

# Gravitational Instantons of Type $D_k$

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## Abstract

We use two different methods to obtain Asymptotically Locally Flat hyperkähler metrics of type  $D_k$ .

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## 1 Introduction

A self-dual gravitational instanton is a hyperkähler manifold of real dimension four. These can be distinguished from each other by the asymptotic behaviour of the metric and their topology.

We give in this paper explicit formulae for asymptotically locally flat (ALF) hyperkähler metrics of type  $D_k$ . The  $A_k$  case has been known for a long time as the multi-Taub-NUT metric of Hawking [1] and is given in terms of one harmonic function  $V(\mathbf{x})$  on  $\mathbf{R}^3$

$$V(\mathbf{x}) = 1/\mu + \sum_1^{k+1} 1/|\mathbf{x} - \mathbf{x}_i|.$$

The metric is

$$ds^2 = V d\mathbf{x} \cdot d\mathbf{x} + V^{-1}(d\theta + \omega)^2. \quad (1)$$

where  $d\omega = *_3 dV$  and  $\omega$  is to be interpreted as a connection form for a principal circle bundle over the points of  $\mathbf{R}^3$  where  $V$  is non-singular. Over a large 2-sphere, this bundle has Chern class  $k+1$ . Now  $V \sim 1/\mu + (k+1)/|\mathbf{x}|$  is invariant under  $\mathbf{x} \mapsto -\mathbf{x}$  and an ALF metric of type  $D_k$  is asymptotic to the quotient of an  $A_{2k-5}$  metric by the action of this reflection. There are also degenerate cases within the family – the  $D_0$  case is the 2-monopole moduli space calculated in [2] and the  $D_2$  case was highlighted in [3] as an approximation to the K3 metric. Both the derivation and formulas for the  $D_0$  metrics benefited from the presence of continuous symmetry groups whereas, as we show here, the general  $D_k$  case has none, which perhaps explains the more complicated features of what follows. The actual manifold on which the metric is defined is, following [4], most conveniently taken to be a hyperkähler quotient by a circle action on the moduli space of  $U(2)$  monopoles of charge 2 with singularities at the  $k$  points  $q_1, \dots, q_k \in \mathbf{R}^3$ .

The original interest in physics of self-dual gravitational instantons (of which these metrics are examples) was motivated by their appearance in the late seventies in the formulation of Euclidean quantum gravity [5, 1, 6]. In this context they play a role similar to that of self-dual Yang-Mills solutions in quantum gauge theories [7]. Since then, however, these objects have appeared in various problems of quantum gauge theory, string theory and M-theory, some of which we now mention. For concreteness, we concentrate on the case of the  $D_k$ -type ALF gravitational instanton:

- compactifications of supergravity, string and M-theory on self-dual gravitational instantons preserve half the amount of supersymmetry of the original theory – M-theory on  $D_k$  ALF spaces, for example, emerges as a strong string coupling limit of type IIA string theory with an O6-plane in the presence of  $k$  D6-branes [8]
- as discussed by Seiberg and Witten [9], quantum moduli spaces of supersymmetric  $\mathcal{N} = 4$   $SU(2)$  gauge theories with  $k$  fundamental hypermultiplets in three dimensions are  $D_k$  ALF spaces
- these spaces can also be considered as moduli spaces of solutions of Bogomolny equations with prescribed singularities, or as moduli spaces of instantons on multi-Taub-NUT spaces that are invariant with respect to the  $S^1$  symmetry [10].

We derive the formulas by two different methods. The first is based on unpublished work of the second author carried out for the ALE case during a visit to the University of Bonn in 1979. It followed the twistor approach to the  $A_k$  case in [11], except that one needed a polynomial solution to  $x^2 - zy^2 = a$  (“Pell’s equation”), where  $z(\zeta)$  is a quartic, instead of the simple factorization  $xy = a$  as in the  $A_k$  case. This was solved by introducing the elliptic curve  $w^2 = z(\zeta)$  and trying to factorize  $a = (x - wy)(x + wy)$  into elliptic functions. There is a divisor class constraint (considered in 1828 by Abel [12]!) to doing this.

Given the more recent interpretation of the ALF solutions in terms of monopoles, the analogous elliptic curve is naturally described as the spectral curve of the monopole, or the equivalent Nahm data. As with all spectral curves, it is subject to a transcendental constraint and the key problem in writing down the metric is to implement analytically this constraint on  $z(\zeta)$ , cutting down the five coefficients of the quartic to give four coordinates on the hyperkähler manifold.

The second method uses the generalized Legendre transform construction of Lindstrom, Ivanov and Roček [13], [14] which has already been successfully used for  $k = 0$  and has begun to be applied to the problem considered here by the first author in [15]. In this case the quartic  $z(\zeta)$  appears in a fundamental way and the constraint is expressed by a differential equation. We show how these two expressions for the constraint coincide.

When the singularities of the monopole lie on a line through the origin, the metric defines an explicit resolution of the  $D_k$  quotient singularity by

a configuration of holomorphic 2-spheres, intersecting according to the  $D_k$  Dynkin diagram. We hope to consider this special case, where there does exist a circle action, in more detail elsewhere.

## 2 Singular monopoles

We shall use certain moduli spaces of singular monopoles to obtain our  $D_k$  metrics, as in [15]. There are two approaches to this: through Nahm's equations, which give us a concrete analytical description of the objects in the space and the  $\mathcal{L}^2$  metric defined on it, and the twistor approach which we use here. The latter describes a generic point in the moduli space via an algebraic curve, and allows us the possibility of an explicit determination of the metric.

### 2.1 The twistor approach to monopoles

It was shown in [4], using the technique developed in [16] as well as results of Kronheimer [17], that a charge 2  $U(2)$  monopole solution to the Bogomolny equations with  $k$  singularities is described by a spectral curve  $\mathbf{S}$  in  $T\mathbf{P}^1$  and two sections  $\pi$  and  $\rho$  of holomorphic line bundles  $L^{-\mu}(k)|_{\mathbf{S}}$  and  $L^{\mu}(k)|_{\mathbf{S}}$  respectively. Viewing  $T\mathbf{P}^1$  as the total space of  $\mathcal{O}(2)$  we denote by  $\zeta$  an affine coordinate on  $\mathbf{P}^1$  and  $\eta$  the standard linear coordinate in the fibre. Then  $L^{\mu}(k) \rightarrow T\mathbf{P}^1$  denotes the line bundle with transition function  $\zeta^{-k} \exp(\mu\eta/\zeta)$ .

There is a real structure  $\sigma$  given by  $\sigma(\eta, \zeta) = (-\bar{\eta}, -1/\bar{\zeta})$ . The spectral curve  $\mathbf{S} \subset T\mathbf{P}^1$  for a charge 2 monopole is then given by an equation

$$\eta^2 - y\eta - z = 0, \quad (2)$$

where  $y$  is a real section of  $\mathcal{O}(2)$  and  $z$  is a real section of  $\mathcal{O}(4)$ . The reality condition for a section  $x$  of  $\mathcal{O}(2n)$  is  $\overline{x(\sigma(\zeta))} = (-1)^n x(\zeta)/\zeta^{2n}$ . Thus in the patch  $\zeta \neq \infty$  the section  $x$  is given by a polynomial  $x(\zeta)$  of degree  $2n$  which satisfies  $\overline{x(-1/\bar{\zeta})} = (-1)^n x(\zeta)/\zeta^{2n}$ .

The position of each of the  $k$  singularities of the monopole configuration can be described by a real section  $q_i$  of  $\mathcal{O}(2)$ ,  $i = 1, \dots, k$ . We denote these sections by  $\mathbf{P}_{q_i}^1 \subset T\mathbf{P}^1$ .

The two sections  $\pi$  and  $\rho$  on the spectral curve are interchanged by the

real structure and satisfy

$$\pi\rho = \prod_{i=1}^k (\eta - q_i). \quad (3)$$

There is a circle action on this data given by

$$(\rho, \pi) \mapsto (\lambda\rho, \lambda^{-1}\pi) \quad (4)$$

Equation (3) says that the intersection of  $\mathbf{S}$  with all the curves  $\mathbf{P}_{q_i}^1$  defines (if  $\mathbf{S}$  does not contain one of them as a component) a divisor on  $\mathbf{S}$  of degree  $4k$ , and the constraint on the spectral curve is expressed by the fact that we can divide these into two sets of  $2k$  points, one of which is a divisor for  $L^{-\mu}(k)|_{\mathbf{S}}$  and the other for  $L^{\mu}(k)|_{\mathbf{S}}$ .

## 2.2 The twistor approach to the moduli space

The spectral curve description fits into the twistor description of the hyperkähler metric on the moduli space, which we recall from [4], and is similar to the case of non-singular monopoles in [2]. Let

$$D = \{(\eta, y, z) \in \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(4) \mid \eta^2 - y\eta - z = 0\}$$

which has a projection  $p_1(\eta, y, z) = \eta$  onto  $T\mathbf{P}^1$  and another  $p_2(\eta, y, z) = (y, z)$  which represents  $D$  as a ramified double covering of  $\mathcal{O}(2) \oplus \mathcal{O}(4)$ . We let  $V^{\mu}$  be the rank 2 holomorphic vector bundle on  $\mathcal{O}(2) \oplus \mathcal{O}(4)$  which is the direct image sheaf

$$V^{\mu} = (p_2)_*(p_1^*L^{\mu}).$$

Then the direct image of the equation (3) defines a subvariety  $Z$  of  $V^{\mu}(k) \oplus V^{-\mu}(k)$  which will be a model of our twistor space. There is some resolution of singularities to be carried out, but that doesn't affect the determination of the metric. In the twistor space approach we need to find the twistor lines, which are sections of  $p : Z \rightarrow \mathbf{P}^1$ . The quadratic and quartic  $y, z$  define a section of  $\mathcal{O}(2) \oplus \mathcal{O}(4)$  and the functorial property of the direct image says that  $\pi, \rho$  define a lifting to  $Z \subset V^{\mu}(k) \oplus V^{-\mu}(k)$ .

Over  $\zeta \neq 0$ , the direct image equation for  $Z$  can be written as

$$(x_1 + \eta x_2)(y_1 + \eta y_2) = \prod_i (\eta - q_i) \pmod{\eta^2 - y\eta - z = 0}$$

or equivalently,

$$\begin{aligned} x_1 y_1 + z x_2 y_2 &= p(y, z) \\ x_2 y_1 + x_1 y_2 + y x_2 y_2 &= q(y, z) \end{aligned} \quad (5)$$

where

$$\prod_i (\eta - q_i) = p + \eta q \pmod{\eta^2 - y\eta - z} = 0.$$

This equation defines a 5-dimensional twistor space for an 8-dimensional hyperkähler manifold – the moduli space of charge 2 singular monopoles. There is a symplectic form along the fibres which can be written as in [15] as

$$\omega = 4 \sum_{j=1}^2 \frac{d\rho(\beta_j) \wedge d\beta_j}{\rho(\beta_j)}, \quad (6)$$

with  $\eta = \beta_j$  being roots of  $\eta^2 - y\eta - z = 0$ , but in coordinates  $x_1, x_2, y, z$  above as

$$\omega = 4 \frac{(x_1 dx_1 + (yx_1 - zx_2) dx_2) \wedge dy + (x_1 dx_2 - x_2 dx_1) \wedge dz}{x_1^2 - zx_2^2 + yx_1 x_2}$$

### 2.3 The hyperkähler quotient

To obtain a 4-dimensional manifold we shall take a hyperkähler quotient by a circle action which at the twistor space level is given by (4) and in the above coordinates is

$$(x_1, x_2, y_1, y_2, y, z) \mapsto (\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2, y, z).$$

In coordinates  $x_1, x_2, y, z$  the vector field generated by this action is

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

so that

$$i(X)\omega = 4dy.$$

**Remark:** The moment map for this action is  $4y$ . From the point of view of monopoles this can be interpreted in terms of the centre of mass, in which case the more natural value would be  $y/2$  which amounts to a rescaling of  $\omega$ .

The hyperkähler quotient from the twistor point of view is just the fibre-wise symplectic quotient, so we set the moment map  $y = 0$  (this means that the centre of mass of the monopole is at the origin) and take the quotient by the  $\mathbf{C}^*$  action. Putting  $y = 0$  in (5), gives

$$\begin{aligned} x_1 y_1 + z x_2 y_2 &= p(y, z) \\ x_2 y_1 + x_1 y_2 &= q(y, z) \end{aligned} \quad (7)$$

We need a smooth 4-manifold so the circle action on the zero set of the hyperkähler moment map must be free. In the nonsingular monopole case this is automatic: up to a finite covering the moduli space is isometric to a product  $M_k^0 \times S^1 \times \mathbf{R}^3$ . The action is not necessarily free in the singular case. For example, the moduli space of charge 1 monopoles with  $k$  singularities is the  $A_{k-1}$  ALF space – multi-Taub-NUT space – and the triholomorphic circle action has fixed points.

Suppose we have a fixed point, then there is a fixed twistor line, so on each fibre of the twistor space a fixed point, which from (4) is where  $x_1 = x_2 = 0$ . This means  $p = q = 0$ . But with  $y = 0$ ,  $p$  and  $q$  are defined by

$$r(\eta) = \prod_i (\eta - q_i) = p + \eta q \pmod{\eta^2 - z} = 0$$

thus

$$p(z) = \frac{1}{2}(r(\eta) + r(-\eta)), \quad q(z) = \frac{1}{2\eta}(r(\eta) - r(-\eta))$$

and  $p$  and  $q$  have a common zero if  $r(\eta)$  and  $r(-\eta)$  have a common zero, i.e.  $q_i = -q_j$  (for all  $\zeta$ ). If  $q_i = 0$ , then  $q(0) = r'(0)$  so since the  $q_i$  are distinct, the action doesn't have a fixed point. Thus, so long as the positions of the singularities of the monopole satisfy  $q_i \neq -q_j$  for any  $i \neq j$ , we shall produce a non-singular 4-manifold as a hyperkähler quotient.

The equation of the twistor space of the quotient can be obtained by using the  $\mathbf{C}^*$ -invariant coordinates

$$P = 2x_1 y_1 - p, \quad Q = 2x_2 y_2 - q$$

and then from (7)

$$\frac{P - p}{Q - q} = z \frac{x_2}{x_1} = z \frac{Q + q}{P + p}$$

which gives

$$P^2 - zQ^2 = \prod_{i=1}^k (z - q_i^2). \quad (8)$$

The holomorphic symplectic form along the fibres is then

$$\omega = d \left( \frac{1}{\sqrt{z}} \log \frac{P + \sqrt{z}Q}{P - \sqrt{z}Q} \right) \wedge dz. \quad (9)$$

**Remark:** Equation (8) defines a subvariety of the rank 2 direct image vector bundle  $V^{2\mu}(2k)$  over  $\mathcal{O}(4)$ . Concretely,  $V^{2\mu}$  has transition matrix

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} \cosh(2\mu\sqrt{z}/\zeta) & \sqrt{z} \sinh(2\mu\sqrt{z}/\zeta) \\ \zeta^2 \sinh(2\mu\sqrt{z}/\zeta)/\sqrt{z} & \zeta^2 \cosh(2\mu\sqrt{z}/\zeta) \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Note that if  $\tilde{z} = z/\zeta^4$  then

$$\tilde{P}^2 - \tilde{z}\tilde{Q}^2 = P^2 - zQ^2$$

which describes a global holomorphic quadratic form  $(u, u)$  on  $V$ , singular on the zero section  $z = 0$ . The null-directions are  $P \pm \sqrt{z}Q$  which are globally defined on the double covering  $\mathcal{O}(2)$  and are the line bundles  $L^{\pm 2\mu}$ . This quadratic form gives a more invariant way of writing (8):

$$(u, u) = \prod_{i=1}^k (z - q_i^2). \quad (10)$$

## 2.4 The $D_k$ singularity

The fibre over  $\zeta \in \mathbf{P}^1$  in the 3-dimensional variety defined by (10) has the equation

$$P^2 - zQ^2 = \prod_{i=1}^k (z + p_i^2) \quad (11)$$

if we set  $p_i = iq_i(\zeta)$ . The universal deformation of the  $D_k$  singularity has the form

$$x^2 - zy^2 = \frac{1}{z} \left( \prod_{i=1}^k (z + p_i^2) - \prod_{i=1}^k p_i^2 \right) + 2i \prod_{i=1}^k p_i y$$

and putting

$$P = iyz - \prod_{i=1}^k p_i, \quad Q = ix$$



we obtain (11). If the  $p_i^2$  are distinct for a fixed  $\zeta$  and if none of the  $p_i$  vanish, (11) defines a smooth surface and  $(x, y, z) \mapsto (P, Q, z)$  is a biholomorphic map.

At the finite number of values of  $\zeta$  at which a  $p_i$  vanishes, the fibre is singular and the actual twistor space  $Z$  for the hyperkähler metric involves a resolution of these singularities. The singular points are replaced by compact rational curves, but since a generic twistor line misses the singularities, we do not need to use this fact to calculate the metric. On the other hand, our formulae are not sufficiently manageable here to use the explicit form of the metric to describe these resolutions, as was done in the  $A_k$  case in [11].

The existence of these compact curves in certain fibres gives us information about the Killing fields for the metric. Any such vector field induces a holomorphic vector field on the twistor space, taking fibres to fibres. The generic fibre is an affine surface in  $\mathbf{C}^3$  and so has no compact subvarieties other than points, since the coordinate functions must be constant. Thus the vector field must preserve those fibres with rational curves. There are then two alternatives. One is that it preserves all fibres and therefore defines a triholomorphic Killing field, but this is impossible: if an ALF space has a triholomorphic isometry, then its metric is exactly equal to the metric given in Eq.(1). The other, since a holomorphic vector field on the sphere has at most two zeros, is that there are only two singular fibres, which is the case when all the  $p_i$  vanish at two antipodal points on  $\mathbf{P}^1$ . In this case the singular fibre is the  $D_k$  singularity  $x^2 - zy^2 = z^{k-1}$ . In another paper we shall explore more explicitly the resolution of this singularity by our twistor lines, but for the present this argument shows that our metrics for a generic choice of  $p_i$  have no Killing fields.

### 3 The equations defining the constraint

The main problem in determining a metric in the twistor approach is to find the twistor lines – holomorphic sections of the projection  $Z \rightarrow \mathcal{O}(4) \rightarrow \mathbf{P}^1$ . From the construction above, we have a rational curve  $C$  in  $\mathcal{O}(4)$  defined by  $z = z(\zeta)$  where  $z(\zeta)$  is a quartic polynomial, and to lift it further we needed a section  $u(\zeta)$  of  $V^{2\mu}(2k)$  over  $C$ , which from the functorial property of the direct image was equivalent to a section  $a = P + \sqrt{z}Q$  of  $L^{2\mu}(2k)$  on the spectral curve. For the section  $u(\zeta)$  to satisfy (10) is equivalent to the existence of sections  $a = P + \sqrt{z}Q, b = P - \sqrt{z}Q$  on the spectral curve  $\mathbf{S}$

satisfying

$$ab = \prod_{i=1}^k (z - q_i^2).$$

The spectral curve of the monopole had sections  $\pi, \rho$  such that

$$\pi\rho = \prod_{i=1}^k (\eta - q_i).$$

Setting

$$a(\eta, \zeta) = \rho(\eta, \zeta)\pi(-\eta, \zeta), \quad b(\eta, \zeta) = (-1)^k \rho(-\eta, \zeta)\pi(\eta, \zeta)$$

gives sections  $a, b$  of  $L^{2\mu}(2k)|_{\mathbf{S}}$  and  $L^{-2\mu}(2k)|_{\mathbf{S}}$  respectively such that

$$ab = \prod_{i=1}^k (z - q_i^2)$$

is satisfied. If the  $\alpha_{ij}$  are roots of  $z(\zeta) - q_i^2(\zeta)$ , then  $a$  vanishes at points  $(\eta, \zeta) = ((-1)^j q_i(\alpha_{ij}), \alpha_{ij}) \in \mathbf{S} \in TP^1$ . The section  $b$  vanishes at  $(\eta, \zeta) = (-(-1)^j q_i(\alpha_{ij}), \alpha_{ij})$ .

A rotation of  $\mathbf{R}^3$  acts as a fractional linear transformation on the coordinate  $\zeta$ . Using such a transformation we can put  $z(\zeta)$  in the form  $r_1\zeta^3 - r_2\zeta^2 - r_1\zeta$  with real  $r_2$  and  $r_1 \geq 0$ . In these coordinates we shall solve the constraint equation for the spectral curve  $\eta^2 = z(\zeta)$  using Weierstrass elliptic functions:  $\eta = \sqrt{r_1}\mathcal{P}'(u)/2$  and  $\zeta = \mathcal{P}(u) + r_2/3r_1$ , where  $u$  is the affine coordinate on  $\mathbf{C}/\Gamma$  representing the torus  $\eta^2 = z(\zeta)$ . Note that  $\eta \rightarrow -\eta$  corresponds to  $u \rightarrow -u$ . In what follows we order the points  $\alpha_{ij}$  so that  $\rho$  vanishes at  $\alpha_{i2}$  and  $\alpha_{i4}$ , while  $\pi$  vanishes at  $\alpha_{i1}$  and  $\alpha_{i3}$ . Let  $u_{ij}$  be the zeros of the sections  $\rho$  and  $\pi$  corresponding to  $(q_i(\alpha_{ij}), \alpha_{ij})$ , then the condition for the sections to be doubly periodic translates into

$$2\mu\sqrt{r_1} + \sum_{i=1}^k \sum_{j=1}^4 (-1)^j u_{ij} = 0, \quad (12)$$

or in terms of  $z$  and  $\zeta$

$$\sum_{ij} (-1)^j \int^{\alpha_{ij}} \frac{d\zeta}{\sqrt{z}} = -4\mu. \quad (13)$$

From the fact that  $a/b$  is a section of  $L^{4\mu}|_{\mathbf{S}}$  with zeroes at  $(-1)^j u_{ij}$  and poles at  $-(-1)^j u_{ij}$  it follows (see [4] and [15]) that with  $a = P + \sqrt{z}Q$ ,  $b = P - \sqrt{z}Q$  and  $\zeta_W$  the Weierstrass zeta function

$$\log \frac{a}{b} = -2\mu\sqrt{r_1} (\zeta_W(u + u_\infty) + \zeta_W(u - u_\infty)) + \log \prod_{ij} \frac{\sigma(u - (-1)^j u_{ij})}{\sigma(u + (-1)^j u_{ij})}. \quad (14)$$

Returning to  $z$  and  $\zeta$

$$\frac{1}{\sqrt{z(\zeta)}} \log \frac{a}{b} = \sum_{ij} (-1)^j \int^{\alpha_{ij}} \frac{d\xi}{(\xi - \zeta)\sqrt{z(\xi)}}. \quad (15)$$

## 4 From the twistor space to the metric

The first calculation uses Penrose's original nonlinear graviton construction [18]. First we calculate the conformal structure and then use the holomorphic form to determine the volume form. The 4-dimensional spacetime  $\mathcal{M}$  is the space of twistor lines, a tangent vector is a holomorphic section of the normal bundle and it is a null vector for the conformal structure if and only if that section vanishes somewhere on the twistor line. This is the infinitesimal version of the statement that two points in  $\mathcal{M}$  are null separated if the twistor lines meet.

### 4.1 The complex conformal structure

The twistor line is given by

$$\eta^2 = z(\zeta) = A \prod_{j=1}^4 (\zeta - a_j) \quad (16)$$

and we define

$$\chi(\zeta) = \frac{1}{\sqrt{z}} \log \frac{a}{b} = \frac{1}{\sqrt{z}} \log \frac{P(\zeta) + \sqrt{z(\zeta)}Q(\zeta)}{P(\zeta) - \sqrt{z(\zeta)}Q(\zeta)}, \quad (17)$$

so that the symplectic form given by equation (9) is

$$\omega = d\chi \wedge dz. \quad (18)$$

We can think of the variables  $a_j$  and  $A$  satisfying the constraint (13) as providing four real coordinates on the  $D_k$  ALF manifold  $\mathcal{M}$ . Later we shall change coordinates to  $z(0)$  and  $Q(0)$ .

Varying the parameters  $A$  and  $a_j$  under the constraint (13) we are seeking the condition for an infinitesimal variation to vanish. From now on we denote the infinitesimal variations by a prime, e.g.  $A', a'_j, z'$  etc. Putting the variation of equation (16) to zero we have

$$\frac{z'}{z} = \frac{A'}{A} - \sum_{j=1}^4 \frac{a'_j}{\zeta - a_j} = 0, \quad (19)$$

while the vanishing of the variation of the constraint (13) gives

$$\sum_{ij} (-1)^j \frac{\alpha'_{ij}}{\sqrt{z(\alpha_{ij})}} = \sum_{ij} (-1)^j \int^{\alpha_{ij}} \frac{d\zeta}{2\sqrt{z(\zeta)}} \left( \frac{A'}{A} - \sum_l \frac{a'_l}{\zeta - a_l} \right), \quad (20)$$

and for the second term on the right hand side we note that

$$\begin{aligned} & \frac{1}{2} \sum_{ij} (-1)^j \int^{\alpha_{ij}} \frac{d\zeta}{(\zeta - a_l)\sqrt{z(\zeta)}} = \frac{1}{2} \chi(a_l) = \\ & = \lim_{\zeta \rightarrow a_l} \frac{1}{2\sqrt{z(\zeta)}} \log \frac{P(\zeta) + \sqrt{z(\zeta)}Q(\zeta)}{P(\zeta) - \sqrt{z(\zeta)}Q(\zeta)} = \frac{Q(a_l)}{P(a_l)}, \end{aligned} \quad (21)$$

as  $z(a_l) = 0$ . Let us introduce a function

$$L(\zeta) = \sum_i \frac{l_i(\zeta)}{z(\zeta) - q_i^2(\zeta)}, \quad (22)$$

where  $l_i$  are cubic polynomials in  $\zeta$  such that  $l_i(\alpha_{ij}) = q_i(\alpha_{ij})$  (Lagrange interpolation polynomials). Also let

$$l = \lim_{\zeta \rightarrow \infty} \zeta L(\zeta). \quad (23)$$

Then using the constraint equation (13) and substituting  $\alpha'_{ij}$  in equation (20) we find

$$\frac{A'}{A} (2\mu - l) = \sum_{l=1}^4 a'_l \left( L(a_l) - \frac{Q(a_l)}{P(a_l)} \right) \quad (24)$$

The variation of  $\chi$  is

$$\chi' = \left( L(\zeta) - \frac{1}{2}\chi(\zeta) \right) \frac{z'}{z} + \sum_{l=1}^4 \frac{a'_l}{\zeta - a_l} \left[ L(a_l) - \frac{Q(a_l)}{P(a_l)} \right]. \quad (25)$$

Requiring the variation of  $\chi$  and  $z$  to vanish gives firstly

$$\sum_{l=1}^4 \frac{a'_l}{\zeta - a_l} \left[ L(a_l) - \frac{Q(a_l)}{P(a_l)} \right] = 0. \quad (26)$$

and secondly, eliminating  $A'$  from the equations (19) and (24) leads to

$$\sum_{l=1}^4 \frac{a'_l}{\zeta - a_l} \left[ 2\mu - l + a_l \left( L(a_l) - \frac{Q(a_l)}{P(a_l)} \right) \right] = 0. \quad (27)$$

Equations (26) and (27) give a system of two equations cubic in  $\zeta$  and linear in  $a'_l$ . Our task is to find the condition on the coefficients of these equations to have a common solution for  $\zeta$ . This condition is provided by the vanishing of the resultant  $R$  of the system (26, 27), which is a polynomial in  $a'_l$  of degree six. However, if  $a'_j = 0$  then  $\zeta = a_j$  solves the equations and so  $R$  must be divisible by  $a'_1 a'_2 a'_3 a'_4$ . This means

$$S(a'_l) = R(a'_l) / (a'_1 a'_2 a'_3 a'_4) \quad (28)$$

is a polynomial which is quadratic in the variations and so is the quadratic form defining the conformal structure.

For convenience we introduce

$$C_i = L(a_i) - \frac{Q(a_i)}{P(a_i)} = L(a_i) - \frac{1}{2}\chi(a_i), \quad D_i = 2\mu - l + a_i C_i. \quad (29)$$

Computing the resultant (see Appendix) we find

$$S(da_i) = \begin{vmatrix} \sum_{i=1}^4 D_i da_i & 0 & a_1^2 D_1 & a_2^2 D_2 & a_3^2 D_3 & a_4^2 D_4 \\ 0 & 0 & a_1 D_1 & a_2 D_2 & a_3 D_3 & a_4 D_4 \\ 0 & -\sum_{i=1}^4 D_i da_i / a_i & D_1 & D_2 & D_3 & D_4 \\ \sum_{i=1}^4 C_i da_i & 0 & a_1^2 C_1 & a_2^2 C_2 & a_3^2 C_3 & a_4^2 C_4 \\ 0 & 0 & a_1 C_1 & a_2 C_2 & a_3 C_3 & a_4 C_4 \\ 0 & -\sum_{i=1}^4 C_i da_i / a_i & C_1 & C_2 & C_3 & C_4 \end{vmatrix} \quad (30)$$

where we use differentials  $da_i$  instead of the primes  $a'_i$ .

## 4.2 The real metric

To impose the reality condition, we note that the map  $\zeta \rightarrow -1/\bar{\zeta}$  takes the set of four roots  $\{a_1, a_2, a_3, a_4\}$  into itself, so that for a permutation  $\sigma : (1, 2, 3, 4) \rightarrow (2, 1, 4, 3)$  we have  $a_{\sigma(i)} = -1/\bar{a}_i$ . Now noting that

$$\overline{L\left(-\frac{1}{\bar{\zeta}}\right)} = \zeta^2 L(\zeta) - l\zeta, \quad (31)$$

$$\overline{\chi\left(-\frac{1}{\bar{\zeta}}\right)} = \zeta^2 \chi(\zeta) - 4\mu\zeta, \quad (32)$$

it follows that

$$\overline{C_{\sigma(i)}} = a_i D_i, \quad \overline{D_{\sigma(i)}} = -a_i C_i \quad (33)$$

$$\overline{a_{\sigma(i)} C_{\sigma(i)}} = -D_i, \quad \overline{a_{\sigma(i)} D_{\sigma(i)}} = C_i \quad (34)$$

$$\overline{a_{\sigma(i)}^2 C_{\sigma(i)}} = \frac{D_i}{a_i}, \quad \overline{a_{\sigma(i)}^2 D_{\sigma(i)}} = -\frac{C_i}{a_i}. \quad (35)$$

With these relations, one can easily check that  $\bar{S} = S/(a_1 a_2 a_3 a_4)$ , and the quantities

$$A_0 = \sum_{i=1}^4 C_i da_i, \quad B_0 = \sum_{i=1}^4 D_i da_i \quad (36)$$

satisfy

$$\overline{A_0} = \sum_{i=1}^4 D_i \frac{da_i}{a_i}, \quad \overline{B_0} = -\sum_{i=1}^4 C_i \frac{da_i}{a_i}. \quad (37)$$

Thus, we find that the metric is proportional to the real symmetric two-form

$$\frac{1}{\sqrt{a_1 a_2 a_3 a_4}} S = (A_0 B_0) G \left( \begin{array}{c} \overline{A_0} \\ \overline{B_0} \end{array} \right), \quad (38)$$

with

$$G = \frac{1}{\sqrt{a_1 a_2 a_3 a_4}} \begin{pmatrix} [a^2 D, aD, aC, C] & -[a^2 D, aD, D, aC] \\ [aD, a^2 C, aC, C] & [aD, D, a^2 C, aC] \end{pmatrix}. \quad (39)$$

Here we use the notation

$$[e, f, g, h] = \det \begin{pmatrix} e_1 & f_1 & g_1 & h_1 \\ e_2 & f_2 & g_2 & h_2 \\ e_3 & f_3 & g_3 & h_3 \\ e_4 & f_4 & g_4 & h_4 \end{pmatrix}. \quad (40)$$

Observing that

$$\begin{aligned} a_1 a_2 a_3 a_4 &= [aD, D, a^2C, aC] / \overline{[aD, D, a^2C, aC]} \\ &= -[a^2D, aD, D, aC] / \overline{[aD, a^2C, aC, C]} \end{aligned} \quad (41)$$

we have  $|[aD, D, a^2C, aC]|G$  given by

$$\begin{pmatrix} [a^2D, aD, aC, C] \overline{[aD, D, a^2C, aC]} & [aD, D, a^2C, aC] \overline{[aD, a^2C, aC, C]} \\ [aD, a^2C, aC, C] \overline{[aD, D, a^2C, aC]} & [aD, D, a^2C, aC] \overline{[aD, D, a^2C, aC]} \end{pmatrix}. \quad (42)$$

Utilizing the identity relating the determinants derived in the Appendix

$$\begin{aligned} [aC, C, a^2D, aD] \overline{[a^2C, aC, aD, D]} &= \\ [C, aC, a^2C, aD] \overline{[C, aC, a^2C, aD]} &+ [C, aC, D, aD]^2, \end{aligned} \quad (43)$$

we have

$$\frac{|[aD, D, a^2C, aC]|}{[aD, D, aC, C]^2} G = \begin{pmatrix} 1 + \gamma\bar{\gamma} & \bar{\gamma}\delta \\ \gamma\bar{\delta} & \delta\bar{\delta} \end{pmatrix}, \quad (44)$$

where

$$\gamma = \frac{[aD, a^2C, aC, C]}{[aD, D, aC, C]}, \quad \delta = \frac{[aD, D, a^2C, aC]}{[aD, D, aC, C]}. \quad (45)$$

Next we identify the differentials  $A_0$  and  $B_0$  in terms of the coordinates  $z(0), \chi(0)$ .

$$\overline{A_0} = (2\mu - l) \frac{dz(0)}{z(0)}, \quad \overline{B_0} = d\chi(0) - \left( L(0) - \frac{1}{2}\chi(0) \right) \frac{dz(0)}{z(0)} \quad (46)$$

From the definition of  $\chi$  (17) we have

$$d\chi = 2 \frac{dQ}{P} + \left( \frac{Q}{P} - \frac{Q}{P} \sum_{j=1}^k \frac{z}{z - q_j} - \frac{1}{2}\chi \right) \frac{dz}{z}, \quad (47)$$

thus

$$\overline{B_0} = 2 \frac{dQ}{P} + \left( \frac{Q}{P} - \frac{Q}{P} \sum_{j=1}^k \frac{z(0)}{z(0) - q_j} - L(0) \right) \frac{dz(0)}{z(0)}. \quad (48)$$

The conformal structure, from (44) can therefore be written as

$$ds^2 \sim A_0 \overline{A_0} + (\bar{\gamma}A_0 + \bar{\delta}B_0)(\gamma\overline{A_0} + \delta\overline{B_0}). \quad (49)$$

The volume corresponding to the expression on the right-hand-side is

$$\text{Vol} = -\frac{(2\mu - l)^2 |\delta|^2}{z\bar{z}P\bar{P}} dz \wedge d\bar{z} \wedge dQ \wedge d\bar{Q}. \quad (50)$$

On the other hand the volume should be equal to  $\frac{1}{4}\omega \wedge \bar{\omega}$  for  $\omega$  given by equation (9). Thus comparing the two expressions for  $\omega = 2dQ \wedge dz/P$  we have the final metric

$$ds^2 = \left| \frac{z}{(2\mu - l)\delta} \right| (A_0 \bar{A}_0 + (\bar{\gamma} A_0 + \bar{\delta} B_0)(\gamma \bar{A}_0 + \delta \bar{B}_0)). \quad (51)$$

## 5 The generalized Legendre transform

We have obtained the hyperkähler metrics above by following the twistor space approach. In this section, we apply a different technique given by generalized Legendre transform [14, 19]. It has been successfully applied in the case of  $D_0$  ALF in [13] to reproduce Atiyah-Hitchin metric, and to study  $D_k$  ALF metrics in [20].

We start with a brief outline of the construction. Using, as earlier,  $\mathbf{z}$  as one of the complex coordinates on the fibres of the twistor space. In the patch  $\zeta \neq \infty$  it has the form

$$\mathbf{z}(\zeta) = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4. \quad (52)$$

The second coordinate  $\chi$  (see (17)) is such that the holomorphic twistorial two-form has the form  $\omega = d\chi \wedge d\mathbf{z}$ . If the section  $\chi$  is represented by the function  $\chi_1$  in the patch  $\zeta \neq \infty$  and  $\chi_2$  in the patch  $\zeta \neq 0$ , then we define a function  $\hat{f}$  and a contour  $C$  such that

$$\oint_C \frac{d\zeta}{\zeta^j} \hat{f} = \oint_0 \frac{d\zeta}{\zeta^{j-2}} \chi_1 - \oint_\infty \frac{d\zeta}{\zeta^j} \chi_2. \quad (53)$$

Given  $\hat{f}$  we define  $G$  such that  $\partial G / \partial \mathbf{z}(\zeta) = \hat{f} / \zeta^2$ . Next, we define the function of the coefficients of  $\mathbf{z}(\zeta)$

$$F(z, v, w, \bar{v}, \bar{z}) = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta^2} G. \quad (54)$$

The generalized Legendre transform construction generates a formula for a Kähler potential  $K(z, \bar{z}, u, \bar{u})$  of the coordinates  $z$  and  $u$ . We impose the



differential constraint  $\partial F/\partial w = 0$  which determines  $w$  as a function of  $z$  and  $v$ , and perform a Legendre transform on  $F$  with respect to coordinates  $v$  and  $\bar{v}$ :

$$K(z, \bar{z}, u, \bar{u}) = F - vu - \bar{v}\bar{u}, \quad u = \frac{\partial F}{\partial v}, \quad \bar{u} = \frac{\partial F}{\partial \bar{v}}. \quad (55)$$

In our case the above procedure produces

$$\oint_C \frac{d\zeta}{\zeta^j} \hat{f} = -4\mu \oint_0 \frac{d\zeta}{\zeta^j} \zeta + \left( \oint_0 - \oint_\infty \right) \frac{d\zeta}{\zeta^{j-2}} \chi_1, \quad (56)$$

since we chose  $u_{ij}$  so that (12) holds, we have

$$\begin{aligned} F(z, v, w, \bar{v}, \bar{z}) &= -4\mu \frac{1}{2\pi i} \oint_0 \frac{d\zeta}{\zeta} \frac{\mathbf{z}(\zeta)}{\zeta^2} \\ &\quad - \frac{1}{2\pi i} \sum_{i=1}^k \sum_{l=0,1} \oint_{C_i^l} \frac{d\zeta}{\zeta} \frac{\sqrt{\mathbf{z}(\zeta)} + (-1)^l q_i(\zeta)}{\zeta} \log(\sqrt{\mathbf{z}(\zeta)} + (-1)^l q_i(\zeta)). \end{aligned}$$

A careful derivation of this formula can be found in [15]. For each  $i$  the pair  $(C_i^0, C_i^1)$  consists of 4 figure-eight shaped contours on the Riemann surface  $\eta^2 = \mathbf{z}(\zeta)$ , each contour surrounding two out of the eight points  $(\eta, \zeta) = (\pm q_i(\alpha_{ij}), \alpha_{ij})$ . The projection of  $C_i^0$  on the  $\zeta$  plane coincides with the projection of  $C_i^1$ . The two contours  $C_i^l$  have figure-eight shapes encircling points  $\alpha_{ij}$  clockwise for odd  $j$  and counterclockwise for even  $j$ . With this in mind we obtain

$$\sum_{l=0,1} \frac{1}{2\pi i} \oint_{C_i^l} \frac{d\zeta}{\sqrt{\mathbf{z}}} f(\zeta) \log(\sqrt{\mathbf{z}} + (-1)^l q_i) = 2 \sum_{j=1}^4 (-1)^j \int^{\alpha_{ij}} \frac{d\zeta}{\sqrt{\mathbf{z}}} f(\zeta). \quad (57)$$

For convenience let us define matrices

$$\mathcal{F} = \begin{pmatrix} F_{vv} & F_{vw} & F_{v\bar{v}} \\ F_{wv} & F_{ww} & F_{w\bar{v}} \\ F_{\bar{v}v} & F_{\bar{v}w} & F_{\bar{v}\bar{v}} \end{pmatrix}, \quad (58)$$

where the subscripts of  $F$  denote partial derivatives (e.g.  $F_{vw} = \partial^2 F/\partial v \partial w$ ), and

$$\mathcal{G} = \begin{pmatrix} G^{vv} & G^{vw} & G^{v\bar{v}} \\ G^{wv} & G^{ww} & G^{w\bar{v}} \\ G^{\bar{v}v} & G^{\bar{v}w} & G^{\bar{v}\bar{v}} \end{pmatrix} = \mathcal{F}^{-1}, \quad (59)$$

where the superscripts merely label the components of  $\mathcal{G}$ . The components of the metric are

$$K_{z\bar{z}} = F_{z\bar{z}} - F_{za}G^{ab}F_{b\bar{z}} \quad (60)$$

$$K_{u\bar{z}} = F_{za}G^{av}, \quad K_{u\bar{u}} = -G^{v\bar{v}} \quad (61)$$

where  $a$  and  $b$  are summed over the values  $(v, w, \bar{v})$ . And the metric is

$$ds^2 = \frac{-1}{G^{v\bar{v}}} \left( -G^{v\bar{v}}(F_{z\bar{z}} - F_{za'}G^{a'b'}F_{b'\bar{z}} + F_{za'}G^{a'\bar{v}}(G^{v\bar{v}})^{-1}G^{vb'}F_{b'\bar{z}})dzd\bar{z} + (F_{zb}G^{b\bar{v}}dz - G^{v\bar{v}}du)(F_{za}G^{av}d\bar{z} - G^{v\bar{v}}d\bar{u}) \right) \quad (62)$$

Since for  $a = v$  as well as for  $b = \bar{v}$  we have  $G^{v\bar{v}}G^{ab} = G^{a\bar{v}}G^{vb}$ , we introduce also an index  $a'$  taking values  $w, \bar{v}$  and  $b'$  with values  $v, w$ . The unwieldy coefficient of  $dzd\bar{z}$  simplifies due to the identity (see Appendix for the proof)

$$\det K_{(z,u)} = \det \begin{pmatrix} F_{z\bar{z}} - F_{za'}G^{a'b'}F_{b'\bar{z}} & F_{za'}G^{a'\bar{v}} \\ G^{vb'}F_{b'\bar{z}} & -G^{v\bar{v}} \end{pmatrix} = 1, \quad (63)$$

and the metric has the following simple form

$$ds^2 = \frac{1}{\beta} \left( dzd\bar{z} + (\alpha dz + \beta du)(\bar{\alpha}d\bar{z} + \bar{\beta}d\bar{u}) \right) \quad (64)$$

where

$$\alpha = F_{zb}G^{b\bar{v}}, \quad \beta = -G^{v\bar{v}}. \quad (65)$$

To find expressions for the exact form of  $\alpha, \beta$ , and  $u$  note that

$$u = F_v = -\frac{1}{2\pi i} \sum_{i=1}^k \oint_{C_i^1} \frac{d\zeta}{\zeta} \frac{1}{\sqrt{\mathbf{z}}} \log(\sqrt{\mathbf{z}} - q_i), \quad (66)$$

$$\bar{u} = F_{\bar{v}} = \frac{1}{2\pi i} \sum_{i=1}^k \oint_{C_i^1} \frac{d\zeta}{\zeta} \frac{\zeta^2}{\sqrt{\mathbf{z}}} \log(\sqrt{\mathbf{z}} - q_i), \quad (67)$$

$$F_w = -4\mu - \frac{1}{2\pi i} \sum_{i=1}^k \oint_{C_i^1} \frac{d\zeta}{\sqrt{\mathbf{z}}} \log(\sqrt{\mathbf{z}} - q_i). \quad (68)$$

If we introduce  $p_n$  defined by

$$p_n = \frac{1}{2\pi i} \sum_{i=1}^k \oint_{C_i^1} \frac{d\zeta}{2} \frac{\zeta^{n+2}}{\mathbf{z}} \left( -\frac{\log(\sqrt{\mathbf{z}} - q_i)}{\sqrt{\mathbf{z}}} + \frac{1}{\sqrt{\mathbf{z}} - q_i} \right) \quad (69)$$

then

$$\mathcal{G}^{-1} = \mathcal{F} = \begin{pmatrix} -p_{-2} & -p_{-1} & p_0 \\ -p_{-1} & -p_0 & p_1 \\ p_0 & p_1 & -p_2 \end{pmatrix}, \quad (70)$$

and

$$F_{zv} = -p_{-3}, \quad F_{zw} = -p_{-2}, \quad F_{z\bar{v}} = p_{-1}. \quad (71)$$

Let  $D = \det F$ , then

$$\alpha = \frac{1}{D} \begin{vmatrix} p_{-3} & p_{-2} & p_{-1} \\ p_{-2} & p_{-1} & p_0 \\ p_{-1} & p_0 & p_1 \end{vmatrix}, \quad \beta = \frac{1}{D} \begin{vmatrix} p_{-1} & p_0 \\ p_0 & p_1 \end{vmatrix}. \quad (72)$$

## 6 Comparison of the two approaches

From the form of the one-forms  $A_0$  and  $B_0$  of equation (46) combined with the expression (15, 17) for  $\chi$  we can identify  $u = \chi(0)$  and

$$\overline{A_0} = (2\mu - l) \frac{dz}{z}, \quad \overline{B_0} = du - (L(0) - u) \frac{dz}{z}. \quad (73)$$

The two expressions (51),(64) for the metric coincide if

$$\alpha = \left( \gamma - (L(0) - u) \frac{\delta}{2\mu - l} \right) e^{i\phi}, \quad (74)$$

$$\beta = z \frac{\delta}{2\mu - l} e^{i\phi} \quad (75)$$

for some real-valued function  $\phi$ . To start, we use the expressions (45) for  $\delta$  and  $\gamma$  to find

$$\frac{\delta}{2\mu - l} = \frac{[1, a, aC, a^2C]}{[(2\mu - l)a + a^2C, 1, aC, C]} \quad (76)$$

$$\gamma - \frac{(L(0) + u)\delta}{2\mu - l} = -\frac{[L(0) + u - C, a, aC, a^2C]}{[(2\mu - l)a + a^2C, 1, aC, C]} \quad (77)$$

On the other hand, introducing  $\Pi_l = \prod_{j \neq l} (a_l - a_j)$  and using (69), one finds

$$p_2 = -\frac{2\mu - l}{\bar{z}} - \frac{1}{\bar{z}} \sum_{l=1}^4 \frac{a_l^4}{\Pi_l} C_l, \quad (78)$$

$$p_n = -\frac{1}{\bar{z}} \sum_{l=1}^4 \frac{a_l^{n+2}}{\Pi_l} C_l, \quad -2 \leq n \leq 1 \quad (79)$$

$$p_{-3} = -\frac{1}{\bar{z}} \frac{1}{a_1 a_2 a_3 a_4} (L(0) - \frac{1}{2} \chi(0)) - \frac{1}{\bar{z}} \sum_{l=1}^4 \frac{1}{a_l \Pi_l} C_l, \quad (80)$$

which leads to the following expressions for the determinants (see Appendix)

$$\bar{z}^3 \begin{vmatrix} p_{-3} & p_{-2} & p_{-1} \\ p_{-2} & p_{-1} & p_0 \\ p_{-1} & p_0 & p_1 \end{vmatrix} = \frac{1}{a_1 a_2 a_3 a_4} \frac{[L(0) - \frac{1}{2} \chi(0) - C, a, aC, a^2 C]}{[1, a, a^2, a^3]}, \quad (81)$$

$$p_{-1} p_1 - p_0^2 = \frac{-1}{\bar{z}^2} \frac{[1, a, aC, a^2 C]}{[1, a, a^2, a^3]}, \quad (82)$$

$$\bar{z}^3 D = -\frac{[(2\mu - l)a + a^2 C, 1, aC, C]}{[1, a, a^2, a^3]}. \quad (83)$$

Indeed, recalling that  $a_1 a_2 a_3 a_4 = z/\bar{z}$  we find

$$\alpha = -\frac{\bar{z}}{z} \frac{[L(0) - \frac{1}{2} \chi(0) - C, a, aC, a^2 C]}{[(2\mu - l)a + a^2 C, 1, aC, C]}, \quad (84)$$

$$\beta = \bar{z} \frac{[1, a, aC, a^2 C]}{[(2\mu - l)a + a^2 C, 1, aC, C]}. \quad (85)$$

Thus the relations (74) indeed hold with  $e^{i\phi} = \bar{z}/z$ .

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## Appendix

### Resultants

Here we find the condition on coefficients  $\mathcal{C}_i$  and  $\mathcal{D}_i$  for the following system of equations to have a solution in  $\zeta$

$$\sum_{j=1}^4 \frac{\mathcal{C}_j}{\zeta - a_j} = 0, \quad \sum_{j=1}^4 \frac{\mathcal{D}_j}{\zeta - a_j} = 0.$$

To compare with the expressions of Section 4 put  $\mathcal{C}_j = C_j da_j$  and  $\mathcal{D}_j = D_j da_j$ . For convenience let us introduce polynomials  $h(\zeta) = \prod_{i=1}^4 (\zeta - a_i)$  and  $g_j(\zeta) = h(\zeta)/(\zeta - a_j)$ . Then for

$$M(\zeta) = \sum_{i=1}^4 \mathcal{C}_i g_i(\zeta), \quad N(\zeta) = \sum_{i=1}^4 \mathcal{D}_i g_i(\zeta), \quad (86)$$

the above system is equivalent to the system of two third order equations  $M(\zeta) = 0 = N(\zeta)$ .  $M$  and  $N$  have no common root if and only if the six polynomials  $M(\zeta), \zeta M(\zeta), \zeta^2 M(\zeta), N(\zeta), \zeta N(\zeta), \zeta^2 N(\zeta)$  are linearly independent. These are fifth order polynomials and can be expanded in the basis formed by e.g.  $h(\zeta), \zeta h(\zeta), g_i(\zeta)$ .

It is convenient to introduce

$$A_0 = \sum_{i=1}^4 \mathcal{C}_i, \quad B_0 = \sum_{i=1}^4 \mathcal{D}_i, \quad A_1 = \sum_{i=1}^4 a_i \mathcal{C}_i, \quad B_1 = \sum_{i=1}^4 a_i \mathcal{D}_i, \quad (87)$$

then

$$\zeta M(\zeta) = A_0 h(\zeta) + \sum a_j \mathcal{C}_j g_j(\zeta) \quad (88)$$

$$\zeta^2 M(\zeta) = A_0 \zeta h(\zeta) + A_1 h(\zeta) + \sum a_j^2 \mathcal{C}_j g_j(\zeta), \quad (89)$$

with analogous expressions for  $N(\zeta)$ .

Now the condition for the existence of a solution is the vanishing of the Sylvester determinant

$$R = \begin{vmatrix} B_0 & B_1 & a_1^2 \mathcal{D}_1 & a_2^2 \mathcal{D}_2 & a_3^2 \mathcal{D}_3 & a_4^2 \mathcal{D}_4 \\ 0 & B_0 & a_1 \mathcal{D}_1 & a_2 \mathcal{D}_2 & a_3 \mathcal{D}_3 & a_4 \mathcal{D}_4 \\ 0 & 0 & \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_4 \\ A_0 & A_1 & a_1^2 \mathcal{C}_1 & a_2^2 \mathcal{C}_2 & a_3^2 \mathcal{C}_3 & a_4^2 \mathcal{C}_4 \\ 0 & A_0 & a_1 \mathcal{C}_1 & a_2 \mathcal{C}_2 & a_3 \mathcal{C}_3 & a_4 \mathcal{C}_4 \\ 0 & 0 & \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{C}_4 \end{vmatrix} \quad (90)$$

One finds  $R$  to be given by

$$R = \epsilon^{ijkl} \mathcal{C}_i \mathcal{D}_j a_k (A_0 \mathcal{D}_k - B_0 \mathcal{C}_k) \left[ a_l^2 (A_0 \mathcal{D}_l - B_0 \mathcal{C}_l) - a_l (B_1 \mathcal{C}_l - A_1 \mathcal{D}_l) \right]. \quad (91)$$

Introducing  $A_2 = \sum_{i=1}^4 C_i/a_i$  and  $B_2 = \sum_{i=1}^4 D_i/a_i$  we have

$$R = \begin{vmatrix} B_0 & 0 & a_1^2 \mathcal{D}_1 & a_2^2 \mathcal{D}_2 & a_3^2 \mathcal{D}_3 & a_4^2 \mathcal{D}_4 \\ 0 & 0 & a_1 \mathcal{D}_1 & a_2 \mathcal{D}_2 & a_3 \mathcal{D}_3 & a_4 \mathcal{D}_4 \\ 0 & -B_2 & \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_4 \\ A_0 & 0 & a_1^2 \mathcal{C}_1 & a_2^2 \mathcal{C}_2 & a_3^2 \mathcal{C}_3 & a_4^2 \mathcal{C}_4 \\ 0 & 0 & a_1 \mathcal{C}_1 & a_2 \mathcal{C}_2 & a_3 \mathcal{C}_3 & a_4 \mathcal{C}_4 \\ 0 & -A_2 & \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{C}_4 \end{vmatrix} \quad (92)$$

### Determinant relations

Computing the determinant of the  $8 \times 8$  matrix

$$\begin{vmatrix} aC & 0 \\ C & C/a \\ a^2D & aD \\ aD & 0 \\ 0 & D \\ D & D/a \\ 0 & C \\ 0 & aC \end{vmatrix} = \begin{vmatrix} aC & 0 \\ 0 & C/a \\ 0 & aD \\ aD & 0 \\ -aD & D \\ 0 & D/a \\ -aC & C \\ -a^2C & aC \end{vmatrix} = \begin{vmatrix} aC & 0 \\ 0 & C/a \\ 0 & aD \\ aD & 0 \\ 0 & D \\ 0 & D/a \\ 0 & C \\ -a^2C & aC \end{vmatrix} = 0. \quad (93)$$

On the other hand the same determinant equals

$$\begin{aligned} & -[aC, C, a^2D, D][aC, C, D, D/a] + [C/a, C, aC, D][aC, a^2D, aD, D] \\ & + [aC, C, aD, D]^2. \end{aligned} \quad (94)$$

Thus

$$\begin{aligned} & [aC, C, a^2D, aD] \overline{[a^2C, aC, aD, D]} = \\ & [C, aC, a^2C, aD] \overline{[C, aC, a^2C, aD]} + [C, aC, D, aD]^2 \end{aligned} \quad (95)$$

which produces equation (43) used in Section 4.

### The determinant of $K_{(z,u)}$

The function  $z(\zeta)$  trivially satisfies the relations  $z(\zeta)_{z\bar{z}} = -z(\zeta)_{v\bar{v}}$ ,  $z(\zeta)_{za'} = z(\zeta)_{v(a'-1)}$ ,  $z(\zeta)_{b'\bar{z}} = -z(\zeta)_{(b'+1)\bar{v}}$ , which implies analogous relations for  $F$ .

$$\det K_{(z,u)} = -G^{v\bar{v}} F_{z\bar{z}} + F_{za'} \left( G^{v\bar{v}} G^{a'b'} - G^{a'\bar{v}} G^{vb'} \right) F_{b'\bar{z}}. \quad (96)$$

For an  $n \times n$  matrix  $A$  we denote by  $A^{(k)}$  the  $k$ -th compound matrix, which is the matrix composed of all order  $k$  minors of  $A$ . The adjugate compound matrix  $\text{adj}^{(k)}$  is obtained from  $A^{(k)}$  by replacing each  $k$ -minor by its complementary minor with the corresponding factor of  $(-1)^l$  and transposition. In other words, due to the Laplace expansion of determinants,  $A^{(k)}\text{adj}^{(k)}A = \det A \mathbf{1}$ . Thus from the Binet-Cauchy theorem

$$G^{(2)} = \left(F^{(2)}\right)^{-1} = \frac{1}{\det F} \text{adj}^{(2)} F. \quad (97)$$

Thus utilizing the relations  $F_{z\bar{z}} = -F_{v\bar{v}}$ ,  $F_{za'} = F_{v(a'-1)}$ , and  $F_{b'\bar{z}} = -F_{(b'+1)\bar{v}}$  we have

$$\det K_{(z,u)} = \frac{1}{\det F} \left( F_{v\bar{v}} F_{v\bar{v}}^{(1)} - F_{v(a'-1)} F_{(b'+1)\bar{v}} \text{adj}^{(2)} F_{(v,a';b',\bar{v})} \right). \quad (98)$$

The expression in brackets above is exactly the Cauchy expansion for  $\det F$ . Thus we have  $\det K_{(z,u)} = 1$ .

## Relations between the $p_n$ and $C_l$ expressions

We observe that

$$[1, a, a^2, a^3] = \epsilon^{ijkl} \Pi_i \Pi_j \frac{a_k - a_j}{a_j - a_i}, \quad (99)$$

without summation over the repeated indices. Then

$$\begin{aligned} (2\bar{z})^2 (p_{-1} p_1 - p_0^2) &= \sum_{i,l} \frac{a_l - a_i}{\Pi_i \Pi_l} a_i C_i a_l^2 C_l = \\ &= \frac{-1}{2} \frac{1}{[1, a, a^2, a^3]} \epsilon^{ijkl} (a_k - a_j) a_i C_i a_l^2 C_l = -\frac{[1, a, aC, a^2 C]}{[1, a, a^2, a^3]}. \end{aligned} \quad (100)$$

Next note the identities

$$\sum_{l=1}^4 \frac{a_l^k}{\Pi_l} = \begin{cases} 1, & k = 3 \\ 0, & 0 \leq k \leq 2 \\ -1/a_1 a_2 a_3 a_4, & k = -1 \end{cases} \quad (101)$$

Using these we have

$$-(2\bar{z})^3 \begin{vmatrix} p_{-3} & p_{-2} & p_{-1} \\ p_{-2} & p_{-1} & p_0 \\ p_{-1} & p_0 & p_1 \end{vmatrix} =$$

$$\begin{aligned}
&= \det \begin{pmatrix} 1/a \\ 1 \\ a \\ a^2 \end{pmatrix} \text{diag} \left( \frac{1}{\Pi_1}, \frac{1}{\Pi_2}, \frac{1}{\Pi_3}, \frac{1}{\Pi_4} \right) (C - L(0) + \frac{1}{2}\chi(0), aC, a^2C, a) = \\
&= -\frac{1}{a_1 a_2 a_3 a_4} \frac{[L(0) - \frac{1}{2}\chi(0) - C, a, aC, a^2C]}{[1, a, a^2, a^3]},
\end{aligned}$$

as well as

$$\begin{aligned}
(2\bar{z})^3 D &= \det \begin{pmatrix} 1 \\ a \\ a^2 \\ a^3 \end{pmatrix} \text{diag} \left( \frac{1}{\Pi_1}, \frac{1}{\Pi_2}, \frac{1}{\Pi_3}, \frac{1}{\Pi_4} \right) (C, aC, a^2C + a(2\mu - l), 1) = \\
&= -\frac{[(2\mu - l)a + a^2C, 1, aC, C]}{[1, a, a^2, a^3]}. \tag{102}
\end{aligned}$$

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