A path-integral representation of the free one-flavour staggered-fermion determinant

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Lattice fermion actions are constructed with path integrals which are equivalent to the free one-flavour staggered fermion determinant. The Dirac operators used are local and have an identical spectrum of states to the staggered theory. Operators obeying a generalised Ginsparg-Wilson relation are developed.

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I. INTRODUCTION

A complete study of QCD on the lattice requires the numerical simulation of dynamical fermions. These Monte Carlo calculations are extremely computationally costly, since the effects of quarks must be included by first integrating out the fermion path integral, and then describing the resulting non-local dynamics of the fermion determinant. For a review of recent developments, see *e.g.* [1, 2]. With present techniques, the most cost-effective means of performing these simulations is to use the staggered fermion formulation of Kogut and Susskind [3]. Recent calculations by the MILC collaboration [4] have demonstrated good agreement between experimentally known strong-interaction measurements and staggered fermion lattice QCD simulation.

The formulation as it stands has a serious deficiency for dynamical simulations. In four dimensions, the staggered fermion determinant describes four flavours of fermion, not one. This means that while it is very simple to simulate four mass-degenerate fermions with the staggered method, the study of one or two flavours must use a fractional power of the fermion determinant. This raises difficult theoretical problems: what are the fermion fields, and what is the local action on these fields which reproduces this determinant? Without a path-integral representation of the fermion determinant, all the standard quantum field theory construction of propagators (which are the two-point functions of the underlying quark fields) is poorly defined. If no local action exists, an even more severe issue arises, since there is then no guarantee that the continuum limit of the lattice simulation is in the same universality class as QCD and the link with physics is lost.

In this paper, we describe a numerical construction of an operator that defines a lattice quantum field theory equivalent to a single, free staggered fermion. Most of the construction is performed in two dimensions to ease the computations, but some suggestive results in four dimensions indicate the same construction works there too. Note that all the work in this paper is for the theory of free fermions, and the question of defining the interacting theory remains open. We do however regard this as a useful starting point for the more difficult problem of finding a path integral representation of the staggered fermion determinant in the presence of background gauge fields and the construction presented does suggest how to proceed further. The paper is organised as follows: Sec. II briefly describes the free staggered fermion, and Sec. III describes the numerical construction of the local operator. Sec. IV then presents a different numerical construction that is seen to obey a modified Ginsparg-Wilson relation. In Secs. V and VI a discussion of our results and conclusions is given.

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II. STAGGERED FERMIONS

In this section, a brief overview of the staggered fermion formulation of Kogut and Susskind [3] is presented, emphasising some critical properties of the resulting free quark operator.

The staggered fermion formalism is constructed by first writing a naive representation of the Dirac operator on the lattice

$$M^{i,j}_{x,y} = am\delta_{x,y}\delta^{i,j} + \frac{1}{2}\sum_{\mu} (\gamma_\mu)^{i,j} \left(\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y}\right), \quad (1)$$

where the Euclidean space indices (x, y) and Dirac algebra indices (i, j) have been included explicitly. This operator has poles not only at zero momentum, but also at the corners of the Brillouin zone. Counting these poles suggests the field coupled through this interaction matrix can be thought of as representing 2^d flavours of fermions in d dimensions. A local change of variable at every site of the lattice, $\chi(x) = T(x)\psi(x)$ with

$$T(x) = \prod_{\mu=1}^{d} (\gamma_\mu)^{x_\mu}, \quad (2)$$

diagonalises the naive operator M over the Dirac algebra. The number of flavours is reduced by discarding all but one of the diagonal components of χ. The operator is then regarded as acting on $n_t = 2^d/2^{d/2} = 2^{d/2}$ flavours of fermions. These flavours, all of which appear in a single instance of the staggered field, are often termed "tastes". The operator on the field is then

$$Q_{x,y} = am\delta_{x,y} + \frac{1}{2}\sum_\mu \eta_\mu(x)\left(\delta_{x+\hat\mu,y} - \delta_{x-\hat\mu,y}\right). \quad (3)$$

where $\eta_\mu(x)$ is the staggered phase, given by

$$\eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}. \quad (4)$$

A natural decomposition for the operator is to break the lattice into hypercubes of side-length $b = 2a$ containing 2^d sites. A site on the full lattice can be labelled with co-ordinates

$$x_\mu = 2N_\mu + \rho_\mu, \quad (5)$$

where N_μ are the co-ordinates of sites on the blocked lattice and $\rho_\mu \in \{0,1\}$. The 2^d staggered variables in a hypercube can be labelled by these hypercubic offset vectors, ρ

$$\chi_\rho(N) = \chi(2N+\rho). \quad (6)$$

Introducing a new spinor field ψ^{ab} on the blocked lattice sites, N

$$\psi^{ab}(N) = \sum_\rho [T_\rho]^{ab}\chi_\rho(N), \quad (7)$$

gives the staggered fermion action in terms of these $2^{d/2}$ tastes of Dirac spinors,

$$S_{\rm stag} = b^4 \sum_{N,N'} \bar\psi(N)Q(N,N')\psi(N'), \quad (8)$$

with

$$Q(N,N') = m(I\otimes I)\delta_{N,N'} + \sum_\mu(\gamma_\mu\otimes I)\Delta_\mu(N,N') + \frac{1}{2}b(\gamma_5\otimes t_\mu t_5)\Box_\mu(N,N'), \quad (9)$$

where Δ_μ and $\Box_\mu$ are the simplest representations of the first and second derivatives on the blocked lattice,

$$\Delta_\mu(N,N') = \frac{1}{2b}(\delta_{N+\mu,N'} - \delta_{N-\mu,N'}), \quad (10)$$

$$\Box_\mu(N,N') = \frac{1}{b^2}(\delta_{N+\mu,N'} + \delta_{N-\mu,N'} - 2\delta_{N,N'}). \quad (11)$$

This representation makes the Dirac and taste structure of the staggered operator more apparent. Taking a direct fractional power of a matrix does not change its structure, so it seems counterintuitive to expect the matrix Q^{1/n_t} to be a sensible representation of the one-flavour Dirac operator, and this operator has been shown to be non-local [5]. To construct a lattice fermion with a more physical interpretation, begin by noting that the operator is γ_5-hermitian, namely

$$Q^\dagger = \gamma_5 Q\gamma_5, \quad (12)$$

so that

$$\det Q^\dagger = \det Q, \quad (13)$$

and

$$\sqrt{\det Q^\dagger Q} = \det Q. \quad (14)$$

The product $\Box = Q^\dagger Q$ is diagonal in the spinor index, and has the form

$$\Box = \sum_\mu \Box_\mu. \quad (15)$$

The lattice interaction $\Box$ thus resembles 2^d distinct copies of the simple discretisation of the continuum Klein-Gordon operator, $-\nabla^2 + m^2$ on each of the 2^d lattices with spacing $b = 2a$.

III. AN EQUIVALENT LOCAL DIRAC OPERATOR

In order to define a theory with a single flavour of fermion, the standard method is to consider the appropriate fractional powers of the fermion determinant. A single flavour of staggered fermion would then be represented by $\det Q^{1/n_t}$. It is important to recognise the significant theoretical difficulty with this prescription: the fractional power of the determinant can no longer be written directly as a path integral over Grassmann fields coupled through a local operator (Q^{1/n_t} is non-local) and hence all the standard quantum field theory mechanisms for generating correlation functions by adding sources to the path integral no longer follow. Locality ensures that interacting theories are in the same universality class of the continuum field theory. In order to define a sensible lattice quantum field theory, an operator with the property

$$\det D = \det Q^{1/n_t}, \quad (16)$$

is required, where D defines local interactions [8]. With this property a path integral representation can be made, namely

$$\det Q^{1/n_t} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left\{-\bar{\psi}D\psi\right\}. \qquad (17)$$

Given this form, correlation functions of the theory can then be constructed by adding source terms and following the standard construction. In this section, an operator D obeying Eq. 16 is defined numerically for the free staggered fermion theory.

One observation in beginning the construction of D is helpful: the operator will not obey the staggered-fermion Dirac algebra (which scatters the spin and taste components over the corners of the unit hypercube) since a counting of degrees-of-freedom suggests there are certain to be too many flavours. Instead, there must be sites on the lattice with no quark field. To construct a fermion field with the correct number of degrees of freedom in a hypercube requires there to be $2^{d/2}$ components, rather than the 2^d components of the staggered field. To begin construction assume there is a single Dirac spinor at one site per hypercube, with no degrees of freedom on the other sites of the cell.

The equivalence property of Eq. 16 is sufficient to define the path integral, but can be trivially satisfied for the free theory: any non-singular matrix can be made to obey this constraint after a rescaling. For the free case a more stringent definition of equivalence must be made, namely that the energy-momentum dispersion relation for fermions in the two theories be related. This will be satisfied if the operator itself squares to the free Klein-Gordon operator on the blocked lattice, *i.e.* if (for massless fermions)

$$D^\dagger D = -\Box. \qquad (18)$$

In this work, this property will be denoted "strong" equivalence, while the condition of Eq. 16, which is trivial for free fermions but non-trivial in the interacting theory is denoted "weak" equivalence.

The properties expected from a well defined lattice Dirac operator D are locality, the correct continuum limit for momenta below the cutoff, π/a and invertibility at all non-zero momenta. Once these properties are satisfied the Nielsen-Ninomiya theorem [6] excludes the possibility of having invariance under continuous chiral transformations. This last issue will be dealt with in the next section. Here, before beginning to describe in detail our proposal of a Dirac operator it is helpful to state what locality means. An action density which has nearest-neighbour interactions or interactions that are identically zero beyond a few lattice units is certainly local, but no physical principle requires this extreme case [9]. On the lattice an action is termed local if its couplings have exponentially decaying tails at large distances. This property is ensured if $D(p)$ is an analytic periodic function of the momenta p_μ with period $2\pi/a$.

The following ansatz for a solution to the "strong" equivalence constraint for the blocked lattice with spacing b is made:

$$D = \gamma_\mu p_\mu - q, \qquad (19)$$

with p_μ and q such that D obeys Eq. 18, so

$$p_\mu p_\mu - q^2 = \Box. \qquad (20)$$

A numerical prescription for constructing an effective representation of the Dirac operator for massless fermions is used. To begin, a sequence of "ultra-local" operators of finite, increasing range is defined. The finite range operator can be described with a number of coefficients weighting each distinct hopping term. The hopping terms are taken from A_r, the set of all vectors $\mathbf{a}$ whose range is less than $\mathbf{r}$. The "taxi-driver" metric is used to define the range of a vector $\mathbf{a}$ with components a_i, so

$$r(\mathbf{a}) = \sum_i |a_i|. \qquad (21)$$

In two dimensions then,

$$\begin{aligned} A_0 &= \{(0,0)\}, \\ A_1 &= \{(0,0),(1,0),(-1,0),(0,1),(0,-1)\}, \ldots \end{aligned} \qquad (22)$$

A general ansatz for both p^μ and q, connecting fields at sites $\mathbf{x}$ and $\mathbf{y}$ is then

$$p^\mu_{\mathbf{x},\mathbf{y}} = \sum_{\mathbf{a}\in A_r} \omega^\mu_p(\mathbf{a})\delta_{\mathbf{x}+\mathbf{a},\mathbf{y}}, \qquad (23)$$

and

$$q_{\mathbf{x},\mathbf{y}} = \sum_{\mathbf{a}\in A_r} \omega_q(\mathbf{a})\delta_{\mathbf{x}+\mathbf{a},\mathbf{y}}. \qquad (24)$$

The coefficients are constrained so that the required symmetries of each operator are preserved to ensure the action is a scalar. This implies that the coefficients ω_q form a trivial representation of the lattice rotation group, while ω^μ_p form a fundamental representation. In two dimensions (where the relevant rotation groups is $C_{4\nu}$) the required irreducible representations are A_1 for q and E for p. This in turn implies the relations

$$\begin{aligned} &\omega_q(a_1,a_2) = \omega_q(a_1,-a_2) = \omega_q(-a_1,a_2) = \omega_q(-a_1,-a_2) = \\ &\quad \omega_q(a_2,a_1) = \omega_q(a_2,-a_1) = \omega_q(-a_2,a_1) = \omega_q(-a_2,-a_1), \end{aligned} \qquad (25)$$

and

$$\omega_p^1(a_1,a_2) = \omega_p^1(a_1,-a_2) = -\omega_p^1(-a_1,a_2) = -\omega_p^1(-a_1,-a_2) = \omega_p^2(a_2,a_1) = -\omega_p^2(a_2,-a_1) = \omega_p^2(-a_2,a_1) = -\omega_p^2(-a_2,-a_1). \quad (26)$$

One further constraint is added to improve the representation of low-momentum states. The coefficient $\omega_q(0,0)$ is chosen such that the operator q vanishes on a zero-momentum plane-wave. The number of free parameters in the operators p and q in two and four dimensions is given for a few low ranges in Table I.

	d=2		d=4	
Range	p^μ	q	p^μ	q
1	1	1	1	1
2	3	2	3	3
3	6	4	7	6
4	10	6	14	11
5	15	9	25	17
10	55	30	189	93

TABLE I: The number of free parameters in the finite-range operators in two and four dimensional lattice actions

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A sequence of lattice Dirac operators, $D_1, D_2, \ldots$ with increasing range is then considered. Each operator in the sequence is chosen to minimise μ_r^2, a positive-definite measure of the difference between the two sides of Eq. 18, namely

$$\mu_r^2 = \frac{1}{4d^2 N_s} \mathrm{Tr}\ (X_r^2), \quad (27)$$

with N_s the number of sites on the blocked lattice and

$$X_r = D_r^\dagger D_r + \Box. \quad (28)$$

Note there are certainly an infinite number of actions obeying the equivalence principle of Eq. 18. Most of these will be non-local but there could well be more than one local action. The following hypothesis is made. If a local action obeying "strong" equivalence exists, then the measure μ_r should fall exponentially, and the operator D_r should have exponentially falling coefficients inside A_r. D_r is the best ultra-local approximation to the solution of the equivalence condition of Eq. 18. The coefficients of the action a long way from the boundary of the operator should also converge as the range is increased.

The sequence of ultra-local actions D_r was computed numerically by finding the minimum of μ_r. The calculations were performed for massless fermions. A short check demonstrated the localisation properties were better for massive fermions.

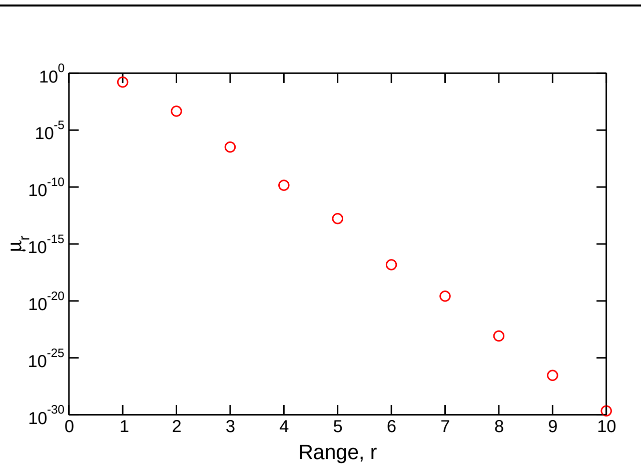

FIG. 1: The error function, μ_r for lattice actions of range up to $10b$.

A. Results

A multi-dimensional Newton-Raphson solver was used, since both the slope and Hessian of μ_r can be computed easily. The GNU multiple precision library (GMP) was used [7] when numerical precision was required beyond 64-bit native arithmetic. Some checks were made to test if the minimum in μ_r was a global one. A range of different starting values of the action parameters were used to seed the Newton-Raphson search and a simulated annealing algorithm was run to search for a minimum at short ranges. A number of local minima were found in many cases making it difficult to determine if the global minima was reached. This issue is discussed later.

1. *Two dimensions*

Fig. 1 shows the dependence of μ_r for the optimal action as a function of the finite range of the action, r. A clear signal for exponential fall-off is displayed: μ_r, the discrepancy between $D_r^\dagger D_r$ and $-\Box$ falls by thirty decades as the action range is increased from b to $10b$. The coefficients in q and p_μ of the action D_{10} are presented in Fig. 2. The on-lattice-axis and diagonal terms are presented. For the operator p_μ, on-axis refers to the terms in p_1 with off-set vector $(a,0)$ and those in p_2 with off-set $(0,a)$. Note that by symmetry, terms in p_1 with off-set $(0,a)$ vanish identically. An exponential fall-off over eighteen decades is observed, providing solid evidence for the existence of a local operator. At 64-bit machine precision, terms with ranges beyond about $6b$ would have uncomputably small contributions to the ac-

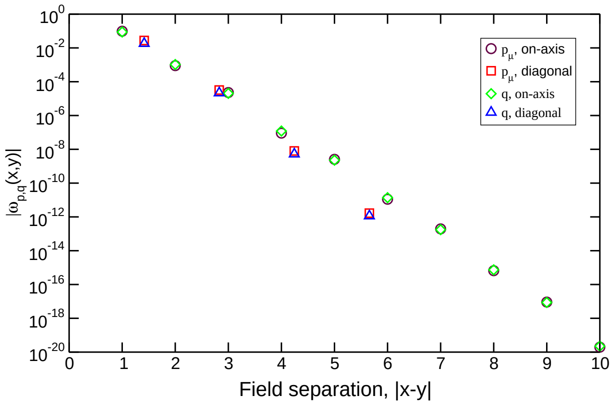

FIG. 2: The coefficients in q and p_μ, the composite operators in the action D_{10}, as a function of the separation between the fields in the bilinear. The field separations are given using the usual 2-norm distance.

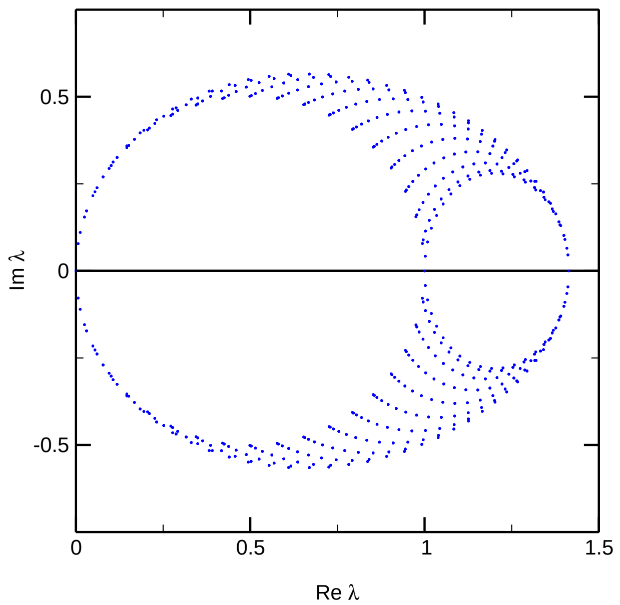

FIG. 3: The eigenvalues of the approximately equivalent Dirac operator, D_{10}. The eigenvalues are scaled so the first doubler, with momentum $(\pi, 0)$ has $\lambda = 1$.

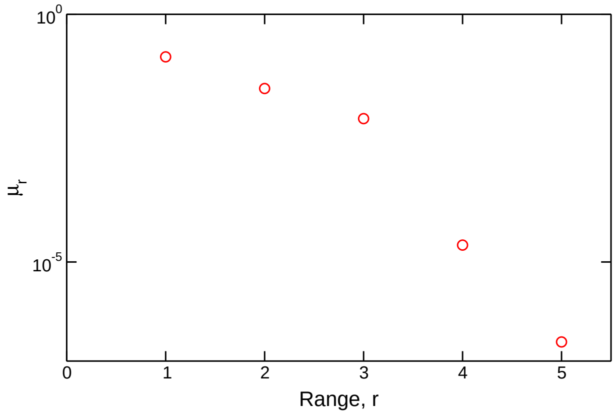

FIG. 4: The error function, μ_r for four-dimensional lattice actions of range up to $5b$.

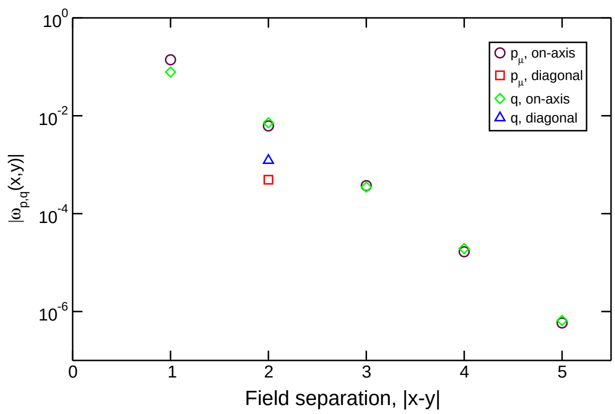

FIG. 5: The coefficients in q and p_μ in the four-dimensional action D_5 as a function of the separation between the fields in the bilinear

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tion of a plane wave state. Notice the terms in the two operators p_μ and q are very similar in magnitude and have similar localisation range. Fig. 3 shows the eigenvalue spectrum of the operator D_{10}. The eigenvalues are purely real when all components of the operator p_μ vanish, *ie.* at the doubling points. The real parts of the eigenvalues for the doublers are close to 0 (for the propagating mode), 1 and $\sqrt{2}$.

2. Four dimensions

A similar construction was followed for the four-dimensional action. A sequence of operators on the four-dimensional lattice with spacing $b = 2a$ was determined. The operators in the action again took the general form of Eqns. 23 and 24, with q transforming trivially under the four-dimensional rotation group and p_μ transforming under the fundamental representation. The computational cost of the four-dimensional calculation means that only actions up to range $r = 5$ have been constructed. Fig. 4 shows the fall-off of μ_r in the sequence of actions. A rapid decay of μ_r as r is increased is again observed. Notice also the fall-off accelerates at range $r = 4$, once the action increases beyond the bounds of the unit hypercube on the spacing-b lattice.

In Fig. 5 the magnitude of the coefficients on the axis and diagonals of the operator D_5 are displayed. A similar pattern to the two-dimensional case is seen, with a five-decade decay over five lattice hops being observed. The data suggest a local operator exists in four dimensions as well as two.

IV. THE GINSPARG-WILSON RELATION

The presence of the operator q in the definition of Eq. 19, means D does not anticommute with γ_5, which would guarantee that the fermionic action is invariant under continuous chiral transformations. This is expected from the Nielsen-Ninomiya theorem. As is now well known, the theorem can be bypassed if one does not insist that the chiral transformations assume their canonical form on the lattice [10, 11]. In particular it was shown that the Ginsparg-Wilson relation,

$$\{\gamma_5, D\} = 2D\gamma_5 RD, \tag{29}$$

implies an exact symmetry of the fermionic action which may be regarded as a lattice form of an infinitesimal chiral rotation.

For a Dirac operator obeying Eq. 19, three properties follow:

$$D^\dagger = \gamma_5 D \gamma_5, \tag{30}$$
$$D^\dagger D = -\Box\, \mathcal{I}, \tag{31}$$
$$D^{-1} = \frac{\gamma_\mu p_\mu + q}{\Box}. \tag{32}$$

The propagator D^{-1} then satisfies the following relation

$$\{\gamma_5, D^{-1}\} = 2R\gamma_5, \tag{33}$$

with

$$R = \frac{q}{\Box}. \tag{34}$$

The construction of Sec. III does not ensure the operator R is local, and so there is no apparent lattice chiral symmetry. Eqn. 34 does suggest the definition of an alternative sequence of actions, $D_r^{(GW)}$ which might lead to a Dirac operator obeying a (generalised) Ginsparg-Wilson relation. Consider an operator of range r with the form

$$D_r^{(GW)} = \gamma_\mu p_r^\mu + \Box R_{r-1}, \tag{35}$$

where R_{r-1} is a local operator with finite range $r-1$. Note this implies the scalar operator $q_r = \Box R_{r-1}$ has range r as before. If the limit of the sequence $D_r^{(GW)}$ is a solution to Eqn. 19 and p^μ and R remain local, then a local operator, equivalent to the staggered fermion and obeying a generalised Ginsparg-Wilson relation will be constructed.

A simple consequence of Eq. 29 is then that chiral symmetry is partly preserved, in particular the lagrangian $L = \overline{\psi} D \psi$ is invariant under the local symmetry transformation:

$$\delta\psi = \gamma_5 \left(1 - \frac{1}{2}RD\right)\psi, \tag{36}$$

$$\delta\overline{\psi} = \overline{\psi}\left(1 - \frac{1}{2}DR\right)\gamma_5. \tag{37}$$

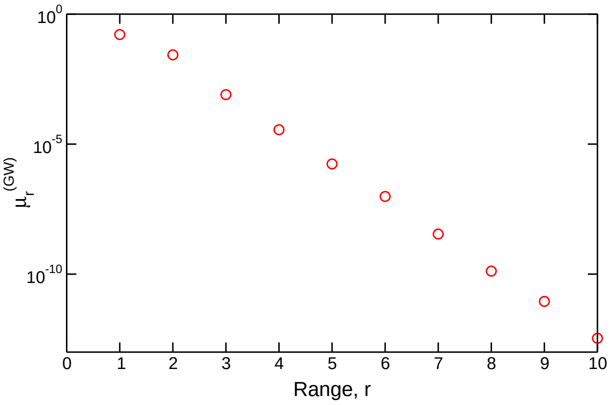

FIG. 6: The error function, μ_r for two-dimensional lattice operators constructed with the Ginsparg-Wilson constraint of Eqn. 35.

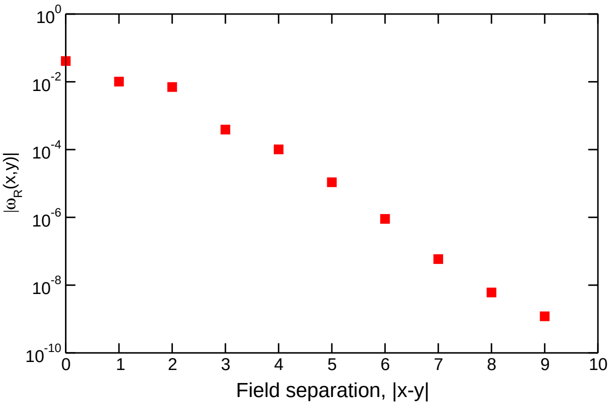

FIG. 7: The coefficients in q and p_μ in the two-dimensional action $D_{10}^{(GW)}$ as a function of the separation between the fields in the bilinear.

A. results

The operator sequence minimising μ_r of Eq. 27 was computed for the two-dimensional lattice. As before, the Newton solver was used for minimisation with the constraint of Eqn. 35. The problem of finding multiple local minima was observed and was more extreme than the initial construction of Sec. III. The dependence of μ_r for the operator obeying the Ginsparg-Wilson constraint is shown in Fig. 6. Exponential decay of the discrepancy measurement, μ_r as the range r is increased is seen again this time over twelve orders-of-magnitude. Fig. 7 shows the fall-off of the symmetry-breaking kernel for the optimised action $D_{10}^{(GW)}$. An exponential fall-off of eight decades is observed. The terms in the operator q are thus exponentially localised. The derivative operator, p_μ also has local coefficients. The eigenvalues of the operator $D_{10}^{(GW)}$ are shown in Fig. 8, along with those for the unconstrained two-dimensional action construction of Sec. III.

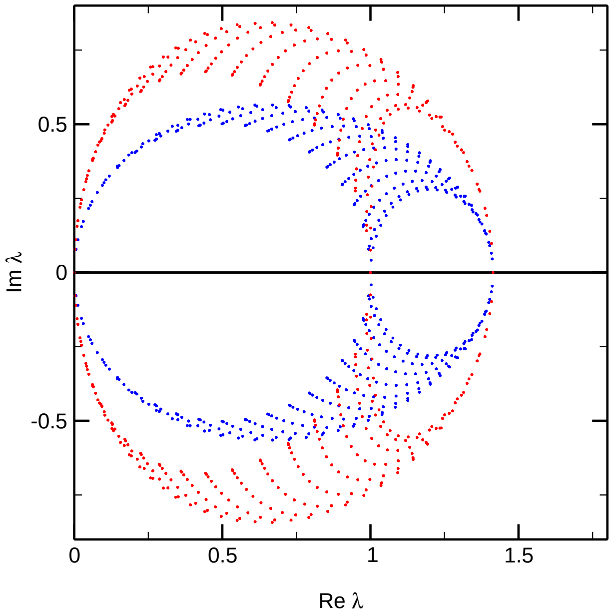

FIG. 8: The eigenvalues of the two operators, D_{10} (blue) and $D_{10}^{(GW)}$ (red). The eigenvalues are scaled such that $\lambda = 1$ for momentum $(\pi, 0)$.

V. DISCUSSION

A numerical construction of ultra-local, approximate actions can never prove the existence of a fermion with a well defined local action (unless the action is itself ultra-local), but the calculations in this paper do present strong evidence for the existence of an equivalent, local theory to the one-flavour free staggered fermion. In two dimensions, the mis-match between the dispersion spectrum of the ultra-local theory and the staggered fermion can be made as small as 10^{-30} with an action of range $10b$. At this range, terms in the action are as small as 10^{-18}. The construction for four-dimensional theories is more difficult, but evidence for locality is seen here too. The numerical construction of the action was performed for massless fermions. Some short tests with massive fermions suggest the localisation properties of these actions are better still; the massless fermion represents the hardest case to reproduce.

The numerical search for a global minima of Eqn. 27, a non-linear function of the action parameters, ω_p and ω_q is made difficult by the presence of local minima. It is difficult to find convincing evidence that the Newton-Raphson solver has found the global minimum for large actions, although searches using different starting points often converged to a common fixed point. An empirical observation is important; the minima with smaller values of μ_r tend to have better localisation properties (their coefficients fall faster). This implies if the searches have not found the global minima, these will represent better actions than those already uncovered, improving the construction rather than spoiling it.

For the constrained construction to build an operator obeying the (generalised) Ginsparg-Wilson relation, the localisation ranges were about twice that of the unconstrained construction and good evidence for exponentially local actions is seen. The problem of finding global minima seems to be exacerbated. Solutions to the standard GW relation (with $R = I$) are now well known [12, 13]. It is important to recognise a solution with $R = I$ is impossible for the equivalent theory; this can be seen easily by considering the doubler momenta, $(\pi, 0)$ and (π, π). For these two points, the eigenvalues of D must be 1 and $\sqrt{2}$ respectively, while $R = I$ would demand they were both unity.

The evidence in this paper (and in Ref. [21]) put staggered-fermion simulations on a more robust footing, but there remain many unanswered questions. For staggered fermion simulations to be correct descriptions of quantum field theory, one must demonstrate two postulates; firstly that a local path-integral representation of the fractional power of the staggered determinant exists and secondly the validity of calculations performed by assuming the propagator of this theory is related to the inverse of the full staggered matrix, Q^{-1}. The work in this paper hints at the right question to address the first issue, but does not address the second point. In using Q^{-1} as the fermion propagator a clear problem arises. In four dimensions, too many pion operators can be constructed for example. The residual symmetry of the staggered matrix ensures these states lie in mass-degenerate multiplets [14, 15, 16], but "taste-breaking" interactions split their masses. These splittings vanish in the continuum limit. Recent work on the low-energy eigenvalue spectrum is beginning to resolve this issue [18, 19, 20].

Future work offers an optimistic possible outcome; if an effective operator can be constructed for QCD, it seems this operator might obey a Ginsparg-Wilson relation and the lattice physicist will have the best of both worlds: cheap dynamical-configuration generation algorithms using the staggered formulation with a theoretically well defined action (possibly with an exact GW chiral symmetry) to compute propagators. This would not be a "mixed action" simulation, where different valence and sea quark actions are employed. The first obstacle to extending the construction of Sec. III to incorporate gauge interactions is that the operator $Q^{\dagger}Q$ is not proportional to $(I \otimes I)$ in Dirac-taste indices and so does not decompose directly into n_t decoupled parts. This is the "strong" definition of equivalence required for the free theory, but a re-definition of μ_r to measure violations in "weak" equivalence can be made. This is under investigation and few conclusions about the success of this programme can be drawn. A number of difficult questions arise immediately, since the two theories would have different apparent symmetries.

VI. CONCLUSIONS

In this paper, a local lattice Dirac operator whose determinant is identical to the free staggered-fermion determinant, and whose energy-momentum dispersion relations are identical (although with different degeneracies) is described as the end-point of a sequence of actions of increasing, finite range. The first few ultra-local actions in the sequence are constructed numerically and convergence of the sequence is demonstrated in both two and four dimensions.

The spectrum of the operator is free from doublers and its low-energy dynamics correctly describes the propagation of free fermions up to corrections of $\mathcal{O}(a^2p^2)$, a property it inherits from its staggered parent. The operator acts on a full Dirac spinor situated only on the sites of a blocked lattice with spacing $b = 2a$.

A constraint is added to the construction to define an operator that obeys a generalised Ginsparg-Wilson relation. In this action, the chiral symmetry breaking in the propagator is described by a local operator, diagonal in the spin index. This implies the existence of a fermion whose path integral is the same as that of one staggered flavour and with an exact chiral symmetry on the lattice.

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