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Pure Spinors are Higher-Dimensional Twistors

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In any even (Euclidean) dimension $d = 2n$, projective pure spinors parameterize the coset space $SO(2n)/U(n)$, which is the space of all complex structures on $\mathbf{R}^{2n}$. For $d = 4$ and $d = 6$, these spaces are $\mathbf{CP}^1$ and $\mathbf{CP}^3$, and the corresponding pure spinors have been interpreted as four and six-dimensional twistor variables. In this paper, we argue that the identification of pure spinors and twistors holds in any even dimension, and we use pure spinors to construct massless solutions in higher dimensions.

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1. Introduction

As defined by Cartan [1] and Chevalley [2], pure spinors in even dimension $d = 2n$ are complex spinors λ^a which satisfy the constraint $\lambda^a(\sigma^{\mu_1..\mu_j})_{ab}\lambda^b = 0$ for $0 \leq j < n$, where $\sigma^{\mu_1...\mu_j}$ is the antisymmetrized product of j Pauli matrices. So $\lambda^a\lambda^b$ can be written as

$$\lambda^a\lambda^b = \frac{1}{n!\ 2^n}\sigma^{ab}_{\mu_1...\mu_n}\ (\lambda^c\sigma^{\mu_1..\mu_n}_{cd}\lambda^d) \tag{1.1}$$

where $\lambda\sigma^{\mu_1...\mu_n}\lambda$ defines an n-dimensional complex plane, and thus complex coordinates on $\mathbf{R}^{2n}$. In Euclidean space, this n-dimensional complex plane is preserved up to a phase by a $U(n)$ subgroup of $SO(2n)$ rotations. So projective pure spinors in $d = 2n$ Euclidean dimensions parameterize the coset space $SO(2n)/U(n)$[1]. (For a more detailed account of this correspondence, we refer the reader to the Appendix.)

In four dimensions, this is the coset space $SO(4)/U(2) = \mathbf{CP}^1$ which is parameterized by a projective chiral spinor λ^a for $a = 1$ to 2. As is well-known, the twistor formalism of Penrose makes use of this $d = 4$ projective pure spinor to construct solutions to $d = 4$ massless equations of motion [3] [4]. In six dimensions, the coset $SO(6)/U(3) = \mathbf{CP}^3$ is parameterized by a projective chiral spinor λ^a for $a = 1$ to 4. Although it is less well-known than its four-dimensional counterpart, this projective pure spinor in six dimensions can similarly be used to construct twistor solutions to the $d = 6$ massless equations of motion, as demonstrated by Hughston [5].

In this paper, these twistor constructions of solutions to massless equations of motion will be generalized for projective pure spinors in any even dimension. Above six dimensions, the construction becomes non-trivial since pure spinors for $d \geq 8$ satisfy non-linear constraints. For example, in eight dimensions, the coset $SO(8)/U(4)$ is parameterized by a projective chiral spinor λ^a for $a = 1$ to 8 satisfying the additional constraint $\lambda^a\lambda^a = 0$. And in ten dimensions, the coset $SO(10)/U(5)$ is parameterized by a projective chiral spinor λ^a for $a = 1$ to 16 satisfying the constraint $\lambda^a\sigma^\mu_{ab}\lambda^b = 0$ where σ^μ_{ab} are the $d = 10$ Pauli matrices. So generalization of the Penrose twistor construction to higher dimensions requires new techniques for integration over these coset spaces.

[1] In Minkowski space, the n-dimensional complex plane is preserved by a $U(n-1)$ subgroup of $SO(d-2)$ rotations and is also preserved by $(2n-1)$ light-like boosts. So projective pure spinors in Minkowski space contain the same number of variables as in Euclidean space, but the coset space is modified to $SO(2n-1,1)/U(n-1) \times \mathbf{R}^{2n-1}$.

Here we limit ourselves to the simple question of solving linearized massless equations of motion in flat spacetime. However, the fact that projective pure spinors provide an elegant higher-dimensional generalization of this twistor construction suggests that pure spinors may also be useful for generalizing other applications of four- dimensional twistors to higher dimensions. Solutions of nonlinear problems, as well as linear problems in non-flat background, are of special interest. For example, four- dimensional twistors have been useful for constructing solutions of self-dual Yang-Mills [6] and self-dual gravity equations [7], [8] , and for constructing Green's functions on multi-Taub-NUT spaces [9]. It might be possible that pure spinors will be useful for generalizing these nonlinear constructions to higher dimensions. In particular, we use pure spinors in this paper to construct self-dual abelian potentials in higher dimensions and one might hope that geometric insight from this construction will lead to the proper formalism for a nonabelian generalization.

Note that ten-dimensional pure spinors have recently been used for covariantly quantizing the superstring [10][11] and many of the techniques described here are generalizations of techniques developed for quantization of the ten-dimensional superstring. So it would not be surprising to find that pure spinors in ten dimensions are useful for constructing solutions to $d = 10$ super-Yang-Mills and supergravity equations [12], which are the low-energy equations of the superstring. However, it is not clear how the higher-dimensional twistors described here can be generalized to higher-dimensional supertwistors.

There have been numerous approaches to generalizing the twistor formalism to higher dimensions, most of which differ from each other and from our approach. For example, Ward presents classes of various nonlinear equations for a nonabelian gauge field in [13] that can be solved using higher-dimensional twistors. Also, twistor-like transforms in higher dimensions have appeared in studies of the superparticle and superstring (e.g. [14][12]). The resemblance of pure spinors and twistors has been noted by many people (e.g. [15][16]), however, the references we are aware of which come closest to the explicit approach presented here are [17] where the properties of twistor space are studied, as well as [18] and [19] where the Penrose transform is constructed and proven to be one- to-one.

In section 2 of this paper, we review the twistor construction of massless solutions using four and six-dimensional pure spinors. In section 3, we show how this twistor construction extends to pure spinors in eight and ten dimensions. And in section 4, we generalize this twistor construction to pure spinors in arbitrary even dimensions.

2. Pure Spinors in Four and Six Dimensions

In four and six dimensions, projective pure spinors parameterize the coset spaces $SO(4)/U(2) = \mathbf{CP}^1$ and $SO(6)/U(3) = \mathbf{CP}^3$, and are therefore described by complex projective two-component and four-component spinors which have been called twistors. As will be reviewed here, these twistors have been used for constructing solutions to massless equations of motion in four [3][4]and six dimensions [5].

2.1. Four dimensions

As is well-known, Penrose has used complex projective two-component spinors to construct twistor solutions to massless equations of motion in four dimensions [3][4]. To describe this method in a manner which will generalize to higher dimensions, consider the massless Klein-Gordon equation $\partial^\mu \partial_\mu \Phi(x) = 0$ for a scalar field $\Phi(x)$ where $\mu = 1$ to 4. It is useful to combine x^μ into a pair of complex coordinates, $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, so that the Klein-Gordon equation (in Euclidean space[2]) is $\partial_{z_j} \overline{\partial}_{\overline{z}_j} \Phi(z, \overline{z}) = 0$. Then if one defines

$$w_1 = z_1 + u\overline{z}_2, \quad w_2 = z_2 - u\overline{z}_1 \tag{2.1}$$

where u is a complex variable, any holomorphic function $f(w_1, w_2)$ will satisfy

$$(\partial_{z_1} \overline{\partial}_{\overline{z}_1} + \partial_{z_2} \overline{\partial}_{\overline{z}_2}) f(w_1, w_2) = \left(\frac{\partial}{\partial w_1} (-u \frac{\partial}{\partial w_2}) + \frac{\partial}{\partial w_2} (u \frac{\partial}{\partial w_1}) \right) f(w_1, w_2) = 0. \tag{2.2}$$

So the massless Klein-Gordon equation has the solution

$$\Phi(z, \overline{z}) = \oint du f(u, w_1, w_2)|_{w_1 = z_1 + u\overline{z}_2, w_2 = z_2 - u\overline{z}_1} \tag{2.3}$$

where $\oint du$ is a contour integral around any region in the complex plane.

This construction of massless $d = 4$ solutions can be made manifestly Lorentz covariant by introducing a bosonic projective spinor λ^a for $a = 1$ to 2 and defining

$$w_{\dot{a}} = \sigma^\mu_{a\dot{a}} x_\mu \lambda^a \tag{2.4}$$

where $\sigma^\mu_{a\dot{a}}$ are the usual $d = 4$ Pauli matrices. Under $d = 4$ conformal transformations, $(\lambda^a, w_{\dot{a}})$ transforms linearly as an $SO(4, 2)$ spinor.

[2] As usual, one can Wick rotate to Minkowski space by replacing x_4 with ix_4 so that $z_2 = x_3 + x_4$ and $\overline{z}_2 = x_3 - x_4$ are independent real variables.

When $\lambda^a = (1, u)$, the relation of (2.4) reduces to (2.1) and solution (2.3) can be written covariantly as

$$\Phi(x) = \oint d\lambda^a \ \lambda_a F(\lambda, w)|_{w=x\lambda} \tag{2.5}$$

where $F(h\lambda^a, hw_{\dot a}) = h^{-2} F(\lambda^a, w_{\dot a})$ so that the contour integral over the projective spinor is well-defined. For example, choosing

$$F(\lambda, w) = \frac{\epsilon_{\dot a \dot b} A_1^{\dot a} A_2^{\dot b}}{(A_1^{\dot c} w_{\dot c})(A_2^{\dot d} w_{\dot d})} \tag{2.6}$$

generates the $d = 4$ Green's function $\Phi(x) = (x^\mu x_\mu)^{-1}$.

One can similarly construct massless $d = 4$ solutions to higher-spin equations by considering functions $F(\lambda^a, w_{\dot a})$ satisfying the condition $F(h\lambda^a, hw_{\dot a}) = h^{-N-2} F(\lambda^a, w_{\dot a})$. If N is positive, one uses the formula

$$\Phi^{(a_1 \ldots a_N)}(x) = \oint d\lambda^b \ \lambda_b \lambda^{a_1} \ldots \lambda^{a_N} F(\lambda, w)|_{w=x\lambda}. \tag{2.7}$$

And if N is negative, one uses the formula

$$\Phi^{(\dot a_1 \ldots \dot a_{-N})}(x) = \oint d\lambda^a \ \lambda_a \left(\frac{\partial}{\partial w_{\dot a_1}} \ldots \frac{\partial}{\partial w_{\dot a_{-N}}} F(\lambda, w) \right) |_{w=x\lambda}. \tag{2.8}$$

Since $\frac{\partial}{\partial x^\mu} F(\lambda, w) = (\lambda \sigma_\mu)_{\dot a} \frac{\partial}{\partial w_{\dot a}} F(\lambda, w)$, one can use $\sigma^\mu_{a \dot a} \sigma_{\mu \ b \dot b} = 2 \epsilon_{ab} \epsilon_{\dot a \dot b}$ to show that $\sigma^\mu_{b \dot b} \frac{\partial}{\partial x^\mu} \Phi^{(b a_2 \ldots a_N)}(x) = 0$ and $\sigma^\mu_{b \dot b} \frac{\partial}{\partial x^\mu} \Phi^{(\dot b \dot a_2 \ldots \dot a_{-N})}(x) = 0$. So (2.7) and (2.8) describe solutions for massless particles of spin $|N|/2$ and helicity $N/2$.

2.2. Six dimensions

Although less familiar than the two-component twistor formulas in four dimensions, projective four-component complex spinors have been used to construct twistor solutions to massless equations of motion in six dimensions [5]. For example, consider the Klein-Gordon massless equation $\partial^\mu \partial_\mu \Phi(x) = 0$ for a scalar field $\Phi(x)$ where $\mu = 1$ to 6. As before, combine x^μ into a triplet of complex coordinates, $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ and $z_3 = x_5 + ix_6$, so that the Klein-Gordon equation (in Euclidean space) is $\partial_{z_j} \overline{\partial}_{\overline{z}_j} \Phi(z, \overline{z}) = 0$. Then if one defines

$$v_j = z_j + u_{jk} \overline{z}_k \tag{2.9}$$

where $u_{jk} = -u_{kj}$ are three independent complex variables, any holomorphic function $f(v_1, v_2, v_3)$ will satisfy

$$\partial_{z_j}\overline{\partial}_{\overline{z}_j} f(v_1, v_2, v_3) = u_{jk} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_k} f(v_1, v_2, v_3) = 0 \tag{2.10}$$

because of the antisymmetry of u_{jk}. So the massless Klein-Gordon equation has the solution

$$\Phi(z, \overline{z}) = (\oint du)^3 f(u_{jk}, v_l)|_{v_j = z_j + u_{jk}\overline{z}_k} \tag{2.11}$$

where the three contour integrals for the u_{jk} variables are chosen arbitrarily.

This construction of massless $d = 6$ solutions can be made manifestly Lorentz covariant by introducing a projective spinor λ^a for $a = 1$ to 4 and defining

$$w_a = \sigma^\mu_{ab} x_\mu \lambda^b \tag{2.12}$$

where $\sigma^\mu_{ab} = -\sigma^\mu_{ba}$ are the $d = 6$ Pauli matrices. Under $d = 6$ conformal transformations, $(\lambda^a, w_{\dot{b}})$ transforms linearly as an $SO(6,2)$ spinor.

When $\lambda^a = (1, u_{23}, u_{31}, u_{12})$, one can check that with a suitable choice for the Pauli matrices,

$$w_a = (\frac{1}{2}\epsilon^{jkl} u_{jk} z_l, \ v_1, \ v_2, \ v_3) \tag{2.13}$$

where v_j is defined in (2.9). Note that (2.13) satisfies $\lambda^a w_a = 0$, as implied by (2.12). Furthermore, the massless solution (2.11) can be covariantly written as

$$\Phi(x) = \oint \epsilon_{abcd} d\lambda^a \wedge d\lambda^b \wedge d\lambda^c \ \lambda^d F(\lambda, w)|_{w = x\lambda} \tag{2.14}$$

where $F(h\lambda^a, hw_b) = h^{-4} F(\lambda^a, w_b)$ so that the integral over the projective spinor is well-defined. For example, choosing

$$F(\lambda, w) = \frac{\epsilon_{abcd} A_1^a A_2^b A_3^c A_4^d}{\prod_{r=1}^4 (A_r^e w_e)} \tag{2.15}$$

generates the $d = 6$ Green's function $\Phi(x) = (x^\mu x_\mu)^{-2}$.

One can similarly construct massless $d = 6$ solutions to higher-spin equations by considering functions $F(\lambda^a, w_b)$ satisfying the condition $F(h\lambda^a, hw_b) = h^{-N-4} F(\lambda^a, w_b)$. When N is positive, one uses the formula

$$\Phi^{(a_1 \dots a_N)}(x) = \oint \epsilon_{bcde} d\lambda^b \wedge d\lambda^c \wedge d\lambda^d \ \lambda^e \lambda^{a_1} \dots \lambda^{a_N} F(\lambda, w)|_{w = x\lambda}. \tag{2.16}$$

And when N is negative, one uses the formula

$$\Phi^{(a_1 \ldots a_{-N})}(x) = \oint \epsilon_{bcde} d\lambda^b \wedge d\lambda^c \wedge d\lambda^d \ \lambda^e \left(\frac{\partial}{\partial w_{a_1}} \ldots \frac{\partial}{\partial w_{a_{-N}}} F(\lambda, w) \right) |_{w=x\lambda}. \qquad (2.17)$$

Since $\frac{\partial}{\partial x^\mu} F(\lambda, w) = (\lambda\sigma_\mu)_a \frac{\partial}{\partial w_a} F(\lambda, w)$, one can use $\sigma^\mu_{ab} \sigma_{\mu\ cd} = 2\epsilon_{abcd}$ to show that $\sigma^\mu_{bc} \frac{\partial}{\partial x^\mu} \Phi^{(ca_2 \ldots a_N)}(x) = 0$ either when N is positive or negative. So the solutions of (2.16) and (2.17) describe a massless spin $\frac{1}{2}$ field when $N = \pm 1$, a self-dual three-form field-strength when $N = \pm 2$, etc.

3. Pure Spinors in Eight and Ten Dimensions

Using the methods of the previous section, it is easy to generalize the non-covariant construction of (2.3) and (2.11) to arbitrary even dimension. To solve the massless Klein-Gordon equation $\partial^\mu \partial_\mu \Phi(x) = 0$ for a scalar field $\Phi(x)$ where $\mu = 1$ to $2n$, first combine x^μ into n complex coordinates, $z_j = x_{2j-1} + i x_{2j}$ for $j = 1$ to n, so that the Klein-Gordon equation in Euclidean space is $\partial_{z_j} \partial_{\overline{z}_j} \Phi(z, \overline{z}) = 0$. Defining

$$v_j = z_j + u_{jk} \overline{z}_k \qquad (3.1)$$

where $u_{jk} = -u_{kj}$ are $n(n-1)/2$ independent complex variables, one finds that any holomorphic function $f(v_j, u_{jk})$ satisfies

$$\partial_{z_j} \partial_{\overline{z}_j} f(v, u) = u_{jk} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_k} f(v, u) = 0. \qquad (3.2)$$

So the massless Klein-Gordon equation has the solution

$$\Phi(z, \overline{z}) = (\oint du)^{n(n-1)/2} f(v, u)|_{v_j = z_j + u_{jk} \overline{z}_k} \qquad (3.3)$$

where the $n(n-1)/2$ contour integrals for u_{jk} are chosen arbitrarily.

To express this solution in a Lorentz-covariant manner using pure spinors, it will be necessary to know how to integrate the pure spinors over the coset space $SO(2n)/U(n)$. When $n = 5$, an integration method for pure spinors was developed in [11] for quantization of the ten-dimensional superstring. As will be shown here, this integration method is easily generalized for arbitrary n, which will allow the massless solution of (3.3) to be expressed in a Lorentz-covariant manner. Before describing this twistor construction for arbitrary even dimension, it will be convenient to first describe the twistor construction for $d = 8$ and $d = 10$.

3.1. Eight dimensions

In eight dimensions, a pure spinor is described by a chiral spinor λ^a for $a = 1$ to 8 which satisfies the additional constraint $\lambda^a\lambda^a = 0$. To covariantize v_j and u_{jk} of (3.1) for $j = 1$ to 4, define the antichiral spinor

$$w_{\dot a} = \sigma^\mu_{a\dot a} x_\mu \lambda^a \tag{3.4}$$

where $\sigma^\mu_{a\dot a}$ are the $d = 8$ Pauli matrices. Note that $w_{\dot a}$ is an antichiral pure spinor and under $d = 8$ conformal transformations, $(\lambda^a, w_{\dot a})$ transforms linearly as an $SO(8,2)$ spinor.

When $\lambda^a = (1,\ u_{jk},\ -\frac{1}{8}\epsilon^{jklm}u_{jk}u_{lm})$, one can choose a representation of the $d = 8$ Pauli matrices such that

$$w_{\dot a} = (v_j,\ \frac{1}{2}\epsilon^{jklm}v_k u_{lm}). \tag{3.5}$$

Note that (3.5) satisfies $\sigma^\mu_{a\dot a}\lambda^a w^{\dot a} = 0$, as implied by (3.4). To covariantize the massless solution of (3.3), one needs to define a suitable integration measure for integrating λ^a over the coset space $SO(8)/U(4)$.

To define such an integration measure, note that

$$[d\lambda]_{d=8} \equiv (C_b\lambda^b)^{-1}\epsilon_{a_1\ldots a_8} d\lambda^{a_1}\wedge \ldots \wedge d\lambda^{a_6}\lambda^{a_7}C^{a_8} \tag{3.6}$$

is independent of the choice of C^b and is therefore Lorentz-invariant. To show independence of C^b, use $\lambda^a\lambda^a = 0$ and $\lambda^a d\lambda^a = 0$ to show that (3.6) is invariant under the transformation

$$\delta C_a = f\lambda_a + gC_a + \epsilon_{abc_1\ldots c_6}\lambda^b h^{c_1\ldots c_6} \tag{3.7}$$

where f, g and $h^{c_1\ldots c_6}$ are arbitrary parameters. Since (3.7) can be used to change C_a in an arbitrary manner, (3.6) is independent of C_a.

Using the measure factor of (3.6), the solution of (3.3) can be written in Lorentz-covariant form as

$$\Phi(x) = \oint [d\lambda]_{d=8} F(\lambda, w)|_{w=x\lambda} \tag{3.8}$$

where $F(h\lambda^a, hw_{\dot a}) = h^{-6}F(\lambda^a, w_{\dot a})$ so that the integral over the projective spinor is well-defined. For example, choosing

$$F(\lambda, w) = \frac{\epsilon_{\dot b_1\ldots\dot b_8}A_1^{\dot b_1}\ldots A_7^{\dot b_7}w^{\dot b_8}}{\prod_{j=1}^7(A_j^{\dot a}w_{\dot a})} \tag{3.9}$$

generates the $d=8$ Green's function $\Phi(x)=(x^\mu x_\mu)^{-3}$.

One can similarly construct massless $d=8$ solutions to higher-spin equations by using the formula

$$\Phi^{(a_1\ldots a_N)}(x)=\oint[d\lambda]_{d=8}\lambda^{a_1}\ldots\lambda^{a_N}F(\lambda,w)|_{w=x\lambda} \quad (3.10)$$

where $F(\lambda^a,w_{\dot a})$ satisfies the condition $F(h\lambda^a,hw_{\dot a})=h^{-N-6}F(\lambda^a,w_{\dot a})$ for N positive. Since

$$\frac{\partial}{\partial x^\mu}F(\lambda,w)=(\lambda\sigma_\mu)_{\dot a}\frac{\partial}{\partial w_{\dot a}}F(\lambda,w),$$

$\sigma^\mu_{a\dot a}\sigma_{\mu\ b\dot b}\lambda^a\lambda^b=0$ implies that $\sigma^\mu_{b\dot b}\frac{\partial}{\partial x^\mu}\Phi^{(ba_2\ldots a_N)}(x)=0$. So (3.10) describes a massless spin $\frac{1}{2}$ field when $N=1$, a self-dual four-form field-strength when $N=2$, etc. But unlike the $d=4$ and $d=6$ cases, one cannot construct massless solutions when N is negative since $\sigma^\mu_{a\dot a}\sigma_{\mu\ b\dot b}\frac{\partial}{\partial w_{\dot a}}\frac{\partial}{\partial w_{\dot b}}$ does not necessarily vanish.

3.2. Ten dimensions

In ten dimensions, a pure spinor is described by a chiral spinor λ^a for $a=1$ to 16 which satisfies the additional constraint $\lambda^a\sigma^\mu_{ab}\lambda^a=0$ where $\sigma^\mu_{ab}=\sigma^\mu_{ba}$ are the $d=10$ Pauli matrices. To covariantize v_j and u_{jk} of (3.1) for $j=1$ to 5, define the antichiral spinor

$$w_a=\sigma^\mu_{ab}x_\mu\lambda^b. \quad (3.11)$$

Note that w_a is an antichiral pure spinor and under $d=10$ conformal transformations, (λ^a,w_a) transforms linearly as an $SO(10,2)$ spinor.

When $\lambda^a=(1,\ u_{jk},\ -\frac{1}{8}\epsilon^{jklmn}u_{jk}u_{lm})$, one can choose a representation of the $d=10$ Pauli matrices such that

$$w_a=(v_j,\ \frac{1}{2}v_{[k}u_{lm]},\ \frac{1}{8}\epsilon^{jklmn}v_ju_{kl}u_{mn}), \quad (3.12)$$

which satisfies $\lambda^a w_a=\lambda^a(\sigma^{\mu\nu})_a{}^b w_b=0$, as implied by (3.11). To covariantize the massless solution of (3.3), one needs to define a suitable integration measure for integrating λ^a over the coset space $SO(10)/U(5)$.

Such a measure was defined in [11] as

$$[d\lambda]_{d=10}\equiv(C_b\lambda^b)^{-3}\epsilon_{a_1\ldots a_{16}}d\lambda^{a_1}\wedge\ldots\wedge d\lambda^{a_{10}}\lambda^{a_{11}}(C\sigma^\mu)^{a_{12}}(C\sigma^\nu)^{a_{13}}(C\sigma^\rho)^{a_{14}}(\sigma_{\mu\nu\rho})^{a_{15}a_{16}}, \quad (3.13)$$

where $[d\lambda]_{10}$ is independent of the choice of C_b and is therefore Lorentz-invariant. A simple way to show independence of the measure of C_b is by using invariance under the $U(1) \times SU(5)$ subgroup which preserves the pure spinor λ^a up to a phase. Under $U(1) \times SU(5)$, the sixteen components of an $SO(10)$ chiral spinor transform as $(1_{5/2}, 10_{1/2}, \overline{5}_{-3/2})$ representations and the sixteen components of an $SO(10)$ antichiral spinor transform as $(5_{3/2}, \overline{10}_{-1/2}, 1_{-5/2})$ representations, where the subscript denotes the $U(1)$ charge. So by our choice of the $U(1) \times SU(5)$ subgroup, λ^a transforms as an $SU(5)$ singlet with $U(1)$ charge $5/2$. Furthermore, since $\lambda^a \sigma^\mu_{ab} d\lambda^b = 0$, $d\lambda^a$ carries either $U(1)$ charge $5/2$ or $1/2$. Therefore, $d\lambda^{[a_1} \wedge ... \wedge d\lambda^{a_{10}} \lambda^{a_{11}]}$ carries $U(1)$ charge $15/2$, which implies by $U(1)$ conservation that only the component of C_b in the $1_{-5/2}$ representation contributes to (3.13). Finally, it is easy to see that (3.13) is invariant under scale transformations of this $1_{-5/2}$ component because there are an equal number of C_b's in the numerator and denominator of (3.13).

Using the measure factor of (3.13), the solution of (3.3) can be written in Lorentz-covariant form as

$$\Phi(x) = \oint [d\lambda]_{d=10} F(\lambda, w)|_{w=x\lambda} \tag{3.14}$$

where $F(h\lambda^a, hw_b) = h^{-8} F(\lambda^a, w_b)$ so that the integral over the projective spinor is well-defined. For example, choosing

$$F(\lambda, w) = \frac{\epsilon_{b_1 \dots b_{16}} A_1^{b_1} \dots A_{11}^{b_{11}} (\sigma^\mu w)^{b_{12}} (\sigma^\nu w)^{b_{13}} (\sigma^\rho w)^{b_{14}} (\sigma_{\mu\nu\rho})^{b_{15} b_{16}}}{\prod_{r=1}^{11} (A_r^a w_a)} \tag{3.15}$$

generates the $d = 10$ Green's function $\Phi(x) = (x^\mu x_\mu)^{-4}$.

One can similarly construct massless $d = 10$ solutions to higher-spin equations by using the formula

$$\Phi^{(a_1 \dots a_N)}(x) = \oint [d\lambda]_{d=10} \lambda^{a_1} \dots \lambda^{a_N} F(\lambda, w)|_{w=x\lambda} \tag{3.16}$$

where $F(\lambda^a, w_b)$ satisfies the condition $F(h\lambda^a, hw_b) = h^{-N-8} F(\lambda^a, w_b)$ for N positive. Since $\frac{\partial}{\partial x^\mu} F(\lambda, w) = (\lambda \sigma_\mu)_a \frac{\partial}{\partial w_a} F(\lambda, w)$, one can use $\sigma^\mu_{ab} \sigma_{\mu\ cd} \lambda^a \lambda^c = 0$ to show that $\sigma^\mu_{bc} \frac{\partial}{\partial x^\mu} \Phi^{(c a_2 \dots a_N)}(x) = 0$. So (3.16) describes a massless spin $\frac{1}{2}$ field when $N = 1$, a self-dual five-form field-strength when $N = 2$, etc. As in the $d = 8$ case, one cannot construct massless solutions when N is negative since $\sigma^\mu_{ab} \sigma_{\mu\ cd} \frac{\partial}{\partial w_b} \frac{\partial}{\partial w_d}$ does not necessarily vanish.

4. Twistor Construction in Higher Dimensions

In this section, we will generalize the constructions of the previous sections to arbitrary even dimension. In dimension $d = 2n$, a pure spinor is defined as a chiral spinor λ^a for $a = 1$ to 2^{n-1} which satisfies the additional constraints

$$\lambda^a \sigma_{ab}^{\mu_1 \ldots \mu_{n-4}} \lambda^b = \lambda^a \sigma_{ab}^{\mu_1 \ldots \mu_{n-8}} \lambda^b = \lambda^a \sigma_{ab}^{\mu_1 \ldots \mu_{n-12}} \lambda^b = \ldots = 0. \tag{4.1}$$

To covariantize u_{jk} and v_j of (3.1) for $j = 1$ to n, define the antichiral pure spinor

$$w_{\overline{b}} = \sigma^{\mu}_{a\overline{b}} x_\mu \lambda^a \tag{4.2}$$

where $\sigma^{\mu}_{a\overline{b}}$ are the $d = 2n$ Pauli matrices and $\overline{b}$ denotes $\overline{b} = \dot{b}$ when n is even and $\overline{b} = b$ when n is odd. Under $d = 2n$ conformal transformations, $(\lambda^a, w_{\overline{b}})$ transforms linearly as an $SO(2n, 2)$ spinor.

When

$$\lambda^a = (1,\ u_{j_1 j_2},\ -\frac{1}{8} u_{[j_1 j_2} u_{j_3 j_4]},\ -\frac{1}{48} u_{[j_1 j_2} u_{j_3 j_4} u_{j_5 j_6]},\ \ldots), \tag{4.3}$$

one can choose a representation of the $d = 2n$ Pauli matrices such that

$$w_{\overline{a}} = (v_{j_1},\ \frac{1}{2} v_{[j_1} u_{j_2 j_3]},\ \frac{1}{8} v_{[j_1} u_{j_2 j_3} u_{j_4 j_5]},\ \ldots) \tag{4.4}$$

which satisfies

$$\lambda \sigma^{\mu_1 \ldots \mu_{n-3}} w = \lambda \sigma^{\mu_1 \ldots \mu_{n-5}} w = \lambda \sigma^{\mu_1 \ldots \mu_{n-7}} w = \ldots = 0, \tag{4.5}$$

as implied by (4.2).

To define integration of pure spinors over the coset space $SO(2n)/U(n)$, a central role will be played by a Lorentz-invariant tensor

$$T^{[a_1 \ldots a_R](b_1 \ldots b_S)} \tag{4.6}$$

which is antisymmetric in its first R indices, symmetric in its last S indices, and satisfies

$$T^{[a_1 \ldots a_R](b_1 \ldots b_S)} \sigma_{b_1 b_2}^{\mu_1 \ldots \mu_{n-4}} = T^{[a_1 \ldots a_R](b_1 \ldots b_S)} \sigma_{b_1 b_2}^{\mu_1 \ldots \mu_{n-8}} = \ldots = 0.$$

When $d = 2n$, $R = 2^{n-1} - 1 - n(n-1)/2$ and $S = (n-2)(n-3)/2$. This tensor can be explicitly constructed by defining

$$T^{[a_1 \ldots a_R](b_1 \ldots b_S)} \theta_{a_1} \ldots \theta_{a_R} \tau_{b_1} \ldots \tau_{b_s} = \tag{4.7}$$

$$(\frac{\partial}{\partial\tau}\sigma^{j_1...j_n}\frac{\partial}{\partial\tau})(\frac{\partial}{\partial\tau}\sigma^{j_{n+1}...j_{2n}}\frac{\partial}{\partial\tau})...(\frac{\partial}{\partial\tau}\sigma^{j_{n(n-2)+1}...j_{n(n-1)}}\frac{\partial}{\partial\tau})$$
$$(\tau\frac{\partial}{\partial\theta})(\tau\sigma_{j_1j_2}\frac{\partial}{\partial\theta})(\tau\sigma_{j_3j_4}\frac{\partial}{\partial\theta})...(\tau\sigma_{j_{n(n-1)-1}j_{n(n-1)}}\frac{\partial}{\partial\theta})\ \ (\theta)^{2^{n-1}}$$

where θ_a is a fermionic spinor and τ_a is a bosonic pure spinor.[3] Note that there are $(2n-2)$ $\frac{\partial}{\partial\tau}$'s, $\frac{n^2-n+2}{2}$ τ's, $\frac{n^2-n+2}{2}$ $\frac{\partial}{\partial\theta}$'s, and 2^{n-1} θ's on the right-hand side of (4.7), which agrees with the powers of τ and θ on the left-hand side of (4.7). When $n=4$, $T^{ab}\tau_b$ is proportional to τ^a, and when $n=5$, $T^{[a_1...a_5](b_1b_2b_3)}\tau_{b_1}\tau_{b_2}\tau_{b_3}$ is proportional to $(\sigma^\mu\tau)^{a_1}(\sigma^\nu\tau)^{a_2}(\sigma^\rho\tau)^{a_3}(\sigma_{\mu\nu\rho})^{a_4a_5}$.

To define integration over λ^a, note that the measure factor

$$[d\lambda]_{d=2n}\equiv(C_f\lambda^f)^{-S}\epsilon_{a_1...a_{n(n-1)/2}bc_1...c_R}d\lambda^{a_1}\wedge...\wedge d\lambda^{a_{n(n-1)/2}}\ \lambda^bT^{[c_1...c_R](e_1...e_S)}C_{e_1}...C_{e_S} \tag{4.8}$$

is independent of the choice of C_b and is therefore Lorentz-invariant. As in the $d=10$ case described in the previous section, the easiest way to prove independence of (4.8) on C_b is to use the invariance under the $U(1)\times SU(n)$ subgroup which preserves the pure spinor λ^a up to a phase. Under $U(1)\times SU(n)$, the components of an $SO(2n)$ chiral spinor transform with $U(1)$ charges $(n/2,(n-4)/2,(n-8)/2,...)$, and λ^a transforms with $U(1)$ charge $n/2$. Furthermore, since $\lambda^a\sigma^{\mu_1...\mu_{n-4}}d\lambda=\lambda^a\sigma^{\mu_1...\mu_{n-8}}d\lambda=...=0$, $d\lambda^a$ carries either $U(1)$ charge $n/2$ or $(n-4)/2$. Therefore, $d\lambda^{[a_1}\wedge...\wedge d\lambda^{a_{n(n-1)/2}}\ \lambda^{b]}$ carries $U(1)$ charge $n(n-2)(n-3)/4$, which implies by $U(1)$ conservation that only the component of C_b with $U(1)$ charge $-n/2$ contributes to (4.8). Finally, it is easy to see that (4.8) is invariant under scale transformations of this $-n/2$ component because there is an equal number of C_b's in the numerator and denominator of (4.8).

Using the measure factor of (4.8), the solution of (3.3) can be written in Lorentz-covariant form as

$$\Phi(x)=\oint[d\lambda]_{d=2n}F(\lambda,w)|_{w=x\lambda} \tag{4.9}$$

where $F(h\lambda^a,hw_{\overline{b}})=h^{2-2n}F(\lambda^a,w_{\overline{b}})$ so that the integral over the projective spinor is well-defined. For example, choosing

$$F(\lambda,w)=\frac{\epsilon_{\overline{b}_1...\overline{b}_M\overline{c}_1...\overline{c}_R}A_1^{\overline{b}_1}...A_M^{\overline{b}_M}T^{[\overline{c}_1...\overline{c}_R](\overline{e}_1...\overline{e}_S)}w_{\overline{e}_1}...w_{\overline{e}_S}}{\prod_{j=1}^M(A_j^{\overline{a}}w_{\overline{a}})} \tag{4.10}$$

[3] It is interesting to note that (4.7) is the state with maximum number of τ's in the cohomology of the nilpotent operator $Q=\tau_a\frac{\partial}{\partial\theta_a}$. When $n=5$, Q is the zero-momentum contribution to the BRST operator for the $d=10$ superparticle.[20]

generates the $d = 2n$ Green's function $\Phi(x) = (x^\mu x_\mu)^{1-n}$ where $M = (n^2 - n + 2)/2$.

One can similarly construct massless $d = 2n$ solutions to higher-spin equations by using the formula

$$\Phi^{(a_1 \ldots a_N)}(x) = \oint [d\lambda]_{d=2n} \lambda^{a_1} \ldots \lambda^{a_N} F(\lambda, w)|_{w=x\lambda} \tag{4.11}$$

where $F(\lambda^a, w_{\bar{b}})$ satisfies the condition $F(h\lambda^a, hw_{\bar{b}}) = h^{-N+2-2n} F(\lambda^a, w_{\bar{b}})$ for N positive. Since $\frac{\partial}{\partial x^\mu} F(\lambda, w) = (\lambda\sigma_\mu)_{\bar{a}} \frac{\partial}{\partial w_{\bar{a}}} F(\lambda, w)$, one can use $\sigma^\mu_{a\bar{b}} \sigma_{\mu\ c\bar{d}} \lambda^a \lambda^c = 0$ to show that $\sigma^\mu_{b\bar{c}} \frac{\partial}{\partial x^\mu} \Phi^{(ba_2 \ldots a_N)}(x) = 0$. So (4.11) describes a massless spin $\frac{1}{2}$ field when $N = 1$, a self-dual n-form field-strength when $N = 2$, etc. Unlike the $d = 4$ and $d = 6$ cases, one cannot construct massless solutions when N is negative since $\sigma^\mu_{a\bar{b}} \sigma_{\mu\ c\bar{d}} \frac{\partial}{\partial w_{\bar{b}}} \frac{\partial}{\partial w_{\bar{d}}}$ does not necessarily vanish.

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Appendix

Here we collect some facts about pure spinors, elucidating their relation to a few descriptions of conventional twistors. In particular, we discuss the relation of pure spinors to complex structures on $\mathbf{R}^{2n}$, as well as to isotropic complex Grassmanians.

4.1. Complex Structures on $\mathbf{R}^{2n}$

Consider all complex structures on $\mathbf{R}^{2n}$ that are compatible with the flat metric. Since these are produced by all orthonormal changes of coordinates modulo the complex changes of coordinates, the moduli space of all complex structures is $SO(2n)/U(n)$.

Let us write this in more detail. Identifying $\mathbf{R}^{2n}$ with $\mathbf{C}^n$, with $z \in \mathbf{C}^n$ given by coordinates z_i so that $z = (z_1, \ldots, z_n)$, the metric is given by

$$ds^2 = (z, \overline{z}) G \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \ G = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

If $M \in SO(2n)$, then M satisfies $MM^\dagger = 1$ and $\overline{M} = GMG$ in the $(z, \overline{z})$ basis. That is, M can be written in a block form

$$M = \begin{pmatrix} T & U \\ \overline{U} & \overline{T} \end{pmatrix},$$

with

$$TT^\dagger + UU^\dagger = 1, \tag{4.12}$$

$$TU^t + UT^t = 0. \tag{4.13}$$

Such a matrix M defines complex coordinates $v = Tz + U\overline{z}$. Unitary transformations

$$\begin{pmatrix} \Lambda & 0 \\ 0 & \overline{\Lambda} \end{pmatrix}, \quad \Lambda\Lambda^\dagger = 1,$$

respect the complex structure, $v \sim \Lambda v$, and generically can be used to put $T = 1_n$. This can be compared to formula (3.1) for v_j in the text.

4.2. Isotropic Grassmanian

Dropping the normalization condition (4.12) and considering now $\Lambda \in GL(n, \mathbf{C})$, we observe that there is a unique solution to (4.12) on the $GL(n, \mathbf{C})$ orbit. Thus $SO(2n)/U(n)$ can be thought of as the space of pairs $(T, U) \sim (\Lambda T, \Lambda U)$ with

$$\begin{aligned} \det(TT^\dagger + UU^\dagger) &\neq 0 \\ TU^t + UT^t &= 0. \end{aligned}$$

The latter is the space of isotropic n-planes in $\mathbf{C}^{2n}$. That is, the $2n \times n$ matrix $\begin{pmatrix} T \\ U \end{pmatrix}$ defines $\mathbf{C}^n \subset \mathbf{C}^{2n}$ that is isotropic since

$$(T, U) G \begin{pmatrix} T^t \\ U^t \end{pmatrix} = 0.$$

Thus

$$SO(2n)/U(n) = G_n^0(\mathbf{C}^{2n}),$$

the isotropic Grassmanian. This space is of real dimension $n(n-1)$.

Given $(T, U)^t$, we consider a coordinate patch with $\det U \neq 0$. Then $(T, U)^t \sim (U^{-1}T, 1)^t$ and the elements of the antisymmetric matrix $U^{-1}T$ provide coordinates in the patch. A different choice of basis amounts to a permutation of rows in $(T, U)^t$. There are exactly 2^{n-1} such permutations respecting the orientation of the space. Thus the isotropic Grassmanian $G_n^0(\mathbf{C}^{2n})$ is covered by a minimum of 2^{n-1} coordinate charts, in other words its Lusternik-Schnirelmann category is 2^{n-1}.

4.3. Pure Spinors

For a pure spinor λ^α the only nonvanishing form is of degree n. Moreover, it is simple, i.e. its coefficients satisfy $\lambda^\alpha \gamma_{\alpha\beta}^{i_1 \ldots i_n} \lambda^\beta = a_1^{[i_1} a_2^{i_2} \ldots a_n^{i_n]}$ for some complex linearly independent vectors $a_1, a_2, \ldots a_n$. Thus each pure spinor defines a complex n-plane in $\mathbf{C}^{2n}$. Moreover, this plane is isotropic since

$$g_{i_1 j_1} (\lambda^\alpha \gamma_{\alpha\beta}^{i_1 \ldots i_n} \lambda^\beta)(\lambda^\gamma \gamma_{\gamma\delta}^{j_1 \ldots j_n} \lambda^\delta) = 0.$$

This correspondence is known as the Cartan map. It is one-to-one for projective pure spinors. Thus the space of all projective pure spinors is the isotropic Grassmanian $G_n^0(\mathbf{C}^{2n})$. We conclude that projective pure spinors parameterize complex structures on $\mathbf{R}^{2n}$.

References

[1] E. Cartan, The Theory of Spinors (Dover, New York, 1981).

[2] C. Chevalley, The Algebraic Theory of Spinors (Columbia University Press, New York, 1954).

[3] R. Penrose, "The nonlinear graviton," General Relativity and Gravitation 7 (1976), no. 2, 171–176.

[4] N. J. Hitchin, "Linear field equations on self-dual spaces," Proc. Roy. Soc. London Ser. **A 370** (1980), no. 1741, 173–191.

[5] R. Penrose and W. Rindler, "Spinors And Space-Time. Vol. 2: Spinor And Twistor Methods In Space-Time Geometry," Cambridge University Press, 1986 (See the Appendix).

[6] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "Construction of instantons," Phys. Lett. A **65**, 185 (1978).

[7] N. J. Hitchin, "Twistor construction of Einstein metrics," Global Riemannian geometry (Durham, 1983), 115–125, Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1984.

[8] S. A. Cherkis and N. J. Hitchin, "Gravitational instantons of type D(k)," arXiv:hep-th/0310084.

[9] M.F. Atiyah "Green's functions for self-dual four-manifolds," Math. Analysis and Applications, Part A Advances in Mathematics Supplementary Studies, Vol 7A (1981) 129-158.

[10] N. Berkovits, "Super-Poincaré covariant quantization of the superstring," JHEP 04 (2000) 018, hep-th/0001035.

[11] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," hep-th/0406055.

[12] P. Howe, "Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions," Phys. Lett. B273 (1991) 90;
P. Howe, "Pure spinors lines in superspace and ten-dimensional supersymmetric theories," Phys. Lett. B258 (1991) 141.

[13] R. S. Ward, "Completely solvable gauge-field equations in dimension greater than four," Nucl. Phys. **B 236** (1984), no. 2, 381–396.

[14] E. Witten, "Twistor-like transform in ten dimensions," Nucl. Phys. B266 (1986) 245;
A. Bengtsson, I. Bengtsson, M. Cederwall and N. Linden, "Particles, superparticles and twistors," Phys. Rev. D36 (1987) 1766;
E. Sokatchev, "Harmonic superparticle," Class. Quant. Grav. 4 (1987) 237;
D. Sorokin, V. Tkach and D. Volkov, "Superparticles, twistors and Siegel symmetry," Mod. Phys. Lett. A4 (1989) 901;

N. Berkovits, "A supertwistor description of the massless superparticle in ten-dimensional superspace," Nucl. Phys. B350 (1991) 193;

E. Bergshoeff, P. Howe, C. Pope, E. Sezgin and E. Sokatchev, "Ten-dimensional supergravity from lightlike integrability in loop superspace," Nucl. Phys. B354 (1991) 113;

A. Galperin, P. Howe and P. Townsend, "Twistor transform for superfields," Nucl. Phys. B402 (1993) 531.

[15] P. Budinich and A. Trautman, The Spinorial Chessboard (Trieste Notes in Physics, Springer-Verlag, Berlin, 1988).

[16] J. P. Harnad and S. Shnider, "Isotropic Geometry, Twistors And Supertwistors. 1. The Generalized Klein Correspondence And Spinor Flags," J. Math. Phys. **33**, 3197 (1992).

[17] M. R. O'Brian and J. H. Rawnsley, "Twistor spaces," Ann. Global. Anal. Geom. **3** (1985) 29-58.

[18] M. K. Murray, "A Penrose transform for the twistor space of an even-dimensional conformally flat Riemannian manifold," Ann. Global Anal. Geom. **4** (1986), no. 1, 71–88.

[19] Y. Inoue, "Twistor spaces of even-dimensional Riemannian manifolds," J. Math. Kyoto Univ. 32 (1992), no. 1, 101–134.

[20] N. Berkovits, "Covariant quantization of the superparticle using pure spinors," JHEP 09 (2001) 016, hep-th/0105050.