The appearence of the resolved singular hypersurface $x_0x_1 - x_2{}^n = 0$ in the classical phase space of the Lie group $SU(n)$

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Abstract

A classical phase space with a suitable symplectic structure is constructed together with functions which have Poisson brackets algebraically identical to the Lie algebra structure of the Lie group $SU(n)$. In this phase space we show that the orbit of the generators corresponding to the simple roots of the Lie algebra give rise to fibres that are complex lines containing spheres. There are $n-1$ spheres on a fibre and they intersect in exactly the same way as the Cartan matrix of the Lie algebra. This classical phase space bundle, being compact, has a description as a variety. Our construction shows that the variety containing the intersecting spheres is exactly the one obtained by resolving the singularities of the variety $x_0x_1 - x_2{}^n = 0$ in $\mathcal{C}^3$. A direct connection between this singular variety and the classical phase space corresponding to the Lie group $SU(n)$ is thus established.

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I Introduction

It has long been known that there is an intriguing algebraic correspondence between the Cartan matrix of simply laced Lie groups and the intersection matrix of spheres that appear when certain simple singularities are resolved[1]. The reason for such a correspondence has also been long known within the framework of algebraic groups [2] . That this correspondence might be more than a mathematical curiosity was established when it was shown that duality in string theory made effective use of such a link[3]. A type 2A string compactified on a $K3$ surface (a four dimensional surface) was conjectured to be dual to a heterotic string compactified on T^4 (the four torus). A test of this conjecture required the zero mass excitations in the two theories to match.The zero mass excitations at the type 2A end came from certain singular points that appear on the $K3$ surface in a certain limit while those at the heterotic string end came from gauge excitations associated with an $SU(n)$ Lie group.The excitations at the type 2A end were "classical" solitonic type excitations while those at the heterotic end were "quantum" gauge excitations. This result suggests that a link between minimally resolved singularities and the "classical limit" of simply laced Lie groups might exist.

In this paper we establish such a link. We demonstrate this link explicitly for the Lie groups $SU(2)$, $SU(3)$ and $SU(4)$. Generalisation to $SU(n)$ is then straightforward. We find that using a Gauss decomposition Z_+HZ_- , where Z_+ is an upper triangular matrix with unit diagonal elements , H is a diagonal matrix, Z_- is a lower triangular matrix with unit diagonal elements that the classical phase space, in which Poisson brackets mirror the Lie algebra structure, can be constructed using a standard coherent state approach [4].We use the Gauss decomposition in order to use the Borel-Weil Theorem. This Theorem shows how irreducilbe representations of a compact Lie group G can be constructed as holomorphic sections over G/T where T is the maximal torus of G. The approach thus describes the group in terms of complex variables and is thus a natural setting for making contact between the group and complex algebraic varieties. We have

Borel – Weil Theorem 1 [5] *The space of holomorphic sections of the line bundle L_λ over G/T is non trivial if λ is the highest weight of an irreducible representation of G. When λ is the highest weight then this space of holomorphic sections is a realisation of the representation space V_λ of G.*

The coherent state approach is an explicit way of implementing the Borel-Weil Theorem. Use is made, in this approach, of the fact that G/T=G_c/B where G_c is the complexification of G and B is the Borel subgroup. In terms of the Gauss decomposition B is generated by H and Z_-. It is in this framework that we search for and identify classical phase space. For $SU(n)$ the phase space is CP^{n-1}. This is a Kähler symplectic manifold. It is identified cleanly in G_c/B as follows.. We first introduce some definitions. Triangular matrices with unit diagonal elements are called unipotent. Z_+,Z_- are unipotent.We claim that by using unipotent group elements of a special kind we can construct the required Kähler symplectic manifold. To do this the unipotent elements used must have two special features. First they must mutually commute . Second the number of elements that commute with the these unipotent elements must equal the rank r of the group. Such elements are called regular.In this coherent state approach the symplectic structure is derived from a Kähler potential which we construct from a scalar product using the group elements described acting on the highest weight vector for the fundamental representation. It is then shown that the expectation value of the generators of the Lie algebra have Poisson brackets, defined in terms of this symplectic structure,isomorphic to the Lie algebra of the group. On this Kähler phase space a fibrebundle structure can be constructed whose fibres are complex curves, containing intersecting spheres. The fibre bundle considered is constructed from the orbit of group elements arising from the simple roots of the corresponding Lie algebra.In our construction we classify the unipotent elements of Z_+ in the following way :

One class have centralizer (elements of the group commuting among themselves) of dimension equal to the rank r of the group. These are the regular elements. The other class comes from the generators of the Lie algebra for simple roots. These are unipotent elements that commute with $r+2$ group elements . Such elements are called subregular . These unipotent subregular elements play a crucial role in our construction. The special unipotent regular elements are used to construct classical phase space. This is done by acting on the highest weight vector with the the elements described. Choosing to work with the highest weight vector is the way the quotienting of G_c by B is implemented in this framework. The unipotent subregular elements acting on a point of this phase space give rise to a fibre containing intersecting spheres. The way these spheres intersect can be summarised in the form of an intersection matrix. This intersection matrix is found to be identical to the negative of the Cartan matrix of the Lie algebra. We show this explicitly for $SU(3)$ where there are two intersecting spheres , for $SU(4)$ where there are three spheres and for $SU(n)$ where there are $n-1$ spheres. Precisely such intersecting spheres also appear when certain singular points are resolved on the hypersurface $x_0x_1 - x_2^n = 0$ where $(x_0, x_1, x_2) \in \mathcal{C}^3$, the space of three complex variables . In section-II we summarise the basic facts we need from the theory of resolution of singularities. In section-III the classical phase space for the Lie groups $SU(2)$, $SU(3)$ and $SU(4)$ is constructed and the link with the resolved singularities established. Finally in section-IV we summarise our conclusions.

II Resolving Singularities

Consider the hypersurface V_n in $\mathcal{C}^3$ defined by the algebraic equation:

$$V_n(x_0, x_1, x_2) = x_0x_1 - x_2^n = 0 \tag{1}$$

where n is an integer ≥ 2 and $(x_0, x_1, x_2) \in \mathcal{C}^3$. We have the following definition[1, 6] :

Definition : *A point $(x_0, x_1, x_2) \in V_n = 0$ is a singular point of the hypersurface if $\partial_{x_i} V_n = 0$ at that point.*

It follows from the definition that the point $(0, 0, 0)$ i.e. the origin is a singular point of the hypersurface $V_n \equiv x_0x_1 - x_2^n = 0$, for $n \geq 2$. Indeed a simple definition of this hypersurface as an orbifold is possible. To see this set $x_0 = \xi^n$, $x_1 = \eta^n$, $x_2 = \xi\eta$ where $(\xi, \eta) \in \mathcal{C}^2$. We note that in terms of these variables ξ, η the equation $x_0x_1 - x_2^n = 0$ is identically satisfied i.e. ξ, η parametrise the hypersurface $V_n = 0$. There is however one restriction on the variables ξ, η when they are on the hypersurface $V_n = 0$ namely the point (ξ, η) must be identified with $(\omega^{\frac{1}{n}}\xi, \omega^{\frac{1}{n}}\eta)$ where ω is an nth root of identity $(\omega^n = 1)$. Thus the hypersurface $V_n = 0$ can be identified with the orbifold $\mathcal{C}^2/\mathcal{Z}_n$ with $\mathcal{Z}_n$ action defined by $(\xi, \eta) \longrightarrow (\omega^{\frac{1}{n}}\xi, \omega^{\frac{1}{n}}\eta)$, $\omega^n = 1$.

There is a standard method of minimally resolving this singularity [1,6] i. e. of constructing a globally well defined hypersurface which is in $1-1$ correspondence with the original hypersurface $V_n = 0$ except at the point $(0,0,0)$. The singular point is "blown up" . We describe this procedure first for the case $n = 2$ and $n = 3$ and then for the general case where $n > 3$.
Let us introduce the space $\mathcal{C}^3 \times \mathcal{P}^2$ where $\mathcal{P}^2$ is the complex projective two space (henceforth we denote $\mathcal{CP}^n$ by $\mathcal{P}^n$). Points in $\mathcal{C}^3 \times \mathcal{P}^2$ can be written as the pair $((x_0,x_1,x_2),[s_0,s_1,s_2])$ where $(x_0,x_1,x_2) \in \mathcal{C}^3$ and $[s_0,s_1,s_2]$ i s an element of $\mathcal{P}^2$ i.e. it represents the equivalence class of points (s_0,s_1,s_2) under the equivalence relation $(s_0,s_1,s_2) \sim \lambda(s_0,s_1,s_2)$ where λ is a complex number $\neq 0$. Next we introduce the space $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$. This is defined as the set:

$$\mathcal{C}^3(\mathcal{P}^2,\mathcal{R}) = \{(x_0,x_1,x_2),[s_0,s_1,s_2]|x_i s_j = x_j s_i, \forall i,j\} \qquad (2)$$

Geometrically the restriction $x_i s_j = x_j s_i$ means that s_i is proportional to x_i. This gives a space consisting of points (x_0,x_1,x_2) in $\mathcal{C}^3$ and lines through the origin and these points. These lines are elements of $\mathcal{P}^2$. Thus for all points in $\mathcal{C}^3$, other than the origin , the element of $\mathcal{P}^2$ is uniquely fixed by (x_0,x_1,x_2). There is thus an $1-1$ correspondence between points in $\mathcal{C}^3$ and the pair of points in $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$ defined by the eq.(2). For the origin however the situation is different. When $x_0 = x_1 = x_2 = 0$, there is no restriction on $[s_0,s_1,s_2]$. Thus the origin of $\mathcal{C}^3$ is replaced by the entire $\mathcal{P}^2$ in $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$; it is "blown up". Let us now study the way the hypersurface $x_0x_1 - x_2^2 = 0$ behaves in $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$. To see the way the singular point in $V_2 = 0$ in $\mathcal{C}^3$ gets mapped in $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$ we approach the origin in $\mathcal{C}^3$. This is done by scaling the points (x_0,x_1,x_2) in $\mathcal{C}^3$ by t and letting $t \longrightarrow 0$. Note that the constraints $x_i s_j = x_j s_i \forall i,j$ imply that $x_i = k s_i$ (where $k = constant$). Thus $(tx_0,tx_1,tx_2) = tk(s_0,s_1,s_2)$ i.e., we get from eqs.(1) and (2)in the $t \longrightarrow 0$ limit , points on $V_2 = 0$ satisfy $s_0s_1 - s_2^2 = 0$ in $\mathcal{P}^2$. We now have the following theorem.
Theorem 2 [7] *A polynomial equation of degree n in $\mathcal{P}^2$ describes a compact Riemann surface of genus g with $g = \frac{1}{2}(n-1)(n-2)$.*
In our case the polynomial equation $s_0s_1 - s_2^2 = 0$ in $\mathcal{P}^2$ is of degree 2 . Hence the surface in $\mathcal{P}^2$ is a genus zero surface i.e., topologically it is a sphere. Thus the singular point of the hypersurface $x_0x_1 - x_2^2 = 0$ in $\mathcal{C}^3$ is replaced by a sphere in $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$. The singularity has been resolved by a process of "blowing up" tuning the singular point into a sphere.
We next consider the case $n = 3$. Repeating the procedure for the $n = 2$ case we find , the points $[s_0,s_1,s_2]$ satisfying eq.(1) for $n = 3$ i.e. $V_3 = 0$ and eq.(2) in the viscinity of the origin now have to satisfy the polynomial equation $t^2(s_0s_1 - kts_2^3) = 0$ i.e., the equation $s_0s_1 = 0$ in the $t \longrightarrow 0$ limit. This gives a pair of spheres $\mathcal{P}^1$'s in $\mathcal{P}^2$ (theorem 2) corresponding to setting $s_0 = 0$, $s_1 = 0$. These two spheres intersect once at the point $(0,0,1)$ in $\mathcal{P}^2$.For $n \geq 4$, we again get the equation $s_0s_1 = 0$ in the limit $t \longrightarrow 0$ and a pair of spheres. However the intersection of these spheres is still a singular point. To see this we choose to describe $\mathcal{C}^3(\mathcal{P}^2,\mathcal{R})$ by first selecting a point in $\mathcal{P}^2$, say , (s_0,s_1,s_2) with $s_2 \neq 0$. Choosing this point does not uniquely fix a point in $\mathcal{C}^3$ but gives a line through the point (s_0,s_1,s_2) and the origin in $\mathcal{C}^3$. Let us set $s_2 = y_2$ $s_0 = y_0y_2$, $s_1 = y_1y_2$, where $(y_0,y_1,y_2) \in \mathcal{C}^3$. Finally set $y_2 = x_2$. Then $x_0 = y_0y_2$, $x_1 = y_1y_2$ and $0 = x_0x_1 - {x_2}^n = {y_2}^2(y_0y_1 - {y_2}^{n-2}) = 0$. By construction $y_2 \neq 0$. So $y_0y_1 - {y_2}^{n-2} = 0$. For $n \geq 4$ this hyperplane has a singularity at the origin $y_0 = y_1 = y_2 = 0$ where $s_0 = 0$, $s_1 = 0$. The process of blowing up has to be repeated . The spheres produced

Figure 1: Dynkin diagram 1

Figure 2: Dynkin diagram 2

by this process of blowing up self intersect in an invariant way.Following a standard procedure it can be shown that the self intersection of the spheres can be taken to be -2 [1]. We summarise the results presented regarding the way the spheres in the "resolved singularity" intersect in the form of a matrix. For $n = 3$, we have the intersection matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. For $n = 4$ we have , $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$. This intersection information can be encoded in the form of a Dynkin diagram shown in fig.1 where each dot denotes a sphere with self intersection -2 and a line joining two dots intersection between two spheres with intersection number one. The construction described here extends to the case of arbitrary n where the diagram is shown in fig.2

III The classical phase space for the Lie groups $SU(2)$,$SU(3)$ and $SU(4)$

We now look at the classical origins of the Lie groups $SU(2)$,$SU(3)$ and $SU(4)$ in the following sense. The groups have a local structure encoded by their Lie algebras. We will call the associated phase space , defined with a suitable symplectic structure , the classical counterpart of the Lie group if functions on the phase space can be constructed which have Poisson brackets algebraically identical to the Lie algebra structure of the Lie group. The construction we will describe involves coherent states associated with the Lie group of interest [4]. We start by quickly summarising the results for $SU(2)$. This simple example contains a crucial ingredient needed for our subsequent analysis. Let us introduce the highest weight representation for $SU(2)$, which we write as the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The coherent state $|\lambda\rangle$ is then defined as:

$$|\lambda\rangle = e^{\lambda J_+} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv e_{12}, \langle\lambda|\lambda\rangle = (1 + \lambda\bar{\lambda}) \tag{3}$$

λ being a complex variable. We then construct the Kähler potential
$V(\lambda, \bar{\lambda}) = k \cdot \log(1 + \lambda\bar{\lambda})$. This gives rise to a symplectic form on the coordinate chart $\lambda_0 \neq 0$ in $\mathcal{P}^1$ as well as the Fubini-Study metric [6,8] where $\lambda = \frac{\lambda_1}{\lambda_0}$ .The symplectic structure is given by

$$\omega_{\lambda\bar{\lambda}} = \partial_\lambda \partial_{\bar{\lambda}} V(\lambda\bar{\lambda}), \omega = \omega_{\lambda\bar{\lambda}} d\lambda \wedge d\bar{\lambda} + \omega_{\bar{\lambda}\lambda} d\bar{\lambda} \wedge d\lambda \tag{4}$$

where the symplectic matrix is given by

$$[\omega] = k \cdot \begin{pmatrix} 0 & \frac{1}{(1+\lambda\bar{\lambda})^2} \\ -\frac{1}{(1+\lambda\bar{\lambda})^2} & 0 \end{pmatrix} \tag{5}$$

k is a constant. We next note that

$$X_+ \;=\; \frac{\langle\lambda|J_+|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{2\lambda}{1 + \lambda\bar{\lambda}}$$

$$
\begin{aligned}
X_- &= \frac{\langle\lambda|J_-|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{2\bar{\lambda}}{1+\lambda\bar{\lambda}} \\
X_0 &= \frac{\langle\lambda|J_0|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}
\end{aligned}
\tag{6}
$$

with $J_- = e_{21}$,$J_+ = e_{12}$, $J_0 = e_{11} - e_{22}$, where e_{ij} stands for the 3×3 matrix with one in the ijth position and zero elsewhere, and X_+,X_-, X_0 are functions on the phase space $\mathcal{P}^1$ described by the complex variable λ and the symplectic form ω given by eq.(4).

Furthermore $X_+X_- + {X_0}^2 = 1$ i.e these functions represent points on S^2. Also

$$
\begin{aligned}
\{X_+, X_-\} &= (\omega^{-1})_{\lambda\bar{\lambda}}\partial_\lambda X_+ \partial_{\bar{\lambda}} X_- + (\omega^{-1})_{\bar{\lambda}\lambda}\partial_{\bar{\lambda}} X_+ \partial_\lambda X_- \\
&= 2iX_0 \\
\{X_0, X_\pm\} &= \pm X_\pm
\end{aligned}
\tag{7}
$$

for suitable choice of k. Thus the expectation values of the generators $J_\pm$, J_0 in the normalised state vector $|\lambda\rangle$ represent the classical functions whose quantisation , achieved by replacing Poisson brackets by commutators leads to the Lie algebra structure. The classical phase space of $SU(2)$ is thus S^2 or $\mathcal{P}^1$. Note that the presence of S^2 could be spotted simply by evaluating $\int \omega d\lambda \wedge d\bar{\lambda} = 4\pi$, where ω is given by eq. (4) and noting that the curvature of the phase space manifold is constant and positive. Also the metric on the phase space derived from the Kähler potential can be seen to be precisely the metric on S^2. The emergence of S^2 for the Lie group $SU(2)$ is the key observation we want to record. For the groups $SU(3)$,$SU(4)$ we will construct appropriate Kähler forms which describe the classical phase space associated with these groups. It will then be demonstrated that the fibre obtained from the orbit of the generators corresponding to simple roots of the Lie algebra acting on this phase space contain intersecting spheres. The spheres can easily be identified by the presence of nontrivial cycles with $\int \omega = 4\pi$ on the fibre and a sphere metric in an appropriate subspace.The intersection properties of these spheres can be determined by using the methods of differential topology [6]. We demonstrate that the spheres described intersect in a manner precisely mirroring the Dynkin diagram of the group. Such a result was established by Tits and Steinberg in a different setting (see the theorem by Tits and Steinberg in the article by Brieskorn E. in ref.[2]).We saw in section II that the spheres present when the singular hypersurface considered there was resolved also contain intersecting spheres of exactly the same kind. We now make use of the following theorems:

Chow's Theorem 3 [6, 7] *A compact hypersurface can always be represented by an algebraic variety in a higher dimensional projective space.*

Theorem 4 [7]: *A complex curve in $\mathcal{C}^n$ can always be embedded in $\mathcal{C}^3$.*

The space constructed here is compact and hence the system can be represented as an algebraic variety and the intersecting spheres, present in $\mathcal{C}^{n-1}$, can be embedded in $\mathcal{C}^3$.

With the help of these theorems we see that the fibres containing the intersecting spheres can be embedded in $\mathcal{C}^3$. Thus the two different mathematical objects: the resolved singular curve and the algebraic curve present in the phase space bundle of $SU(n)$ both live in $\mathcal{C}^3$ and both contain the same number of spheres that intersect in the same way.

Now for the details. For $SU(3)$ we again work with the fundametal representation and introduce the coherent state

$$|\nu_1, \nu_2\rangle = e^{\nu_1 e_{13}} e^{\nu_2 e_{23}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ 1 \end{pmatrix} \tag{8}$$

Note that $g = e^{\nu_1 e_{13} + \nu_2 e_{23}} = \begin{pmatrix} 1 & 0 & \nu_1 \\ 0 & 1 & \nu_2 \\ 0 & 0 & 1 \end{pmatrix}$ has generators e_{13} , e_{23} that commute . Furthermore g commutes with e_{13} and e_{23} so that dimension of the centre of g is 2 = rank of $SU(3)$. It is thus a regular element in $SU(3, c)$. The element is also unipotent as it is an upper triangular matrix with unit diagonal elements.

The Kähler potential is given by

$$V(\nu_1, \nu_2, \bar{\nu}_1, \bar{\nu}_2) = k \cdot \log \langle \nu_1, \nu_2 | \bar{\nu}_1, \bar{\nu_2} \rangle = k \cdot \log(1 + \nu_1 \bar{\nu}_1 + \nu_2 \bar{\nu}_2) = k \cdot \log \langle \nu | \nu \rangle \tag{9}$$

and the symplectic structure determined by

$$\omega_{i\bar{j}} = k \cdot \partial_{\nu_i} \partial_{\bar{\nu_j}} V, \omega = \omega_{i\bar{j}} d\nu_i \wedge d\bar{\nu_j} \tag{10}$$

is precisely that on $\mathcal{P}^2$ [6,8] in the coordinate chart ${\nu'}_0 \neq 0$, $({\nu'}_0, {\nu'}_1, {\nu'}_2)$ being the homogeneous coordinates and ν_1, ν_2 stand for $\frac{{\nu'}_1}{{\nu'}_0}$ and $\frac{{\nu'}_2}{{\nu'}_0}$. Also the symplectic structure (10) can be proved to be global [6] . It is then easy to verify that the commutation relations of $SU(3)$ are reflected in the Poisson brackets between the functions $\langle e_{ij} \rangle$ and $\langle e_{kl} \rangle$ where $\langle e_{ij} \rangle = \frac{\langle \nu_1 \nu_2 | e_{ij} | \nu_1 \nu_2 \rangle}{\langle \nu_1 \nu_2 | \nu_1 \nu_2 \rangle}$

Note that

$$[\omega] = \begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b' & 0 \\ 0 & b' & 0 & c \\ -b & 0 & -c & 0 \end{pmatrix} \tag{11}$$

$$[\omega^{-1}] = \frac{1}{ac - bb'} \begin{pmatrix} 0 & -c & 0 & b' \\ c & 0 & -b & 0 \\ 0 & b & 0 & -a \\ -b' & 0 & a & 0 \end{pmatrix} \tag{12}$$

where

$$a = \partial_{\nu_1} \partial_{\bar{\nu_1}} \log \langle \nu | \nu \rangle, b = \partial_{\nu_1} \partial_{\bar{\nu_2}} \log \langle \nu | \nu \rangle, c = \partial_{\nu_2} \partial_{\bar{\nu_2}} \log \langle \nu | \nu \rangle \tag{13}$$

and prime stands for complex conjugate .

We now proceed to construct fibres on this phase space:

The generators corresponding to the simple roots of $SU(3)$ are e_{12} , e_{23} and the group elements $\in Z_+$ are $e^{\mu e_{12}}$, $e^{\lambda e_{23}}$. They are unipotent subregular elements $\in Z_+$. To see this set $x = e^{\mu e_{12}}$. Then $y = e^{\alpha e_{13} + \beta(h_1 + 2h_2) + \gamma e_{32} + \delta e_{12}} \in SU(3, c)$ where α, β, γ, δ are complex parameters ,commutes with x. So dimension of the centre of x in $SU(3, c)$ is 4 = 2+2 = rank of $SU(3)$ + 2 . Thus x is subregular in $SU(3, c)$. Similarly the element

$e^{\lambda e_{23}}$ is subregular.We now consider the orbits of $e^{\mu e_{12}}$ and $e^{\lambda e_{23}}$ at the base point on the phase space $\begin{pmatrix} \nu_1 \\ \nu_2 \\ 1 \end{pmatrix}$.

We have then μ-orbit and λ-orbit as the fibre elements as $\begin{pmatrix} \nu_1 + \mu\nu_2 \\ \nu_2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \nu_1 \\ \nu_2 + \lambda \\ 1 \end{pmatrix}$. Relabelling the first orbit as $\begin{pmatrix} z_1 \\ \nu_2 \\ 1 \end{pmatrix}$ we can associate with it a Kähler potential $\log(1+z_1\bar{z}_1)$. This is a sphere $\mathcal{P}^1$. Similarly the λ orbit is also a sphere. To demonstrate that these two spheres intersect with intersection number 1 we consider $e^{\mu e_{12}} e^{\lambda e_{23}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ 1 \end{pmatrix}$. The common z_1-z_2 (the first two coordinates) plane has an associated Kähler potential $\log(1+z_1\bar{z}_1+z_2\bar{z}_2)$. The symplectic structure is then given by equation (10) with $i, j = 1, 2$. We now note the following theorems:

Theorem 5 [6] *The de Rham cohomology and Dolbeault cohomology groups for $\mathcal{P}^n$ are related*:

$$H^{p,p}_{\bar{\partial}}(\mathcal{P}^n) \cong H^{2p}_{DR} \cong C$$

Theorem 6 [6] *The intersection of the two surfaces $\mathcal{C}_i$ and $\mathcal{C}_j$ are given by $\mathcal{C}_i \cdot \mathcal{C}_j = \frac{1}{(4\pi)^2} \int \omega_i \wedge \omega_j$ where the integration is in the space containing the surface $\mathcal{C}_i$ and $\mathcal{C}_j$ and is a space of dimension four and ω_i , ω_j are (de Rham cohomology) elements of $H^2_{DR}(M, R)$.*

The cohomology groups associated with the symplectic form constructed are Dolbeault cohomology groups while the intersection formula is valid for de Rham cohomology groups. However for $\mathcal{P}^n$ they are equivalent (theorem 5). We can thus determine intersection of spheres by simply evaluating $\frac{1}{(4\pi)^2} \int \omega \wedge \omega$, where the relevant four dimensional manifold (complex dimension 2) is the common z_1-z_2 plane . This is precisely seen to be one. Thus the two spheres on the fibre intersect once with intersection number one.

The procedure outlined can be repeated for $SU(4)$. This time

$$|\nu\rangle = |\nu_1, \nu_2, \nu_3\rangle = e^{\nu_1 e_{14}} e^{\nu_2 e_{24}} e^{\nu_3 e_{34}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ 1 \end{pmatrix} \tag{14}$$

and the Käler potential is

$$V(\nu_1, \nu_2, \nu_3, \bar{\nu_1}, \bar{\nu_2}, \bar{\nu_3}) = k \cdot \log \langle \nu | \nu \rangle = k \cdot \log(1 + \nu_1\bar{\nu_1} + \nu_2\bar{\nu_2} + \nu_3\bar{\nu_3}) \tag{15}$$

Here again

$$\omega_{i,\bar{j}} = \partial_{\nu_i} \partial_{\bar{\nu}_j} \log \langle \nu | \nu \rangle, \omega = \omega_{i,\bar{j}} d\nu_i \wedge d\bar{\nu}_j \tag{16}$$

is a symplectic structure on $\mathcal{P}^3$.The corresponding $[\omega]$ an d $[\omega^{-1}]$ matrices are given by

$$[\omega] = \begin{pmatrix} 0 & a & 0 & b & 0 & c \\ -a & 0 & -b' & 0 & -c' & 0 \\ 0 & b' & 0 & d & 0 & f \\ -b' & 0 & -d & 0 & -f' & 0 \\ 0 & c' & 0 & f' & 0 & g \\ -c & 0 & -f & 0 & -g & 0 \end{pmatrix} \tag{17}$$

where the primed entries stand for complex conjugates and

$$[\omega^{-1}] = \frac{1}{N} \begin{pmatrix} 0 & dg - ff' & 0 & fc' - gb' & 0 & b'f' - dc' \\ ff' - dg & 0 & bg - cf' & 0 & cd - bf & 0 \\ 0 & cf' - bg & 0 & ag - cc' & 0 & bc' - af' \\ gb' - fc' & 0 & cc' - ag & 0 & af - b'c & 0 \\ 0 & bf - cd & 0 & b'c - af & 0 & ad - bb' \\ dc' - b'f' & 0 & af' - bc' & 0 & bb' - ad & 0 \end{pmatrix} \tag{18}$$

where

$$\begin{aligned} N^2 = det[\omega], a = \partial_{\nu_1}\partial_{\bar{\nu_1}} \log \langle\nu|\nu\rangle, b = \partial_{\nu_1}\partial_{\bar{\nu_2}} \log \langle\nu|\nu\rangle, c = \partial_{\nu_1}\partial_{\bar{\nu_3}} \log \langle\nu|\nu\rangle \\ d = \partial_{\nu_2}\partial_{\bar{\nu_2}} \log \langle\nu|\nu\rangle, f = \partial_{\nu_2}\partial_{\bar{\nu_3}} \log \langle\nu|\nu\rangle, g = \partial_{\nu_3}\partial_{\bar{\nu_3}} \log \langle\nu|\nu\rangle \end{aligned} \tag{19}$$

and the prime denotes complex conjugate . Using $[\omega^{-1}]$ from eq.(18) it is again straightforward to verify that the commutation relations of $SU(4)$ are reflected in the Poisson bracket between the functions $\langle e_{ij}\rangle$ and $\langle e_{kl}\rangle$ where again $\langle e_{ij}\rangle \equiv \frac{\langle\nu|e_{ij}|\nu\rangle}{\langle\nu|\nu\rangle}$.

Finally we look at the fibre . The unipotent subregular elements corresponding to the simple roots are $e^{\mu e_{12}}$, $e^{\lambda e_{23}}$ and $e^{\rho e_{34}}$. So we have μ , λ and ρ orbits on the fibre at the base point $\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ 1 \end{pmatrix}$. Each element acts on a 2-complex dimensional subspace. It is easy to determine , as we have shown for the $SU(3)$ case that an individual orbit is $\mathcal{P}^1$ on the fibre. Two generators intersect if they act on a common subspace i.e group elements corresponding to e_{12} and e_{23} both act on the space labelled by 2, while e_{34} and e_{12} have no common subspace. A differential topology way of spotting this is to construct the vector obtained by acting $e^{\mu e_{12}}$ $e^{\lambda e_{23}}$, say , on the base point. The variables μ,λ appear in the 1,2 position. Setting the variables corresponding to 1 ,2 as z_1, z_2 the symplectic structure can be constructed from the corresponding Kähler potential and $\frac{1}{(4\pi)^2}\int \omega\wedge\omega = \frac{1}{(4\pi)^2}\int 2(\omega_{z_1\bar{z}_1}\omega_{z_2\bar{z}_2} - \omega_{z_1\bar{z}_2}\omega_{z_2\bar{z}_1})dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2$ gives the intersection. Similarly e_{23} , e_{34} will give rise to intersection in the subspace z_2-z_3. The intersection corresponding to e_{12}, e_{34} will be zero since they have no subspace in common. Here we have to evaluate $\int\omega\wedge\omega$ in the subspace 1,2 or 3,4 in each of which $\omega\wedge\omega$ vanishes. Hence there is no intersection. The intersection properties of the orbits described easily generalises to the case of $SU(n)$.For $SU(n)$ the symplectic structure will come from the group elements constructed from the generators e_{12}, e_{13},..e_{1n} while the simple roots are

e_{12},e_{23},..$e_{n,n-1}$. It is the orbit of the group elements generated by these simple roots acting at any point on phase space that give fibres containing $n-1$ intersecting spheres.

IV Conclusions

We have shown in two examples , i.e for $SU(3)$ and $SU(4)$ how a classical phase space can be associated with these groups. In this phase space the orbit of the generators corresponding to the simple roots of the Lie algebra gives rise to intersecting spheres as fibres. For $SU(n)$ the fibre of the bundle consists of $n-1$ (equal to the rank of $SU(n)$) intersecting spheres. These spheres intersect precisely as the negative of the Cartan matrix of $SU(n)$. A simple understanding of how this happens is provided in our work. The structure of $SU(n)$, contained in the commutation properties of the simple roots of its Lie algebra, is exactly the structure used to construct the orbit of these generators in phase space. This structure in phase space gives rise to the resolved variety associated with a singular variety. An algebraic group demonstration of the relationship between simple singularities of the ADE type and simply laced Lie groups of ADE type was proved by Brieskorn [2] where the role of the unipotent subregular elements of the groups was stressed and the earlier result regarding intersecting spheres of Tits and and Steinberg (theorem by Tits and Steinberg discussed in the article by Brieskorn E. in ref.[2]) stated. Our explicit construction uses unipotent regular elements of a special kind (viz. ones involving mutually commuting generators) to construct classical phase space and confirms the role played by subregular unipotent elements for making contact with resolved singularity. In our physically motivated approach it is geometrically very clear why unipotent subregular elements are crucial: they are the group elements that come from the simple roots of the Lie algebra. The classical phase space for $SU(n)$ is shown explicitly to be contained in G_c/B as CP^{n-1}. The orbit of the generators of the simple roots of the corresponding Lie algebra then provide a local trivialisation of a bundle contained in G_c/B with the classical phase space as the base. It is in the fibre of this space that the variety corresponding to the resolved singularity is contained. Our construction extends easily to $SU(n)$. Extention of the construction described here to the D,E groups should be straightforward. It is pleasing that classical phase space is where the resolved singular variety corresponding to $x_0x_1 - x_2{}^n = 0$ makes its appearence. Our work thus provides confirmation of the classical/quantum correspondance discovered in string theory between groups and singularities.

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