

Gauge Theories on a 2+2 Anisotropic Lattice

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(Dated: March 26, 2003)

Abstract

The implementation of gauge theories on a four-dimensional anisotropic lattice with two distinct lattice spacings is discussed, with special attention to the case where two axes are finely and two axes are coarsely discretized. Feynman rules for the Wilson gauge action are derived and the renormalizability of the theory and the recovery of the continuum limit are analyzed. The calculation of the gluon propagator and the restoration of Lorentz invariance in on-shell states is presented to one-loop order in lattice perturbation theory for $SU(N_c)$ on both 2+2 and 3+1 lattices.

I. INTRODUCTION

The anisotropic lattice is a popular tool in numerical simulations. The usual approach is to make the temporal lattice spacing fine while keeping the spatial directions relatively coarse. The extra temporal resolution in this 3+1 discretization scheme is useful when measuring two- and three-point correlation functions of particles which decay rapidly. This is particularly useful for heavy hadrons and glueball states, for which the signal to noise ratio decreases rapidly with time. The glueball spectrum [1, 2] was an early success for this approach and, more recently, heavy quark systems have been studied using anisotropic lattices [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In this paper we consider a generalization of the anisotropic approach to include lattices with a 2+2 discretization. The temporal and one spatial direction are made fine, keeping the remaining two spatial directions coarse. The motivation for this choice is to explore the feasibility of calculating decays which produce high-momentum daughter particles. These include the phenomenologically interesting transitions $B \rightarrow \pi l \nu$ and $B \rightarrow K^* \gamma$. CKM matrix elements are determined from such exclusive decay processes through a combination of experimentally measured branching fractions and theoretical calculations of form factors. The light daughter hadrons in these decays have non-zero momentum and so in a lattice calculation of the nonperturbative form factors there are cutoff effects proportional to this momentum in units of the lattice spacing. Therefore $B \rightarrow \pi l \nu$ has discretisation errors proportional to ap_π where a is the lattice spacing. However, the range of momenta reliably reached by current experimental and lattice data do not overlap. Lattice calculations work best with $p_\pi \leq 1\text{GeV}$ but the bulk of experimental data for the exclusive decay, $B \rightarrow \pi l \nu$ lie at $p_\pi \geq 1.5\text{GeV}$. To avoid model-dependence kinematic cuts can be applied, restricting the range of lattice data to momenta where the calculation is reliable [15], awaiting improved experimental results. Alternatively, the lattice data can be extrapolated to match experiment [16, 17, 18] but this introduces model-dependence and increases substantially the systematic error in lattice calculations and therefore in $|V_{ub}|$. For $B \rightarrow K^* \gamma$ the situation is more acute since this decay happens at the maximum recoil momentum of the daughter meson, far from the range accessible to current lattice calculations. Therefore it is necessary to extrapolate well outside the range of reliable data [19, 20], once again introducing model-dependence and increasing the systematic errors. For this reason it has not been

widely studied using lattice methods and calculations to date have used isotropic lattices. In both cases the difficulty for lattice calculations is that errors proportional to the momentum of the order ap , grow quickly. These must be controlled to make phenomenologically relevant lattice calculations. The advantages then of the 2+2 lattice are twofold. Firstly, the fine temporal lattice spacing serves the same purpose as in the 3+1 case: correlation functions should be accurately determined while keeping computational costs modest. Secondly, making one spatial direction fine and injecting all momentum along that direction keeps discretization errors of $\mathcal{O}(ap)$ small for high momenta.

The transition $B \rightarrow \pi l\nu$ has been calculated using a 3+1 discretisation scheme and the improved resolution in the temporal direction led to higher-momentum ($0 \leq p_\pi \leq 1.5\text{GeV}$) particles being reliably simulated [21]. Both this calculation and the isotropic calculations which were reviewed in Ref. 22 find that one of the largest systematic errors in the range of accessible momenta is due to the chiral extrapolation. The 2+2 discretization does not address this issue but it is hoped that it will further extend the range of momentum available to lattice calculations and that statistical precision will be enhanced.

The paper is organized as follows. In Section II a general anisotropic formulation for $SU(N_c)$ Yang-Mills theory in four dimensions is presented. The differences between 3+1 and 2+2 discretisations are discussed in terms of the lattice symmetries and the parameter tuning required. Section III contains the framework of the perturbative calculation, already outlined in Ref. 23. Although the goal is a 2+2 discretization, the Feynman rules and the analytic procedure given are completely general and allow an exact treatment of the calculations. In Section IV we present our results and include a comparison with other work. The challenging algebraic manipulations were performed on a computer using a symbolic code which handles the dependence on the anisotropy analytically. The core of the code is similar to that used in Ref. 24, while the treatment of the lattice integrals follows from Ref. 25. Section V contains our conclusions, while some technical details are given in Appendices A and B.

II. YANG-MILLS THEORIES ON A 2+2 ANISOTROPIC LATTICE

In this section, formulations of $SU(N_c)$ Yang-Mills theory on general orthogonal lattice types are considered. A generalisation of the Wilson action for $SU(N_c)$ Yang-Mills theory

on the lattice is given by

$$S_W = \beta \sum_{n,\mu\nu} c_{\mu\nu} \left(1 - \frac{1}{2N_c} \text{Tr}(P_{\mu\nu}(x) + P_{\mu\nu}^\dagger(x)) \right), \quad (1)$$

where $\beta = 2N_c/g^2$ with g^2 , the lattice coupling constant and N_c the number of colors. $P_{\mu\nu}$ is the plaquette in the (μ, ν) plane;

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x). \quad (2)$$

The six coefficients, $c_{\mu\nu}$ in Eq. (1) will describe the anisotropy class.

If some sub-sets of these parameters are given identical values, the lattice action may have symmetries under the interchange of axes. In particular, if the two identities

$$c_{12} = c_{23} = c_{31} (= c_{cc}) \quad \text{and} \quad c_{41} = c_{42} = c_{43} (= c_{cf}) \quad (3)$$

are imposed, the lattice action is symmetric under the cubic point group, and the 3+1 anisotropy class is realised. Similarly, if the identity

$$c_{13} = c_{14} = c_{23} = c_{24} (= c_{cf}) \quad (4)$$

is imposed, the lattice action transforms trivially under elements of the group $C_{4\nu} \otimes C_{4\nu}$, with the first constituent group comprising the rotations and reflections in the (1,2) plane and the second being those in the (3,4) plane. This anisotropy class is denoted 2+2.

For phenomenologically motivated reasons, as discussed in Section I, consideration will be restricted to these cases where only two distinct lattice spacings are permitted. The coarse and fine lattice spacing will be denoted a_c and a_f respectively. In this paper the overall scale a is chosen to be a_f . The implementation of a 3+1 anisotropy, in which the lattice spacing in the temporal direction is made fine keeping the spatial lattice spacing coarse, has been widely discussed in the literature [26, 27].

In the continuum, the corresponding sub-groups of $O(4)$, the rotations in Euclidean space that are generated by demanding the same symmetries in the coefficients of the continuum dimension-four operators, are $O(3)$ for the class described in Eq. (3) and $O(2) \otimes O(2)$ for Eq. (4). If the operators which transform trivially under these symmetries are enumerated there is one operator for the Euclidean group, $(\text{Tr } F_{\mu\nu}F_{\mu\nu})$, two operators for the $O(3)$ spatial rotation group (most conveniently denoted $\text{Tr } E_iE_i$ and $\text{Tr } B_iB_i$ for chromoelectric

and chromomagnetic) and three operators for the $O(2) \otimes O(2)$ case ($\text{Tr } F_{ff'}F_{ff'}$, $\text{Tr } F_{fc}F_{fc}$ and $\text{Tr } F_{cc'}F_{cc'}$).

The recovery of Euclidean invariance in the low-energy physics of an anisotropic lattice requires a parameter tuning, in contrast to the isotropic case. Additional differences arise between the 3+1 lattices, for which a one-parameter tuning suffices and 2+2 lattices which necessitate a two-parameter tuning. The importance of parameter tuning, in particular for the 2+2 case is emphasised in subsection II A.

Finally, it is interesting to note that taking the anisotropy $\xi \rightarrow \infty$ yields, up to a gauge transformation, the Hamiltonian limit of the theory on a 3+1 lattice. For the 2+2 anisotropy class, this limit can not be taken naively; a theory in which only two of the four dimensions are discretised is not regularised.

A. 2+2: the need for tuning

As the continuum limit is approached for the 3+1 lattice theory, the physics of the Euclidean invariant Yang-Mills theory is reproduced, provided care is taken to account for the different grid spacings, a_c and a . This can be achieved by tuning the relative weights of the two sets of coefficients in Eq. (3) to ensure the ratio of scales, $\xi = a_c/a$ measured by a physical probe takes its desired value or alternatively, an arbitrary choice of the two parameters c_{cc} and c_{cf} can be taken and the ratio of scales measured *post-hoc*. The 3+1 anisotropic theory is certain to be in the same universality class as the desired theory since the single free parameter (the relative weight of c_{cc} and c_{cf}) determines the ratio of scales ξ .

For the 2+2 case, an important distinction arises; while there are only two distinct dimension-four continuum operators in the 3+1 class, there are three for the 2+2 case. As a result, the recovery of a Euclidean invariant continuum theory is not guaranteed since there are two free parameters and only a single ratio of scales. As a result, the general 2+2 lattice theories can lie in a larger universality class than the continuum four-dimensional Yang-Mills theory.

For this reason, care must be taken to ensure the recovery of Euclidean invariance in the continuum limit for a 2+2 simulation. The relative weights of the three operators in the action must be determined to ensure Lorentz invariance in on-shell Green's functions. This tuning can be achieved perturbatively or by restoring symmetries in a non-perturbative

calculation of, for example the static inter-quark potential. In the remainder of this paper, perturbation theory is used to determine the parameters in the action. A paper describing the nonperturbative tuning of the parameters is in preparation [28].

B. $c_{\mu\nu}$ from perturbation theory

At tree level, it is straightforward to determine the values of the coefficients in the action. For the 3+1 lattice action, they are $c_{cc} = 1/\xi$, $c_{cf} = \xi$ and for the 2+2 case, they are $c_{cc} = 1/\xi^2$, $c_{cf} = 1$ and $c_{ff} = \xi^2$. The one-loop definitions of $c_{\mu\nu}$ can then be parameterised as

$$c_{\mu\nu}^{(2+2)} = \begin{cases} \xi^2 (1 + \eta_{ff}^{(1)} g^2 + O(g^4)) & \mu, \nu \text{ fine} \\ 1 (1 + \eta_{cf}^{(1)} g^2 + O(g^4)) & \mu \text{ coarse(fine), } \nu \text{ fine(coarse)} \\ \frac{1}{\xi^2} (1 + \eta_{cc}^{(1)} g^2 + O(g^4)) & \mu, \nu \text{ coarse} \end{cases} \quad (5)$$

and

$$c_{\mu\nu}^{(3+1)} = \begin{cases} \xi (1 + \eta_{cf}^{(1)} g^2 + O(g^4)) & \mu \text{ coarse(fine), } \nu \text{ fine(coarse)} \\ \frac{1}{\xi} (1 + \eta_{cc}^{(1)} g^2 + O(g^4)) & \mu, \nu \text{ coarse} \end{cases} \quad (6)$$

In both cases, an overall multiplicative weight can be absorbed into a redefinition of the field integration variables, and this is redundant in the action-tuning procedure since a symmetry is being enforced in on-shell Green's functions. This change of variables will become important once matrix-element matching of gluon fields is being performed.

III. PERTURBATION THEORY

We can now proceed to discuss Feynman rules for a general four-dimensional anisotropic Wilson-like action, already outlined in Ref. 23. We follow the notation given in Ref. 29 and the interpretation of the anisotropy as a difference in momentum cut-offs will be our guideline. The matching of the lattice gluon action in Eq. (1) with its continuum counterpart is made clearer by expressing the link variables $U_\mu(n) = e^{i\phi_\mu(n)}$ in terms of dimensionless fields

$$\phi_\mu(n) = \phi_\mu^b(n) T^b, \quad (7)$$

where T^b are the $SU(N_c)$ generators in the fundamental representation satisfying the relations, $[T^a, T^b] = if^{abc} T^c$ and $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. The dimensionful gluon fields, $A_\mu(x)$ can

then be reintroduced in Eq. (7) using the relation

$$\phi_\mu^b(n) = g a \xi_\mu A_\mu^b(x), \quad (8)$$

where

$$\xi_\mu = \begin{cases} 1 & \mu \text{ fine} \\ \xi & \mu \text{ coarse} \end{cases} \quad (9)$$

is the anisotropy index. By taking into account the Jacobian

$$\sum_n = \frac{1}{\xi^d a^d} \sum_x, \quad (10)$$

where d denotes the number of coarse directions, one easily gets Eqs. (5) and (6). The continuum limit of the Eq. (1) is then written as

$$S_W = \sum_n \frac{1}{4} c_{\mu\nu} \hat{F}_{\mu\nu}^b(n) \hat{F}_{\mu\nu}^b(n) + \mathcal{O} = \frac{1}{4} \sum_x F_{\mu\nu}^b(x) F_{\mu\nu}^b(x) + O(a^2), \quad (11)$$

where \mathcal{O} is an irrelevant operator in the continuum involving terms with three or more gluon fields from which the non-trivial contribution to the Feynman rules will arise, while

$$\hat{F}_{\mu\nu}^b(n) = \hat{\partial}_\mu^R \phi_\nu^b(n) - \hat{\partial}_\nu^R \phi_\mu^b(n) - g f^{bcd} \phi_\mu^c(n) \phi_\nu^d(n), \quad (12)$$

and

$$\hat{\partial}_\mu^R \varphi(n) = \varphi(n + \hat{\mu}) - \varphi(n) \quad (13)$$

are the dimensionless field strength and lattice right derivative, whereas

$$F_{\mu\nu}^b(x) = \partial_\mu^R A_\nu^b(x) - \partial_\nu^R A_\mu^b(x) - g f^{bcd} A_\mu^c(x) A_\nu^d(x), \quad (14)$$

and

$$\partial_\mu^R \varphi(x) = \frac{1}{a \xi_\mu} (\varphi(x + a \xi_\mu \hat{\mu}) - \varphi(x)) \quad (15)$$

are their dimensionful equivalents.

As is well known, the Wilson action has $O(a^2)$ discretization errors in the evaluation of various physical quantities. In addition, the lattice regularization gives rise to finite renormalization coefficients when compared to other continuum schemes. These effects can be reduced by adding irrelevant operators in the action which reduce discretization artefacts, or by improving the convergence of renormalization coefficients to better match continuum quantities [30, 31]. In any of these cases an exact and completely algebraic treatment of the

Feynman rules is always viable, even at orders higher than one [32, 33, 34, 35, 36]. As for the lattice integrals, while a completely numerical evaluation presents no problem at first order, any one-plaquette action will change the structure of the $O(a^2)$ (irrelevant) terms in the four and higher gluon vertices but will not change the propagators, the three-gluon vertex, the measure or the gauge-fixing terms, which are all fixed by the Haar measure and the naive matching with the continuum limit [37]. This means that once the first-order analytic technique for the Wilson action is developed it can be applied to any other one-plaquette action. For gauge actions with Wilson loops which extend two grid spacings, the technique given in Ref. 25 (for fermion propagators) can be adapted while for actions with even larger loops, suitable techniques can be developed in the same spirit. Using coordinate-space methods, analytic results are also available for higher orders [32, 38, 39]. Finally, mean-link improvement schemes [40] are straightforward to implement once the corresponding Feynman diagram is written down.

A. The gauge fixing and the gluon propagator

For perturbation theory it is necessary to choose a gauge fixing and proceed to define the other parts of the action needed to write the Feynman rules. In analogy with the isotropic case we choose the gauge-fixing term such that

$$S_{gf} = \frac{1}{\alpha} \sum_x \text{Tr} (\partial_\mu^L A_\mu)^2, \quad (16)$$

where $\partial_\mu^L \varphi(x) = 1/(a \xi_\mu) (\varphi(x) - \varphi(x - a \xi_\mu \hat{\mu}))$. It is straightforward to show that

$$S_{gf} = \frac{1}{\tilde{\alpha}} \sum_n \text{Tr} \mathcal{F}^2 \text{ for } \mathcal{F} = b_\mu^{(l)} \hat{\partial}_\mu^L \phi_\mu(n), \quad (17)$$

where now $\hat{\partial}_\mu^L \varphi(n) = \varphi(n) - \varphi(n - \hat{\mu})$ and

$$b_\mu^{(l)} = \begin{cases} \xi^{l+1} & \mu \text{ fine} , \\ \xi^{l-1} & \mu \text{ coarse} , \end{cases} \quad (18)$$

gives the desired form once $\alpha = \xi^{-2l+d-2} \tilde{\alpha}$. The choice $l = (d - 2)/2$ would be easiest but we shall see in the following it is not the most convenient. The gluon propagator can now be calculated from the two field terms in Eq. (12) and the definition of the gauge fixing in

Eq. (17). The functional form is similar to the isotropic case, namely

$$\mu, a \text{ \scriptsize k } \text{ \scriptsize ν, b } = \frac{\delta^{ab}}{\hat{k}^2} \left(\delta_{\mu\nu} - (1 - \alpha) \frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right), \quad (19)$$

where

$$\hat{k}_\mu = \frac{2}{a\xi_\mu} \sin\left(\frac{a\xi_\mu}{2} k_\mu\right), \quad \hat{k}^2 = \sum_\mu \hat{k}_\mu^2, \quad k_\mu \in \left(-\frac{\pi}{a\xi_\mu}, \frac{\pi}{a\xi_\mu}\right), \quad (20)$$

are the dimensionful lattice momenta spanning the anisotropic Brillouin zones. All the anisotropy is now encapsulated in this form of the Brillouin zone. This will prove to be a constant pattern: for any vertex or propagator which has a continuum analogue, the form in terms of the (anisotropic) dimensionful momenta will resemble the isotropic ones, since the asymmetry now lies solely in the different momenta cut-offs.

B. The Haar measure

On the lattice the functional integral is obtained by integrating over the dimensionless gauge links, but to do perturbation theory it must be expressed in terms of the dimensionful gauge fields. The Jacobian resulting from the expression of the Haar measure in terms of Eq. (7) is [29]

$$J = \sqrt{\det\left(\frac{1}{2}MM^\dagger\right)}, \quad M(\tilde{\phi}_\mu(n)) = \frac{1 - e^{-i\tilde{\phi}_\mu(n)}}{i\tilde{\phi}_\mu(n)}, \quad \tilde{\phi}_\mu(n) = \phi_\mu^a(n)t^a, \quad (21)$$

where the $\phi_\mu^a(n)$ are the same as in Eq. (7) and the t^a are the adjoint generators of $SU(N_c)$. Reexpressing J as $e^{-S_{\text{meas}}}$ gives

$$S_{\text{meas}} = -\frac{1}{2} \sum_{n,\mu} \text{Tr} \log \left(\frac{2 \left(1 - \cos \tilde{\phi}_\mu(n)\right)}{\tilde{\phi}_\mu^2(n)} \right). \quad (22)$$

When expanded to lowest order in the fields (which is all that is needed for one-loop two-point function calculations) this reads

$$S_{\text{meas}} \simeq \frac{1}{4!} \sum_{n,\mu} N_c \delta^{ab} \phi_\mu^a(n) \phi_\mu^b(n), \quad (23)$$

which putting back the dimensionful fields gives the vertex

$$\mu, a \text{ \scriptsize k } \text{ \scriptsize k' } \text{ \scriptsize ν, b } = -(2\pi)^4 \delta^4(k + k') \frac{g^2}{a^2} \frac{N_c}{12} \delta_{\mu\nu} \delta^{ab} \frac{\xi_\mu^2}{\xi^d}. \quad (24)$$

By convention all gluon momenta are incoming.

C. Fadeev-Popov ghost fields

The Faddeev-Popov determinant, which forces the integration only on a section of the gauge orbits, must also be included in the action. Using \mathcal{F} in Eq. (17) to enforce the gauge condition on ϕ_μ and using the response of the gauge fields on the lattice to (infinitesimal) gauge transformations [29]

$$\hat{D}_\mu(\phi) = M^{-1}(\tilde{\phi}_\mu)\hat{\partial}_\mu^R + i\tilde{\phi}_\mu, \quad (25)$$

where $\tilde{\phi}_\mu$ and the inverse adjoint-valued operator, M are given in Eq. (21), we obtain

$$S_{\text{gh}} = - \sum_{n,\mu} \bar{c}^a(n) b_\mu^{(l)} \hat{\partial}_\mu^L \hat{D}_\mu^{ab}(\phi) c^b(n), \quad (26)$$

where \hat{c} and $\bar{\hat{c}}$ are the dimensionless lattice ghost fields, introduced to make Eq. (26) local. From the expansion of M^{-1}

$$M^{-1}(\tilde{\phi}_\mu) = \mathbb{I}_{\text{adj}} + \frac{i}{2}\tilde{\phi}_\mu - \frac{1}{12}\tilde{\phi}_\mu^2 + \dots \quad (27)$$

and by reintroducing dimensions, Eq. (26) can be written as

$$S_{\text{gh}} \simeq -\xi^{l-1} \sum_x \left(\bar{c}^a(x) \delta^{ab} \partial_\mu^L \partial_\mu^R c^b(x) + g f^{abc} \bar{c}^a(x) \partial_\mu^L [A_\mu^c(x) (1 + \frac{a\xi_\mu}{2} \partial_\mu^R)] c^b(x) + \frac{1}{2!} \frac{g^2 a^2 \xi_\mu^2}{12} \delta_{\mu\nu} \{t^c, t^d\}_{ab} (\partial_\mu^R \bar{c}^a(x)) (\partial_\mu^R c^b(x)) A_\mu^c(x) A_\nu^d(x) \right),$$

by including only terms which are relevant for one-loop two-point functions. The factor ξ^{l-1} at the front of S_{gh} gives rise to spurious coefficients with no continuum analogue which however, cancel in any graph with no ghost outer legs. Alternatively the coefficients can be absorbed in a redefinition of the ghost fields, $\xi^{(l-1)/2} c \rightarrow c$ or by setting $l = 1$, thus rescaling the lattice gauge-fixing parameter $\tilde{\alpha} = \xi^{4-d} \alpha$. For each of these alternatives the ghost propagator and the two ghost one- and two-gluon vertices are

$$a \cdots \cdots \begin{array}{c} \blacktriangleright \\ k \\ \blacktriangleleft \end{array} \cdots \cdots b \quad = \quad \delta^{ab} \frac{1}{\hat{k}^2}, \quad (28)$$

$$\begin{array}{c} p', b \\ \blacktriangleright \\ \blacktriangleleft \\ p, a \end{array} \cdots \cdots \begin{array}{c} k \\ \text{wavy} \\ \mu, c \end{array} \quad = \quad ig(2\pi)^4 \delta^4(k+p-p') f^{abc} \hat{p}'_\mu \tilde{p}_\mu, \quad (29)$$

$$\begin{array}{c} p, b \\ \blacktriangleright \\ \blacktriangleleft \\ p', a \end{array} \cdots \cdots \begin{array}{c} k \\ \text{wavy} \\ \mu, c \end{array} \quad \begin{array}{c} p', a \\ \blacktriangleright \\ \blacktriangleleft \\ k', \nu, d \end{array} \quad = \quad \frac{1}{12} g^2 a^2 (2\pi)^4 \delta^4(k+k'+p-p') \{t^c, t^d\}_{ab} \delta_{\mu\nu} \xi_\mu^2 \hat{p}'_\mu \hat{p}'_\nu, \quad (30)$$

with $\tilde{p}_\mu = \cos \frac{1}{2} p_\mu a \xi_\mu$. As usual the vertex which has a continuum analogue carries no explicit dependence on the anisotropy.

D. Gluon vertices

Taking into account the three-gluon terms arising from Eq. (1), one of which comes from the irrelevant operator \mathcal{O} defined in Eq. (11) gives

$$S_W^{(3)} = g \sum_x f^{abc} \delta_{\lambda\nu} \left(A_\mu^a(x) + \frac{a \xi_\mu}{2} \partial_\nu^R A_\mu^a(x) \right) (\partial_\mu^R A_\nu^b(x)) A_\lambda^c(x), \quad (31)$$

which, after a Fourier transformation, leads to an expression equivalent to the isotropic case

$$\begin{array}{l}
 k, \mu, a \\
 \text{wavy line} \\
 k', \nu, b
 \end{array}
 \begin{array}{l}
 \text{wavy line} \\
 \text{wavy line} \\
 \text{wavy line}
 \end{array}
 k'', \lambda, c = ig(2\pi)^4 \delta^4(k + k' + k'') f^{abc} \left(\delta_{\nu\lambda} (\widehat{k'' - k'})_\mu \tilde{k}_\nu \right. \\
 \left. + \delta_{\mu\lambda} (\widehat{k - k''})_\nu \tilde{k}'_\lambda + \delta_{\mu\nu} (\widehat{k' - k})_\lambda \tilde{k}''_\mu \right). \quad (32)$$

Taking the four-gluon terms arising from Eq. (1), mostly coming from \mathcal{O} , (see Ref. 29 for details), and rechecking the cancellations which must occur due to Bose symmetry, one finds the four-gluon vertex whose expression is given in Appendix A.

E. One-loop correction vertex

The one-loop corrections to the action coming from Eqs. (5) and (6) give rise to an extra vertex which reads

$$\begin{array}{l}
 \mu, a \\
 \text{wavy line}
 \end{array}
 \begin{array}{c}
 \overset{k}{\curvearrowright} \\
 \bullet \\
 \overset{k'}{\curvearrowleft}
 \end{array}
 \begin{array}{l}
 \text{wavy line} \\
 \text{wavy line}
 \end{array}
 \nu, b = -(2\pi)^4 \delta^4(k + k') g^2 \delta^{ab} (\delta_{\mu\nu} \sum_\rho \eta_{\mu\rho}^{(1)} \hat{k}_\rho^2 - \eta_{\mu\nu}^{(1)} \hat{k}_\mu \hat{k}_\nu), \quad (33)$$

where

$$\eta_{\mu\nu}^{(1)} = \begin{cases} \eta_{ff}^{(1)} & \mu, \nu \text{ fine} \\ \eta_{cf}^{(1)} & \mu \text{ coarse(fine) and } \nu \text{ fine(coarse)} \\ \eta_{cc}^{(1)} & \mu, \nu \text{ coarse} \end{cases} \quad (34)$$

In the 3+1 case one can either set $\eta_{ff}^{(1)}$ to zero or leave it free, as by the traceless property of the vertex it will always cancel.

F. The continuum limit and anisotropic renormalization

The calculation of the one-loop corrections to the gluon self-energy, $\Sigma_{\mu\nu,ab}(p)$, using the Wilson action involves five Feynman diagrams [41] and Eq. (33). Each diagram is a function of the external momenta p and can be written as

$$G(p) = \int \frac{d^4k}{(2\pi)^4} F(k, p), \quad (35)$$

where k is the integration momenta. Since we are interested in the continuum limit of Eq. (35), if the integral is ultraviolet convergent we can simply substitute the function $F(k, p)$ with its continuum equivalent. Otherwise, if Eq. (35) is divergent and contains only massive propagators so that $F(k, p)$ is finite for any set of momenta going to zero, one can use the lattice version of the BPHZ technique [42, 43, 44] by writing

$$\begin{aligned} G(p) &= \int \frac{d^4k}{(2\pi)^4} \left[F(k, p) - \sum_{n=0}^{n_F} \frac{1}{n!} p_{\mu_1} \cdots p_{\mu_n} \left(\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_n}} F(k, p) \right)_{p=0} \right] \\ &\quad + \int \frac{d^4k}{(2\pi)^4} \sum_{n=0}^{n_F} \frac{1}{n!} p_{\mu_1} \cdots p_{\mu_n} \left(\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_n}} F(k, p) \right)_{p=0} \\ &\equiv G^c(p) + G^L(p), \end{aligned} \quad (36)$$

where n_F is the degree of divergence of the diagram. The first term in Eq. (36) is ultraviolet finite [43, 44] and therefore its continuum limit can be taken, making it independent of ξ . All the effects of the lattice regularization remain in the second term, which is simply a polynomial in the external momenta with coefficients given by zero-momentum lattice integrals.

On the other hand, if a diagram contains massless propagators, as in our case, more care is needed: indeed an expansion around $p = 0$ can give rise to infrared divergences. A simple recipe is to introduce an intermediate infrared regularization. Given an anisotropic cut-off the introduction of a mass, m in the propagators is the most suitable solution [25, 32, 45], while dimensional regularization is popular in the literature for the isotropic case [33, 41]. $G^c(p)$ and $G^L(p)$ are then divergent for $m \rightarrow 0$ separately but the divergences cancel in the sum.

IV. RESULTS

In this section we present the results of the one-loop correction to the gluon propagator using the Wilson action in the Feynman gauge for a general anisotropic lattice in four dimensions. Applying the procedure explained in Section (IV A) we calculate the values of the coefficients which restore on-shell Lorentz invariance, where the definition and treatment of the lattice integrals \mathcal{B}_ξ is given in Appendix (B).

A. On-shell Lorentz invariance

As explained in Section II, the one-loop propagator obtained from the Feynman rules given in Section III will not in general satisfy Lorentz invariance. The free parameters in Eq. (34) must be tuned to restore the symmetry [26, 27, 46]. From Eq. (33) it is clear that not every Lorentz-breaking term can be cancelled, as expressions of the form $\delta_{\mu\nu}/a^2$ and $\delta_{\mu\nu}p_\mu^2$ arise from each diagram. The first non-trivial result of our calculation is that, for any value of d and ξ , such terms cancel exactly when the diagrams are summed up, just as in the isotropic case, independently of the tuning procedure.

The remaining Lorentz-breaking artefacts arising from the mixing of longitudinal and transverse field components have the correct structure and can be cured by tuning. We choose to fix the parameters in the action by demanding the recovery of Lorentz invariance for on-shell physical soft gluon modes, imposing that the two physical eigenvalues of the one-loop propagator vanish for $E^2 = p^2$. Since this is a gauge-invariant condition we restrict ourselves to a particular gauge. We have chosen the Feynman gauge for which $\alpha = 1$. By injecting the gluon momentum in any possible direction and calculating the eigenvalues and eigenvectors of the propagator we obtain a general condition for $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$, independent of the direction of p_μ . The value of $\eta_{cf}^{(1)}$ remains unconstrained as it can be reabsorbed in a one-loop β -shift.

To identify the physical eigenmodes, the momenta directions with residual symmetry are first identified; In the case of the 2+2 lattice, this corresponds to any momenta in the coarse-coarse or fine-fine planes. Then for more general momenta, the axis of propagation was smoothly varied away from these symmetric cases, and the eigenmodes continuously traced. This investigation lead to a generalisation of the polarisation condition, $p_\mu \epsilon_\mu = 0$

for on-shell gluon polarisation vector ϵ_μ . The resulting 2+2 lattice polarisation condition is

$$p_\mu Z_{\mu\nu} \epsilon_\nu = 0, \quad (37)$$

with $Z_{\mu\nu}$ is the diagonal matrix $Z_{\mu\nu} = \delta_{\mu\nu} (Z_c \delta_{\mu c} + Z_f \delta_{\mu f})$ and Z_c, Z_f the coarse and fine gluon field renormalisation coefficients. $\delta_{\mu f}$ and $\delta_{\mu c}$ are one(zero) if μ is fine(coarse).

B. The one-loop coefficients for the restoration of Lorentz invariance

For 2+2 anisotropic lattice the one-loop coefficients for the restoration of Lorentz invariance are

$$\begin{aligned} \eta_{cc}^{(1)} - \eta_{cf}^{(1)} = & -\frac{1}{2N_c} \left[\mathcal{B}_\xi^c(1, 1) - \frac{1}{4} \right] + N_c \left[-\frac{1}{16} + \frac{\mathcal{B}_\xi(1)}{6} \left(\frac{7}{2} + \frac{1}{\xi^2} \right) + \frac{\mathcal{B}_\xi^c(1, 1)}{4} \right. \\ & \left. - \frac{\mathcal{B}_\xi^c(2, 1)}{3} \left(2 + \frac{5}{2\xi^4} + \frac{11}{2\xi^2} \right) + \frac{\mathcal{B}_\xi^f(2, 1, 1)}{6} \left(\frac{1}{2} + \xi^2 \right) \right], \end{aligned} \quad (38)$$

and

$$\begin{aligned} \eta_{ff}^{(1)} - \eta_{cf}^{(1)} = & \frac{1}{2N_c} \left[\frac{1}{4} - \frac{1}{2\xi^2} + \frac{\mathcal{B}_\xi^c(1, 1)}{\xi^4} \right] - \frac{N_c}{2} \left[\frac{1}{4} \left(\frac{1}{2} - \frac{1}{\xi^2} \right) + \mathcal{B}_\xi(1) \left(\frac{1}{2} + \frac{1}{3\xi^2} \right) \right. \\ & \left. + \frac{\mathcal{B}_\xi^c(1, 1)}{2\xi^4} - \mathcal{B}_\xi^c(2, 1) \left(\frac{5}{3\xi^4} + \frac{1}{\xi^2} + \frac{1}{6} \right) - \frac{\mathcal{B}_\xi^f(2, 1, 1)}{3\xi^2} \right], \end{aligned} \quad (39)$$

which diverge logarithmically with ξ , while for 3+1 we have

$$\begin{aligned} \eta_{cc}^{(1)} - \eta_{cf}^{(1)} = & \frac{N_c}{\xi} \left[\frac{\mathcal{B}_\xi(1)}{6} \left(\frac{\xi^2}{3} + \frac{19}{6} + \frac{7}{2\xi^2} \right) + \frac{\mathcal{B}_\xi^c(1, 1)}{4} \left(1 + \frac{1}{\xi^2} \right) \right. \\ & \left. - \mathcal{B}_\xi^c(2, 1) \left(\frac{1}{3} + \frac{11}{6\xi^2} + \frac{5}{2\xi^4} \right) - \frac{1}{8} \right] - \frac{1}{2\xi N_c} \left[\mathcal{B}_\xi^c(1, 1) \left(1 + \frac{1}{\xi^2} \right) - \frac{1}{2} \right], \end{aligned} \quad (40)$$

which agrees with the general result of Ref. 27 and the $N_c = 3$ result of Ref. 46. The techniques used by both groups involve extrapolations in some suitable parameter, absent in our treatment. The functions \mathcal{B}_ξ are defined in Appendix B. Using the results given there the Hamiltonian limit, $\xi \rightarrow \infty$ can be treated analytically also and agrees again with Ref. 27. The difference in the parameters, $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$ appears as a polynomial in N_c with terms proportional to $1/N_c$ and N_c only, as expected. Numerical values for the coefficients in this polynomial for a range of anisotropies are given to ten decimal places in Tables (I,II).

C. The gluon self-energy in the Feynman gauge

Once the values in Eqs. (38), (39) and (40) are inserted, the one-loop correction to the gluon self-energy reads

$$\Sigma_{\mu\nu,ab}^{(1)}(p) = g^2 \delta^{ab} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \left\{ A - \eta_{cf}^{(1)} - B (\delta_{\mu c} \delta_{\nu c} - \delta_{\mu f} \delta_{\nu f}) \right\}. \quad (41)$$

For 2+2 we have

$$A = -\frac{1}{8N_c} + N_c \left[\frac{1}{16} + \frac{\mathcal{B}_\xi(1)}{2} \left(\frac{1}{3} - \frac{1}{4\xi^2} \right) + \frac{\mathcal{B}_\xi^c(2,1)}{4\xi^2} \left(\frac{7}{3\xi^2} - 1 \right) - \frac{\mathcal{B}_\xi^f(2,1,1)}{12} + \frac{5}{3} X(\xi) \right] \quad (42)$$

$$B = N_c \left[\frac{\mathcal{B}_\xi(1)}{12} \left(1 + \frac{1}{2\xi^2} \right) - \frac{\mathcal{B}_\xi^c(2,1)}{4\xi^2} \left(1 + \frac{1}{\xi^2} \right) \right], \quad (43)$$

while for 3+1

$$A = \frac{1}{2N_c \xi} \left[\frac{\mathcal{B}_\xi^c(1,1)}{\xi^2} - \frac{1}{2} \right] + \frac{N_c}{\xi} \left[\frac{1}{8} + \frac{5\xi}{3} X(\xi) - \frac{\mathcal{B}_\xi^c(1,1)}{4\xi^2} + \frac{\mathcal{B}_\xi^c(2,1)}{4\xi^2} \left(\frac{7}{\xi^2} + \frac{1}{3} \right) + \frac{\mathcal{B}_\xi(1)}{6} \left(\frac{1}{3} - \frac{5}{2\xi^2} \right) \right], \quad (44)$$

$$B = \frac{N_c}{\xi} \left[\frac{\mathcal{B}_\xi(1)}{6} \left(\frac{1}{2} + \frac{1}{\xi^2} \right) - \frac{\mathcal{B}_\xi^c(2,1)}{4\xi^2} \left(1 + \frac{3}{\xi^2} \right) \right], \quad (45)$$

where

$$X(\xi) = \frac{1}{16\pi^2} (-\log(a^2 p^2) + F_0(\xi) - \gamma_E + \frac{28}{9}). \quad (46)$$

We stress that the term proportional to B in Eq. (41) does not break Lorentz invariance but arises from the difference in the renormalizations of the fine and coarse fields A_μ . This difference must be taken into account when calculating physical quantities such as the Λ parameter or matrix elements. After including the difference, the physical modes will still be transverse.

Setting $\xi = 1$ we recover from Eq. (41) the well-known value of the gluon self-energy [32, 33, 41]

$$\Sigma_{\mu\nu,ab}(p) = \delta^{ab} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \left[1 - g^2 \left\{ \frac{N_c}{16\pi^2} \left(-\frac{5}{3} \log(p^2 a^2) + \frac{28}{9} \right) - \frac{1}{8N_c} + N_c \left(\frac{7}{72} Z_0 + \frac{5}{48\pi^2} (F_0 - \gamma_E) + \frac{1}{16} \right) \right\} \right] + O(g^4), \quad (47)$$

where Z_0 and F_0 are standard integrals in perturbation theory on the isotropic lattice, and are defined in Ref. 45. In order to calculate coefficients at the one-loop level in mean-link

improved perturbation theory, the evaluation of either the plaquette, $\langle \text{Tr} P_{\mu\nu} \rangle$ or the link trace in Landau gauge, $\langle \text{Tr} U_\mu \rangle$ is required. We find

$$\frac{1}{N_c} \langle \text{Tr} P_{cc} \rangle = 1 - \frac{g^2}{2\xi^{d-2}} (N_c - \frac{1}{N_c}) \mathcal{B}_\xi^c(1, 1), \quad (48)$$

$$\frac{1}{N_c} \langle \text{Tr} P_{fc} \rangle = 1 - \frac{g^2}{4\xi^d} (N_c - \frac{1}{N_c}) \left(\xi^2 \mathcal{B}_\xi^f(1, 1) + \mathcal{B}_\xi^c(1, 1) \right), \quad (49)$$

$$\frac{1}{N_c} \langle \text{Tr} P_{ff} \rangle = 1 - \frac{g^2}{2\xi^d} (N_c - \frac{1}{N_c}) \mathcal{B}_\xi^f(1, 1), \quad (50)$$

and

$$\frac{1}{N_c} \langle \text{Tr} U_c \rangle = 1 - \frac{g^2}{4\xi^d} (N_c - \frac{1}{N_c}) \left[\xi^2 \mathcal{B}_\xi(1) - \mathcal{B}_\xi^c(2, 1) \left(1 - \frac{1}{\alpha} \right) \right], \quad (51)$$

$$\frac{1}{N_c} \langle \text{Tr} U_f \rangle = 1 - \frac{g^2}{4\xi^d} (N_c - \frac{1}{N_c}) \left[\mathcal{B}_\xi(1) - \mathcal{B}_\xi^f(2, 1) \left(1 - \frac{1}{\alpha} \right) \right], \quad (52)$$

where $\mathcal{B}_\xi^f(q, 1)$ satisfies Eq. (B4). Numerical values of the N_c -polynomial coefficients, $\eta_{cc}^{(1)}$, $\eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)}$ for a range of anisotropies and including these improvement terms are given in Tables (III,IV,V,VI).

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$		$\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c	$1/N_c$	N_c
1	0	0	0	0
2	-0.1161042701	0.0958601237	0.0775690169	-0.0741548219
3	-0.1886258210	0.1529850132	0.1010941459	-0.1044390683
4	-0.2391230676	0.1920838712	0.1107973557	-0.1217185361
5	-0.2774799637	0.2215115752	0.1156439679	-0.1335092263
6	-0.3083219799	0.2450412380	0.1183899089	-0.1424129673
7	-0.3340937184	0.2646297423	0.1200891686	-0.1495669092
8	-0.3562244833	0.2814071688	0.1212112364	-0.1555537098
9	-0.3756169823	0.2960809268	0.1219898822	-0.1607076435
10	-0.3928761275	0.3091218866	0.1225517876	-0.1652371442

TABLE I: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 2+2 anisotropic lattice.

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c
1	0	0
2	-0.0853420430	0.0663515978
3	-0.1202379609	0.0892976526
4	-0.1389282597	0.1007025039
5	-0.1504977336	0.1074686585
6	-0.1583451793	0.1119335310
7	-0.1640119208	0.1150960459
8	-0.1682939409	0.1174518583
9	-0.1716426764	0.1192740369
10	-0.1743328788	0.1207251404

TABLE II: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 3+1 anisotropic lattice.

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$		$\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c	$1/N_c$	N_c
1	-1/16	1/16	1/16	-1/16
2	-0.2366564052	0.2164122588	0.1981211519	-0.1947069547
3	-0.3454387314	0.3097979237	0.2579070564	-0.2612519788
4	-0.4211846015	0.3741454050	0.2928588896	-0.3037800700
5	-0.4787199456	0.4227515571	0.3168839498	-0.3347492082
6	-0.5249829699	0.4617022280	0.3350508989	-0.3590739573
7	-0.5636405777	0.4941766016	0.3496360278	-0.3791137684
8	-0.5968367250	0.5220194105	0.3618234781	-0.3961659514
9	-0.6259254734	0.5463894180	0.3722983733	-0.4110161346
10	-0.6518141912	0.5680599503	0.3814898513	-0.4241752080

TABLE III: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 2+2 anisotropic lattice using mean (coarse) plaquette improvement.

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$		$\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c	$1/N_c$	N_c
1	$-3Z_0/8$	$3Z_0/8$	$3Z_0/8$	$-3Z_0/8$
2	-0.2304051667	0.2101610203	0.1918699134	-0.1884557162
3	-0.3391678963	0.3035270885	0.2516362213	-0.2549811436
4	-0.4150758989	0.3680367024	0.2867501870	-0.2976713674
5	-0.4727526305	0.4167842420	0.3109166347	-0.3287818930
6	-0.5191216331	0.4558408912	0.3291895621	-0.3532126205
7	-0.5578576084	0.4883936322	0.3438530585	-0.3733307991
8	-0.5911125215	0.5162952070	0.3560992746	-0.3904417479
9	-0.6202462174	0.5407101620	0.3666191173	-0.4053368786
10	-0.6461700016	0.5624157607	0.3758456617	-0.4185310183

TABLE IV: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ and $\eta_{ff}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 2+2 anisotropic lattice using mean (coarse) link improvement. Z_0 is defined in Section IV B.

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c
1	$-1/16$	$1/16$
2	-0.1694788603	0.1504884150
3	-0.2118450434	0.1809047350
4	-0.2337180290	0.1954922732
5	-0.2468908748	0.2038617997
6	-0.2556482395	0.2092365912
7	-0.2618777619	0.2129618870
8	-0.2665309579	0.2156888753
9	-0.2701369251	0.2177682856
10	-0.2730125217	0.2194047833

TABLE V: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 3+1 anisotropic lattice using mean (coarse) plaquette improvement.

ξ	$\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$	
	$1/N_c$	N_c
1	$-3Z_0/8$	$3Z_0/8$
2	-0.1643550433	0.1453645980
3	-0.2068380749	0.1758977666
4	-0.2288323401	0.1906065843
5	-0.2420838939	0.1990548187
6	-0.2508913882	0.2044797400
7	-0.2571539394	0.2082380645
8	-0.2618297768	0.2109876942
9	-0.2654518384	0.2130831989
10	-0.2683392407	0.2147315023

TABLE VI: N_c -polynomial coefficients of $\eta_{cc}^{(1)} - \eta_{cf}^{(1)}$ for $\xi = 1, \dots, 10$ on a 3+1 anisotropic lattice using mean (coarse) link improvement. Z_0 is defined in Section IV B.

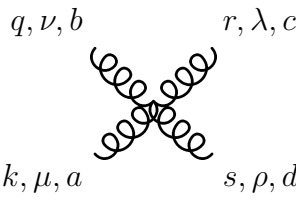
V. CONCLUSIONS

In this paper, a generalization of the Wilson discretization to an anisotropic lattice with two coarse and two fine directions has been described. In particular, an important distinction between the 2+2 anisotropy and the well-established 3+1 case has been emphasized. The difference is that the coefficients in the 2+2 action must be determined before simulation to ensure Lorentz invariance, while any 3+1 action leads to a Lorentz-invariant theory once the ratio of scales, ξ is determined. We are currently investigating non-perturbative techniques for computing these coefficients in the 2+2 case [28].

The main result of the paper was to compute these coefficients to first order in perturbation theory. While the focus of the calculation was on determining the Feynman rules for the 2+2 lattice Wilson gauge action, a more general prescription was developed to allow the 3+1 case to be investigated as well. This allowed us to check our results against previously published work. The results in a mean-link improvement scheme were presented.

The usefulness of this scheme arises from the need to make accurate calculations of form-factors and matrix elements at high momentum. This paper establishes the tools for perturbation theory calculations on the 2+2 anisotropic lattice, which will be important later when computing the matching factors to link calculations of weak-decay matrix elements to their continuum counterparts. As part of this program, quark fields on 2+2 anisotropic lattices are under consideration.

APPENDIX A: THE FOUR GLUON VERTEX



$$\begin{aligned}
&= -g^2 \left\{ \sum_e f^{abe} f^{cde} \left\{ \delta_{\mu\lambda} \delta_{\nu\rho} \left[\widetilde{(q-s)}_\mu \widetilde{(k-r)}_\nu - \frac{a^4}{12} \xi_\mu^2 \xi_\nu^2 \hat{k}_\nu \hat{q}_\mu \hat{r}_\nu \hat{s}_\mu \right] \right. \right. \\
&\quad \left. \left. - \delta_{\mu\rho} \delta_{\nu\lambda} \left[\widetilde{(q-r)}_\mu \widetilde{(k-s)}_\nu - \frac{a^4}{12} \xi_\mu^2 \xi_\nu^2 \hat{k}_\nu \hat{q}_\mu \hat{r}_\mu \hat{s}_\nu \right] \right. \right. \\
&\quad \left. \left. + \frac{a^2}{6} \delta_{\nu\lambda} \delta_{\nu\rho} \xi_\nu^2 \widetilde{(s-r)}_\mu \hat{k}_\nu \tilde{q}_\mu - \frac{a^2}{6} \delta_{\mu\lambda} \delta_{\mu\rho} \xi_\mu^2 \widetilde{(s-r)}_\nu \hat{q}_\mu \tilde{k}_\nu \right. \right. \\
&\quad \left. \left. + \frac{a^2}{6} \delta_{\mu\nu} \delta_{\mu\rho} \xi_\rho^2 \widetilde{(q-k)}_\lambda \hat{r}_\rho \tilde{s}_\lambda - \frac{a^2}{6} \delta_{\mu\nu} \delta_{\mu\lambda} \xi_\lambda^2 \widetilde{(q-k)}_\rho \hat{s}_\lambda \tilde{r}_\rho \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{a^2}{12} \delta_{\mu\nu} \delta_{\mu\lambda} \delta_{\mu\rho} \xi_\mu^2 \sum_\sigma (\widehat{q-k})_\sigma (\widehat{s-r})_\sigma \Big\} \\
& + (b \leftrightarrow c, \nu \leftrightarrow \lambda, q \leftrightarrow r) + (b \leftrightarrow d, \nu \leftrightarrow \rho, q \leftrightarrow s) \Big\} \\
& + \frac{g^2}{12} a^4 \left\{ \frac{2}{N_c} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \right. \\
& + \left. \sum_e (d^{abe} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce}) \right\} \\
& \times \left\{ \delta_{\mu\nu} \delta_{\mu\lambda} \delta_{\mu\rho} \xi_\mu^2 \sum_\sigma \xi_\sigma^2 \hat{k}_\sigma \hat{q}_\sigma \hat{r}_\sigma \hat{s}_\sigma - \delta_{\mu\nu} \delta_{\mu\lambda} \xi_\mu^2 \xi_\rho^2 \hat{k}_\rho \hat{q}_\rho \hat{r}_\rho \hat{s}_\mu \right. \\
& - \delta_{\mu\nu} \delta_{\mu\rho} \xi_\mu^2 \xi_\lambda^2 \hat{k}_\lambda \hat{q}_\lambda \hat{s}_\lambda \hat{r}_\mu - \delta_{\mu\lambda} \delta_{\mu\rho} \xi_\mu^2 \xi_\nu^2 \hat{k}_\nu \hat{r}_\nu \hat{s}_\nu \hat{q}_\mu - \delta_{\nu\lambda} \delta_{\nu\rho} \xi_\mu^2 \xi_\nu^2 \hat{q}_\mu \hat{r}_\mu \hat{s}_\mu \hat{k}_\nu \\
& \left. + \delta_{\mu\nu} \delta_{\lambda\rho} \xi_\mu^2 \xi_\lambda^2 \hat{k}_\lambda \hat{q}_\lambda \hat{r}_\mu \hat{s}_\mu + \delta_{\mu\lambda} \delta_{\nu\rho} \xi_\mu^2 \xi_\nu^2 \hat{k}_\nu \hat{r}_\nu \hat{q}_\mu \hat{s}_\mu + \delta_{\mu\rho} \delta_{\nu\lambda} \xi_\mu^2 \xi_\nu^2 \hat{k}_\nu \hat{s}_\nu \hat{q}_\mu \hat{r}_\mu \right\}
\end{aligned}$$

APPENDIX B: BOSONIC INTEGRAL EVALUATION

Apart from the denominators in Eq. (19) and (28), all the ξ dependence in the evaluation of Feynman diagrams is polynomial in ξ and $1/\xi$. Having changed integration variables to the asymmetric Brillouin zone, only lattice zero-momentum integrals of the form

$$\mathcal{B}_\xi(q, n_1, n_2, n_3, n_4) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{\hat{k}_x^{2n_1} \hat{k}_y^{2n_2} \hat{k}_z^{2n_3} \hat{k}_t^{2n_4}}{D_B(k, m)^q} \quad (\text{B1})$$

remain. In Eq. (B1), q and n_i are positive integers, $\hat{k}_\mu = 2 \sin(k_\mu/2)$ and the ξ dependence is in the inverse bosonic propagator

$$D_B(k, m) = \sum_{\mu \in f} \hat{k}_\mu^2 + \frac{1}{\xi^2} \sum_{\mu \in c} \hat{k}_\mu^2 + m^2. \quad (\text{B2})$$

alone. To evaluate these integrals, we adapt a technique given in Refs. 25, 45. In the following, when one of the n_i is zero it is omitted as an argument of \mathcal{B}_ξ , while, where no confusion can arise, the index in n_μ is dropped and \mathcal{B}_ξ^f , \mathcal{B}_ξ^c or \mathcal{B}_ξ^{fc} denotes integrals whose numerators have a fine, coarse or mixed momentum. In Refs. 25, 45, a set of recursion relations was defined to reduce every relevant integral to a linear combination of three basic integrals. Although similar relations exist in the asymmetric case, the lack of a complete symmetry among the indices, n_i makes this reduction more difficult. The relations can still be used to reduce the number of integrals which must be calculated considerably and to prove the cancellations of Lorentz-breaking terms. A first set can be obtained by expanding

the trivial identity

$$\mathcal{B}_\xi(q - n, n_a, \dots, n_b) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{\hat{k}_a^{2n_a} \dots \hat{k}_b^{2n_b} D_B(k, m)^n}{D_B(k, m)^q}, \quad (\text{B3})$$

which gives relations of the type

$$(4 - d)\mathcal{B}_\xi^f(q, 1) + \frac{d}{\xi^2}\mathcal{B}_\xi^c(q, 1) = \mathcal{B}_\xi(q - 1) - m^2\mathcal{B}_\xi(q), \quad (\text{B4})$$

$$(4 - d)\mathcal{B}_\xi^f(q, 2) + (4 - d)(3 - d)\mathcal{B}_\xi^f(q, 1, 1) + \frac{2d(4 - d)}{\xi^2}\mathcal{B}_\xi^{fc}(q, 1, 1) + \frac{d}{\xi^4}\mathcal{B}_\xi^c(q, 2) + \frac{d(d - 1)}{\xi^4}\mathcal{B}_\xi^c(q, 1, 1) = \mathcal{B}_\xi(q - 2) - 2m^2\mathcal{B}_\xi(q - 1) + m^4\mathcal{B}_\xi(q) \quad (\text{B5})$$

and so on. Furthermore, when $r > 1$, using the identity

$$\frac{(\hat{k}_\mu^2)^r}{D_B(k, m)^q} = 4\frac{(\hat{k}_\mu^2)^{r-1}}{D_B(k, m)^q} + 2\xi_\mu^2\frac{(\hat{k}_\mu^2)^{r-2}}{q - 1} \sin k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{D_B(k, m)^{q-1}}, \quad (\text{B6})$$

and integrating by parts, we obtain the relation

$$\mathcal{B}_\xi(q, \dots, r_\mu, \dots) = \xi_\mu^2\frac{r - 1}{q - 1}\mathcal{B}_\xi(q - 1, \dots, r_\mu - 1, \dots) - \xi_\mu^2\frac{4r - 6}{q - 1}\mathcal{B}_\xi(q - 1, \dots, r_\mu - 2, \dots) + 4\mathcal{B}_\xi(q, \dots, r_\mu - 1, \dots), \quad (\text{B7})$$

which depends on the index $\mu = f, c$. A final set of relations is found by using the trivial fact that the numerator cannot have more than four different arguments, e.g.

$$\mathcal{B}_\xi(q, 1, 1, 1, 1) = (d - 4)\mathcal{B}_\xi(q + 1, 2_f, 1, 1, 1) + \frac{d}{\xi^2}\mathcal{B}_\xi(q + 1, 1, 1, 1, 2_c) + m^2\mathcal{B}_\xi(q + 1, 1, 1, 1, 1). \quad (\text{B8})$$

The 2+2 case has the bonus relation, $1/\xi\mathcal{B}_\xi = \xi\mathcal{B}_{\frac{1}{\xi}}$ which is relevant to the numerical evaluation of a range of integrals. A similar relation would map 3+1 integrals to 1+3. With the help of these relations we can reduce the propagator calculation to four converging integrals, $\mathcal{B}_\xi(1)$, $\mathcal{B}_\xi^c(1, 1)$, $\mathcal{B}_\xi^c(2, 1)$ and $\mathcal{B}_\xi^f(2, 1, 1)$, the last of which vanishes for 3+1, and one infrared diverging integral, $\mathcal{B}_\xi(2)$.

For numerical calculation, setting $n = \sum_i n_i$ and using the well-known Schwinger representation [41], the integrals are rewritten as

$$\mathcal{B}_\xi(q, n_1, n_2, n_3, n_4) = \frac{(-1)^n}{2^{q-n}\Gamma(q)} \int_0^\infty d\lambda \lambda^{q-1} e^{-m^2\lambda/2} \prod_{\mu=1}^4 \left[\frac{d^{n_\mu}}{dx^{n_\mu}} \exp^{-x} I_0(x) \right]_{x=\frac{\lambda}{d_\mu^2}}, \quad (\text{B9})$$

ξ	$2 + 2$	$3 + 1$
1	0.15493339023106021408483720810745	0.15493339023106021408483720810745
2	0.27309284159872576605248340464151	0.39746855267384293273601515359921
3	0.34489265148380372504682107682350	0.63909144650207181551150685543306
4	0.39476479072982726208009053119503	0.87599824804148552213829648851546
5	0.43269826887073920594692758485023	1.10989753717092264667311285543088
6	0.46325286965085016997908881189910	1.34197554748773397230022959566745
7	0.48882308585244885903279712081054	1.57290464634371239724254250061935
8	0.51080767194929084936132635888190	1.80307278894183022754245621178021
9	0.53009088018361126653278475943989	2.03271449270324135251760022607330
10	0.54726616387832592956739420406785	2.26197832054321458404301146488387

TABLE VII: $\mathcal{B}_\xi(1)$ for $2 + 2$ and $3 + 1$ as a function of ξ to 10^{-32} precision.

with

$$\frac{d^n}{dx^n} \exp^{-x} I_0(x) = \frac{e^{-x}}{2^{n-1}} \sum_{k=0}^n (-1)^{n-k} M_3^{n,k} I_k(x), \quad (\text{B10})$$

where $M_3^{n,k}$ are partition multinomial coefficients and $I_k(x)$ are modified Bessel functions of the first kind [47, 48]. Since we want to focus attention on the massless case ($m^2 \rightarrow 0$) and keeping only the non-vanishing terms either divergent or finite, power-counting shows that the integrals, $\mathcal{B}_\xi(q, n_1, n_2, n_3, n_4)$ for $q - n \leq 1$, are infrared finite. As a result their value can be directly calculated from Eq. (B9) by setting $m^2 = 0$. Numerical evaluation to, say, 32 digits becomes trivial with the use of a numerical integration package available with programs like Mathematica or Maple (see Table VII). Furthermore, simple dimensional arguments give $\mathcal{B}_\xi(q, n_f, n_c) = O((\xi \log \xi)^{q-n_f}/\xi)$ for $q > n_f$ and $\mathcal{B}_\xi(q, n_f, n_c) = O(\xi^{q-n_f})$ for $q \leq n_f$ for the $2+2$ case while for $3+1$, $\mathcal{B}_\xi(q, n_f, n_c) = O(\xi^{q-n_f})$. From this, the asymptotic behaviour of the one-loop corrections can be easily obtained, by noting $\lim_{\xi \rightarrow \infty} \mathcal{B}_\xi^c(2, 1)/(\xi^2 \mathcal{B}_\xi(1)) = 1/6$ for $3+1$.

We can now described the computation of the relevant part of $\mathcal{B}_\xi(q, n_1, n_2, n_3, n_4)$, $q - n \geq 2$, which is infrared divergent. Using the asymptotic expansion for large x of $I_\nu(x)$

$$I_\nu(x) \approx \frac{\exp^x}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu + k + 1/2)}{(2x)^k k! \Gamma(\nu - k + 1/2)}, \quad (\text{B11})$$

the leading and sub-leading behaviour is easily determined. This can be reexpressed in terms

of $\Gamma(i, m^2)$, $i = 0 \dots q - n - 2$ whose integral representation can be directly subtracted from the integrand, leaving an $m^2 \rightarrow 0$ converging integral. For example, in the case $n = 0$, the constants b_i are defined as the expansion of $I_0(x)^4$, which are rational numbers multiplied by π^{-2} . The divergent part of $\mathcal{B}_\xi(q)$ is then given by

$$\frac{\xi^d}{\Gamma(q)} \sum_{i=2}^{q-1} \frac{b_{i-2} \Gamma(q-i)}{2^i (m^2)^{q-i}} - \frac{\xi^d b_{q-2}}{2^p \Gamma(q)} \log m^2. \quad (\text{B12})$$

Following the literature, the finite contribution to

$$\mathcal{B}_\xi(2) = \frac{\xi^d}{16\pi^2} (-\log m^2 - \gamma_E + F_0(\xi)) + O(m^2) \quad (\text{B13})$$

is calculated first and the finite part of the other integrals is defined up to it. Using

$$\Gamma(0, m^2) = -\log m^2 - \gamma_E - \sum_{n=1}^{\infty} \frac{(-m^2)^n}{nn!} \quad (\text{B14})$$

and taking the $\lim_{m^2 \rightarrow 0}$, yields

$$F_0(\xi) = \frac{4\pi^2}{\xi^d} \int_0^2 d\lambda \lambda \exp^{-(4-d+\frac{d}{\xi^2})\lambda} I_0^{4-d}(\lambda) I_0^d\left(\frac{\lambda}{\xi^2}\right) + \int_2^\infty d\lambda \left(\frac{4\pi^2}{\xi^d} \lambda \exp^{-(4-d+\frac{d}{\xi^2})\lambda} I_0^{4-d}(\lambda) I_0^d\left(\frac{\lambda}{\xi^2}\right) - \frac{1}{\lambda} \right).$$

For example

$$\mathcal{B}_\xi(3) = \frac{\xi^d}{32\pi^2 m^2} + \frac{\xi^d}{128\pi^2} (\log m^2 + \gamma_E - F_0(\xi)) + f\mathcal{B}_\xi(3), \quad (\text{B15})$$

$$\mathcal{B}_\xi^f(3, 1) = \frac{\xi^d}{64\pi^2} (-\log m^2 - \gamma_E + F_0(\xi)) + f\mathcal{B}_\xi^f(3, 1) \quad (\text{B16})$$

$$\mathcal{B}_\xi^c(3, 1) = \frac{\xi^{d+2}}{64\pi^2} (-\log m^2 - \gamma_E + F_0(\xi)) + f\mathcal{B}_\xi^c(3, 1). \quad (\text{B17})$$

which using Eq. B5 and taking $\lim_{m^2 \rightarrow 0} m^2 \mathcal{B}_\xi(3)$, must satisfy

$$(4-d)f\mathcal{B}_\xi^f(3, 1) + \frac{d}{\xi^2} f\mathcal{B}_\xi^c(3, 1) + \frac{\xi^d}{32\pi^2} = 0. \quad (\text{B18})$$

Numerical values for $\mathcal{B}_\xi(1)$ and F_0 are presented for a range of anisotropies in Tables (VII, VIII).

ξ	$2 + 2$	$3 + 1$
1	4.3692252338747587180021767477	4.3692252338747587180021767478077
2	3.1818862274504847285875097585	3.0013807239614706751354255882375
3	2.2336373613788360098511237787	2.0763302182432342962706829473432
4	1.5519382904759206782491885601	1.4343462876751234663331108882378
5	1.0353940883292892812839278453	0.9504439561174006940006958674003
6	0.6235592513202110800048370509	0.5632462682606760634612041417967
7	0.2823597812415587497126918746	0.2405570583083765791881447111856
8	-0.0084789916018349072700020647	-0.036178339243301342993891362519
9	-0.2617758427992475365501484308	-0.278533198254716148526058151986
10	-0.4860763765593393395653061782	-0.494190516782956896533394268235

TABLE VIII: $F_0(\xi)$ for $2 + 2$ and $3 + 1$ as a function of ξ to 10^{-28} precision.

ACKNOWLEDGMENTS

This work was funded by Enterprise-Ireland grants SC/2001/306 and SC/2001/307. We would like to thank H. Panagopoulos for helpful discussions.

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