The space of pairs $(\mathcal{C}_u, L_u)$, where $u \in B - \Sigma$, $L_u \in J_u$ is an algebraically integrable system.

Proof. We can view the curves $\mathcal{C}_u$ as compact algebraic curves embedded in $S = T^*\mathbf{CP}^1$ by rewriting the equation of the curve in the homogeneous form:

$$z_0^2 + z_1^2 - z_0 z_1 P_u(x) = 0$$

The symplectic form ω_S is equal to $\frac{1}{2\pi i} dx \wedge \frac{dz}{z}$. We are in the situation of the lemma from the basic example where the homology class β is equal to

$$\beta = (r+1)\left[\{x = 0\}\right] + \left[\{z_0 = 0\}\right] + \left[\{z_1 = 0\}\right]$$

Lemma. In this example the map ρ can be written explicitly as:

$$a^i(p) = \frac{1}{2\pi i}\oint_{\alpha_i} x\frac{dz}{z}, \quad a_{D,i}(p) = \frac{1}{2\pi i}\oint_{\beta^i} x\frac{dz}{z},$$

where now α_i and β^i denote A- and B-cycles on the curve $\mathcal{C}_u$ defined as follows:

Let $x_i^{\pm}$ be the roots of the equation $P_u(x^{\pm}) = \pm 2$. Of course there is no natural ordering for x_i^+'s and x_i^-'s, so our construction is canonical up to the action of $W \times W$:
$\diamondsuit$ the cycle β^i is represented by the curve surrounding the cut in the x-plane going from x_{i+1}^+ to x_i^+,
$\diamondsuit$ $\alpha_i = e_{i+1} - e_i$, e_i is the path going from x_i^- to x_i^+, $i = 1, \ldots, r$.

• It is clear from the construction that the monodromy around the locus where at least one $a^i \to \infty$ generates the subgroup of Γ isomorphic to W.

Special coordinates on $\mathcal{S}$

Strategy. For any Γ-invariant Lagrangian submanifold $\mathbf{L}_t$ of $\mathbf{C}^{2r}$ which is sufficiently close to $\mathbf{L} \equiv \mathbf{L}_0$ define a distinguished basis f_k^t in the space $\mathbf{T}_t$ of Γ-invariant functions. Then the special coordinates T_k and the deformed generating function $\mathcal{F}(a,T)$ as a function of a^i and special coordinates are defined by the partial differential equations:

$$\frac{\partial \mathcal{F}(a,T)}{\partial T_k} = f_k^{t(T)}(a)$$

Conditions on f_k^t

1. f_k^t extends to a Γ-equivariant holomorphic function in the neighbourhood of $\mathbf{L}_t$ in $\mathbf{C}^{2r}$;

2. as $a^i \to \infty$ f_k^t can be viewed as a function of a^i. Then $f_k^t(sa^1, \ldots, sa^r) = s^{d_k} I_k(a) + \mathrm{o}(s^{-1})$ for $s \to \infty$;

Conjecture. These conditions are sufficient for determining T_k.

At the moment we can prove that the conditions above define the basis f_k^t unambiguously at least in the case where $d_k - 2 < 2h$.

Integrability

The system of equations defining T_k is integrable and generalizes to higher dimensions the Whitham hierarchy.

• Let us assign to the special coordinates T_k degree $d_k - 2$, and to a_i degree zero.

♢ One can show that the definition of the special coordinates agrees with the homogeneity properties of the prepotential $\mathcal{Z}_A(T)$, and that it predicts correct terms (determined by blowup arguments) in $\mathcal{F}_t(T)$ whose total degree does not exceed $2h$.

♣ To prove our conjecture one has to show that the special coordinates defined above do realize the four dimensional mirror symmetry described in the next lecture.

LECTURE 4

FOUR DIMENSIONAL MIRROR SYMMETRY

AND EXAMPLES

♠ Assume that we are given Γ-invariant deformed Lagrangian submanifold $\mathbf{L}_t \subset \mathbf{C}^{2r}$ of the type described in the previous lecture.

♢ Take its Zariski closure in $\mathbf{C}^{2r}$, $\bar{\mathbf{L}}_t$. It is Γ-invariant.

♣ Denote by L_t the quotient $\bar{\mathbf{L}}_t/\Gamma$ and by $\Sigma_t = \left(\bar{\mathbf{L}}_t \backslash \mathbf{L}_t\right)/\Gamma$.

♡ For a 4-fold X let $\mathbf{l}_t(X)$ denote the supermanifold: $\mathbf{l}_t(X) = \left[\Pi \mathcal{T}_{L_t} \otimes \mathrm{H}^1(X, \mathbf{R})\right] \times \mathrm{H}^2(X, \Lambda)$, fibered over L_t.

Let $\mu_X(t)$ be a measure on $\mathbf{l}_t$ which is the sum

• of the "bulk" term

• and the "boundary " Seiberg-Witten contributions of the discriminant loci.

♠ Both will be described below

Then $4d$ mirror is the equality:

4d mirror formula

$$\mathcal{Z}_A(T^k_\alpha) = \int_{\mathbf{l}_t(X)} \mu_X\left(t\left(T^k_\alpha e^\alpha\right)\right)$$

Bulk contribution to $\mu_X(t)$

• Let ψ denote the (fermionic) coordinate on $\Pi \mathrm{H}^1(X, \mathbf{t})$ (= the fiber of $\Pi \mathcal{T}_{L_t} \otimes \mathrm{H}^1(X, \mathbf{R})$), and $\lambda \in \mathrm{H}^2(X, \Lambda)$. Then

$$\mu_X(t) = \mathcal{D}a\mathcal{D}\psi\, \Delta(t)^{\frac{\sigma}{8}} \varpi(t)^{\frac{\chi}{2}} \exp\left(\int_X \mathcal{F}_t(a + \psi + \lambda) + \bar{\partial}(\mathcal{R})\right)$$

• ϖ - ratio of a suitably transported (from $t = 0$) r-form on L_t to the r-form $\mathcal{D}a \equiv da^1 \wedge \ldots \wedge da^r$,

• $\Delta(t)$ - function on L_t whose divisor of zeroes is Σ_t and has the same asymptotics as $a^i \to \infty$ as Δ.

• The form $\mathcal{R}$ can be written given $\mathcal{F}_t$. One does not need the explicit form of $\mathcal{R}$ if the measure μ_X is considered as a holomorphic top form which is to be integrated over a $(r|rb_1)$- dimensional submanifold of $\mathbf{l}_t(X)$

Seiberg-Witten contributions

- to $\mu_X(t)$: involve Parshin residues at Σ_t of the form

$$\mathcal{D}a\mathcal{D}\psi \left(\frac{\Delta(t)}{\prod_i a^i}\right)^{\frac{\sigma}{8}} \varpi(t)^{\frac{\chi}{2}}$$

$$\sum_\lambda \int_{\mathcal{M}_{SW}(\lambda)} \frac{1}{\prod_i (a^i + c_1(\mathcal{L}_i))} \exp(\int_X \widetilde{\mathcal{F}}_t(a+\psi+\lambda))$$

- "renormalized generating function" : $\widetilde{\mathcal{F}} = \mathcal{F} - \sum_i \frac{1}{2}\left(a^i\right)^2 \log a^i$

♠ The space $\mathcal{M}_{SW}(\lambda)$ is the moduli space of solutions to the generalized Seiberg-Witten equations:

1. $F_A^+ = \bar{M}\Gamma M$
2. $DM = 0$

- A - a connection in the $\mathbf{T}$ bundle $\tilde{\mathcal{L}}$ (actually, $Spin_c \otimes \mathbf{T}$ structure) over X with $c_1 = \lambda$,

- M - a section of $S_+ \otimes \tilde{\mathcal{L}}$,

- $\Gamma : S_+ \otimes S_+ \to \Lambda^{2,+}T^*X$ is the intertwiner, and the solutions are identified if they differ by a gauge transformation.

- $\mathcal{L}_i$ is the $U(1)$ bundle over $\mathcal{M}_{SW}(\lambda)$ which consists of all the solutions to the equations above up to the gauge transformations whose i'th $U(1)$ part is identity at some marked point $x \in X$.

EXAMPLES

Different X's, different $\mathbf{G}$'s.....

Answers on the A side, answers on the B side...

Comparison with the two dimensional mirror symmetry....

If $b_2^+(X) > 1$ then the **bulk** contribution vanishes

If X supports a metric of positive scalar curvature then **boundary** contribution vanishes.

$X = \mathbf{S}^2 \times \mathbf{S}^2$, $\mathbf{G} = SU(2)$

♠ Let us denote by $u = -\frac{1}{8\pi^2}\mathrm{Tr}\phi^2$ (recall the notations from the lecture **2**).

♢ $\mathrm{H}^*(X, \mathbf{R}) = \mathbf{R}^4$, with basis
$e_0 = 1, e_1 = W\left(\mathbf{S}^2_1\right), e_2 = W\left(\mathbf{S}^2_2\right), e_3 = e_1 e_2 = W(pt)$

Specialization of the 4d mirror formula to this case

$$\langle \exp\left(T^3_1 u + \int_{\mathbf{S}^2_2} T^1_1 \mathcal{O}^{(2)}_u + \int_{\mathbf{S}^2_1} T^2_1 \mathcal{O}^{(2)}_u + T^0_1 \int_X \mathcal{O}^{(4)}_u \right)\rangle =$$

$$= \oint \sum_{N \in \mathbf{Z}} \frac{(du)^2}{N da + T^1_1 du} e^{T^1_1 T^2_1 G(u) + T^3_1 u}$$

• the contour is around $u = \infty$,

$$a(u) = \int_{-\Lambda}^{\Lambda} dx \frac{\sqrt{x-u}}{\sqrt{x^2 - \Lambda^4}} = \sqrt{u} + \ldots, \quad u \to \infty$$

• $\Lambda = \exp T^0_1, \qquad G(u) = a\frac{du}{da} - 2a$

♠ The asymmetry between T^1_1 and T^2_1 in this case is a reflection of the **non-invariance** of Donaldson invariants under the changes of metric in the $b^+_2(X) = 1$ case: one must specify the relative position of the lattice $\mathrm{H}^2(X, \mathbf{Z})$ and the real line $\mathrm{H}^{2,+}(X)$ (**period point**) - we take $\mathbf{S}^2_1 \ll \mathbf{S}^2_2$.

♢ The formula agrees with the computations of Göttche and Zagier, Moore and Witten.

$X = K3$, $\mathbf{G} = SU(2)$

- $\mathrm{H}^*(X, \mathbf{R}) = \mathbf{R}^{24}$, with the basis:

$e_0 = 1, e_{24} = W(pt), \gamma_i = W(\Sigma_i) \in \mathrm{H}^2(X, \mathbf{Z}), i = 1, \ldots, 23$

$$\langle \exp\left(T_1^{24} u + \frac{1}{2} \int_{\Sigma_i} T_1^i \mathcal{O}_u^{(2)} + T_1^0 \int_X \mathcal{O}_u^{(4)} \right) \rangle =$$

$$2 \mathrm{cosh} \Lambda^2 \left(T_1^{24} + \frac{1}{2} \sum_{i,j} T_1^i T_1^j (\gamma_i, \gamma_j) \right)$$

♢ in agreement with the results of Kronheimer and Mrowka.

♡ In this case the **bulk** contribution vanishes while the **boundary** contribution is non-trivial only for $\lambda = 0$.

$X = \mathbf{S}^2 \times \mathbf{S}^2$, $\mathbf{G} = SU(r+1)$

In the case $r > 1$ there is no mathematical computation at this point.

♠ Here is our prediction: for $u^i = \mathrm{Tr}_{\Lambda^{i+1}\mathbf{C}^{r+1}} \phi$

$$\langle \exp\left(T_i^3 \mathcal{O}_{u^i}^{(0)} + \int_{\mathbf{S}_2^2} T_i^1 \mathcal{O}_{u^i}^{(2)} + \int_{\mathbf{S}_1^2} T_i^2 \mathcal{O}_{u^i}^{(2)} + T_1^0 \int_X \mathcal{O}_{u^1}^{(4)} \right) \rangle =$$

$$\oint \sum_{\vec{N} \in \mathbf{Z}^r} \frac{du^1 \wedge \ldots \wedge du^r}{\frac{\partial W}{\partial a^1} \cdots \frac{\partial W}{\partial a^r}} \exp\left(\frac{1}{2} T_i^1 T_j^2 G^{ij}(u) + T_i^0 u^i \right)$$

• a^i are α_i the periods of the $x\frac{dz}{z}$ differential from the **Periodic Toda System** of the last lecture.

• $G^{ij} = \frac{\partial u^i}{\partial a^l} \frac{\partial u^j}{\partial a^k} \frac{d}{d\tau_{kl}} \log\Theta(\tau)$

$$\Theta(\tau) = \sum_{\vec{\lambda} \in \mathbf{Z}^r} (-1)^{\sum_{i=1}^r (r+1-2i)\lambda_i} \exp\left(\pi i \sum_{k,l} \tau_{kl} \lambda_k \lambda_l \right)$$

• τ_{kl} is the period matrix of the **Toda** spectral curve, and finally

$$W = \sum_{i=1}^r N_i a^i + T_i^1 u^i$$

$X = \mathbf{S}^2 \times \Sigma$, $\mathbf{G} = SU(2)$

• Σ is the genus $g > 1$ Riemann surface.

In the chamber where $\Sigma \ll \mathbf{S}^2$ the moduli space of $SU(2)$ instantons contains as an open dense subset the moduli space of holomorphic maps $\mathbf{S}^2 \to \mathcal{M}_g$ to the moduli space of $\mathbf{G}$-flat connections on Σ.

♢ Instanton $\Rightarrow$ stable holomorphic bundle $\mathcal{E}$. Restrict $\mathcal{E}$ onto a fiber Σ over a point $w \in \mathbf{S}^2$. For generic w we get a semi-stable bundle over it $\Rightarrow$ a point $m_m \in \mathcal{M}_g$.

♠ The map $w \mapsto m_w$ is holomorphic

♡ However, for special $w = w_*$ the restriction is unstable - we get a **freckle** of the lecture 1.

♠ Some correlators **are not** affected by freckles $\Rightarrow$

4d mirror $\Rightarrow$ 2d mirror

♠ Most of the correlators **are** affected by freckles $\Rightarrow$

4d mirror does not follow from 2d mirror

A compactification of the moduli space of instantons on X

via stable maps from $\mathbf{S}^2$ to $\mathcal{M}_g$ does not seem to provide us with a way of computing the refined Donaldson-Witten invariants of X.

Nevertheless one may deduce some useful information using Witten-Dijkgraaf-Verlinde-Verlinde equations applied to $\mathcal{M}_g$.

is not sensitive to the details of the compactification, here is the answer from the 4d theory: for $\mathbf{G} = SO(3)$ case with $(w_2, [\Sigma]) \neq 0$.

• The classical cohomology ring of $\mathcal{M}_g$ is generated by the observables in the two dimensional Yang-Mills theory:

$$a = \int_\Sigma \mathcal{O}^{(2)}_{\mathrm{Tr}\phi^2}, \quad b = \mathcal{O}^{((0)}_{\mathrm{Tr}\phi^2}$$

$$c = \sum_{i=1}^{g} \int_{A_i} \mathcal{O}^{(1)}_{\mathrm{Tr}\phi^2} \int_{B^i} \mathcal{O}^{(1)}_{\mathrm{Tr}\phi^2}$$

$$\langle \exp\left(\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c\right)\rangle =$$

$$\oint \frac{du dz}{(u^2-1)^g z^{g+1}} e^{2\varepsilon_2 u + \left(\varepsilon_1 u + \varepsilon_3 \left(u^2-1\right)\right) z} \frac{\sigma_3(\varepsilon_1 + z)}{\sigma(\varepsilon_1)\sigma_3(z)}$$

• $\sigma_3(z) = 1 + \frac{u}{24} z^2 + \ldots$, $\sigma(z) = z + \ldots$

are the Weierstraß elliptic functions associated to the curve:

$$y^2 = 4x^3 - \frac{x}{4}\left(\frac{u^2}{3} - \frac{1}{4}\right) - \frac{1}{48}\left(\frac{2u^3}{9} - \frac{u}{4}\right)$$

The last illustrative example: freckled instantons in 2d

In Lecture 1 we looked at the charge 1 freckled instantons in the $\mathbf{CP}^2$ sigma model. We shall conclude these lectures by carefully studying this example in details.

• Recall: $V = \mathbf{CP}^2 = \{ \left(Q^0 : Q^1 : Q^2 \right) \}$.

• $\mathcal{M}_1$ - moduli space of holomorphic degree 1 maps $\mathbf{P}^1 \to V$, $\overline{\mathcal{M}}_1$ - freckled instantons of charge 1.

• $\overline{\mathcal{M}}_1 = \mathbf{P}^5 = \{ \left(Q^0_0 : Q^0_1 : Q^1_0 : Q^1_1 : Q^2_0 : Q^2_1 \right) \}$.

♠ Let L_k, $k = 1, 2, 3$ denote the lines in V. Each line is the set of solutions to the linear equation:

$$L_k \leftrightarrow \sum_{m=0}^{2} Q^m \ell^k_m = 0$$

♠ Let P_k, $k = 1, 2$ denote the points in V. Each point is the set of solutions to the system of linear equations:

$$P_k \leftrightarrow \sum_{m=0}^{2} Q^m \rho^{k,a}_m = 0, \quad a = 1, 2$$

In Lecture 1 we defined the submanifolds

$$\mathcal{M}^0_{1,L_k}(z), \mathcal{M}^2_{1,P_k} \subset \mathcal{M}_1$$

and their closures $\overline{\mathcal{M}}^0_{1,L_k}(z), \overline{\mathcal{M}}^2_{1,P_k} \subset \overline{\mathcal{M}}_1$:

$\diamondsuit$ hyperplane $\overline{\mathcal{M}}^0_{1,L_k}(z)$: $\quad \sum_{m=0}^{2} \sum_{c=0}^{1} Q_c^m z^c \ell_m^k = 0$

$\diamondsuit$ quadric $\overline{\mathcal{M}}^2_{1,P_k}$: $\mathrm{Det}_{ac} \| \sum_{m=0}^{2} \rho^k_{m,a} Q_c^m \| = 0$

The intersection

$$\overline{\mathcal{M}}^0_{1,L_1}(0) \cap \overline{\mathcal{M}}^0_{1,L_2}(1) \cap \overline{\mathcal{M}}^0_{1,L_3}(\infty) \cap \overline{\mathcal{M}}^2_{P_1} \cap \overline{\mathcal{M}}^2_{P_2}$$

consists of $2 \times 2 = 4$ points (product of the degrees).

How many of these points correspond to the actual maps?

How many are freckles?

• Freckles: $Q_a^m = q^m p_a$:

$$q = (q^0 : q^1 : q^2) \in V, \quad p = (-p_1 : p_0) \in \mathbf{P}^1$$

the image of the degree 0 map and the location of the freckle respectively.

Hence $\overline{\mathcal{M}}_1 = \mathcal{M}_1 \cup \mathbf{P}^1 \times V$,

with (q, p) parameterizing the second piece

♠ The point (q, p) obviously belongs to $\overline{\mathcal{M}}^2_{1,P_k}$ for any k.

The point (q, p) belongs to $\overline{\mathcal{M}}^0_{1,P_k}(z)$ iff either $z = p$, or $q \in L_k$

♢ Hence we find the following **three** freckles in the intersection of the five submanifolds:

$$(L_2 \cap L_3, 0) \quad (L_1 \cap L_3, 1) \quad (L_1 \cap L_2, \infty)$$

The rest $4-3=1$ must come from the regular maps:

Indeed,

there is exactly one straight line passing through two generic points in $\mathbf{P}^2$.

This line L crosses the fixed lines L_1, L_2, L_3 at the points $z_1, z_2, z_3 \in L$.

There exists a **unique** parameterization of L in which

$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty$$

Q.E.D.