# Grid Approximation of Singularly Perturbed Elliptic Convection-Diffusion Equations in Unbounded Domains*

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**Abstract**

In the quarter plane $\{(x, y) : \ x, \, y \geq 0\}$, we consider the Dirichlet problem for a singularly perturbed elliptic convection-diffusion equation. The highest derivatives of the equation and the first derivative along the $y$-axis contain respectively the parameters $\varepsilon_1$ and $\varepsilon_2$, which take arbitrary values from the half-open interval $(0, 1]$ and the segment $[-1, 1]$. For small values of the parameter $\varepsilon_1$, a boundary layer appears in a neighbourhood of the domain boundary. Depending on the ratio between the parameters $\varepsilon_1$ and $\varepsilon_2$, this layer may be regular, parabolic or hyperbolic. Besides the boundary-layer scale controlled by the perturbation parameters, one can observe a resolution scale, which is specified by the "width" of the domain on which the problem is to be solved on a computer. It turns out that, for solutions of the boundary value problem and of a formal difference scheme (i.e., a scheme on meshes with an infinite number of nodes) considered on the bounded subdomains of interest (referred to as the resolution subdomains), the domains of essential dependence, i.e., such domains outside which the finite variation of the solution causes relatively small disturbances of the solution on the resolution subdomains, are bounded uniformly with respect to the vector-parameter $\overline{\varepsilon} = (\varepsilon_1, \, \varepsilon_2)$. Using the conception of "region of essential solution dependence", we design a constructive finite difference scheme (i.e., a scheme on meshes with a finite number of nodes) that converges $\overline{\varepsilon}$-uniformly on the bounded resolution subdomains.

## 1. Introduction

The direct use of discrete methods developed for solving boundary value problems in bounded domains leads to a contradiction in the case of unbounded domains. In the case of elliptic or parabolic equations on unbounded domains, for solutions considered at a point, the domains of their dependence (on the problem data which determine the solution) are unbounded. To solve such problems numerically, discrete sets with an infinite number of mesh points is required. However, the solution can be computed (even on "powerful" computers) only on meshes with a finite number of nodes. Thus, the construction of effective numerical methods for boundary value problems in unbounded domains seems to be a problem of extreme importance at present. In the case of singularly perturbed problems the problem is complicated by boundary and transition layers arising for small values of the perturbation parameters.

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Difficulties appearing in the numerical solution of singularly perturbed equations in bounded domains are well known (see, e.g., [1, 3, 4, 6, 10, 11, 14]).

For a number of problems in unbounded domains whose solutions are considered in some bounded subdomains, it turns out that the domain of "essential" dependence of a solution (i.e., such a domain outside which the finite variation of the solution causes relatively small disturbances of the solution on the bounded subdomains of interest) is bounded (see, e.g., the statement of Theorem 4.2 in Section 4 in the case of the boundary value problem (2.2), (2.1) for a singularly perturbed elliptic equation). A similar effect of the boundedness of the domain of essential solution dependence is also observed in the case of discrete problems on meshes with an infinite number of nodes, which approximate boundary value problems in unbounded domains (see the statements of Theorems 5.1 and 5.2 in the case of the difference scheme (3.2) on meshes (3.1) and (3.4)). Motivated by the boundedness property of domains of essential dependence of solutions to differential and discrete problems, interest arises in the development of constructive numerical methods. Constructive methods are those that make use of meshes with a finite number of nodes, which allows us to approximate solutions of the boundary value problems on prescribed bounded domains. For singularly perturbed problems, there exists a strong interest in constructive numerical methods whose solutions converge on the prescribed subdomains uniformly with respect to the perturbation parameter, moreover, the size of these subdomains is independent of the value of the parameter.

In the case of convection-diffusion equations with two perturbation parameters $\varepsilon_1$ and $\varepsilon_2$ multiplying the diffusion and convection terms, the nature of arising boundary layers depends on the relation between the parameters $\varepsilon_1$, $\varepsilon_2$. For problems of this type, special numerical methods, with errors in solutions being independent of values of the vector-parameter $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, are available only for problems in bounded domains and only for some particular sets of the parameters $\varepsilon_1$ and $\varepsilon_2$: $\varepsilon_1 \in (0, 1]$ and $\varepsilon_2 = 0$, or $|\varepsilon_2| = 1$ (see, e.g., [4, 10, 11, 14]). Thus, an important task is the development of numerical methods for problems in both bounded and unbounded domains that approximate solutions with errors independent of the vector-parameter $\overline{\varepsilon}$ for any relations between $\varepsilon_1$ and $\varepsilon_2$ (i.e., methods convergent $\overline{\varepsilon}$-uniformly).

In the present paper we consider discrete approximations of the Dirichlet problem for a singularly perturbed elliptic equation in an unbounded domain, i.e., in the first quarter plane. The differential equation is of convection-diffusion type. The highest derivatives of the equation and the first derivative along the vertical axis contain respectively the parameters $\varepsilon_1$ and $\varepsilon_2$ taking arbitrary values from the half-open interval $(0, 1]$ and the segment $[-1, 1]$. Depending on the parameter $\varepsilon_2$, the vertical component of the flow velocity may be both positive (for $\varepsilon_2 < 0$) and negative (for $\varepsilon_2 > 0$), or equal to zero (for $\varepsilon_2 = 0$); the horizontal component is strongly negative. For $\varepsilon_1 \to 0$ a boundary layer appears in a neighbourhood of the domain boundary (or of its part); the layer is regular in a neighbourhood of the vertical boundary. In a neighbourhood of the horizontal boundary, the type of the layer is defined by the parameter $\varepsilon_2$ and can be regular, parabolic or hyperbolic; no boundary layer arises for $\varepsilon_2 \le -m < 0$ (see, e.g., Remark 4 to Theorem 7.1 in Section 7).

Let us mention some singularities of the problem being investigated in the paper. Errors in solutions of the difference scheme on a mesh with an infinite number of its nodes (formal difference scheme) essentially depend on the values of $\varepsilon_1$ and $\varepsilon_2$ (see estimates (3.5), (3.7) for the difference scheme (3.2), (3.4) in Section 3). The domains of essential dependence for solutions of the boundary value problem under consideration and of the formal difference scheme are bounded (see the definition of the domain of essential dependence in Sections 4 and 5). This property of the domains of essential dependence makes it possible to develop constructive difference schemes that are convergent on the prescribed subdomains, whose sizes can grow with increasing the number of mesh points (see, for example, estimate (6.15) in Section 6). When there are no restrictions imposed on the distribution of mesh points, the error in the numerical solution (on the prescribed subdomain) depends on the value of the component-parameters $\varepsilon_1$, $\varepsilon_2$ and, in general, grows (up to a full loss of accuracy) for $\varepsilon_1$, $\varepsilon_2 \to 0$ (see, e.g., estimates (6.3), (6.5), (6.6) in Section 6).

When constructing $\overline{\varepsilon}$-uniformly convergent constructive schemes, we use piecewise uniform meshes condensing in the boundary layer region; the rule of mesh refinement is different in a neighbourhood of the vertical and horizontal boundaries and is determined by both parameters $\varepsilon_1$ and $\varepsilon_2$ (see scheme (6.2), (6.12), which converges $\overline{\varepsilon}$-uniformly).

**About the contents**. Problem formulation, the aim of research and *a priori* estimates are given in Sections 2 and 7. Formal difference schemes are investigated in Section 3. Domains of essential dependence for solutions of the boundary value problem and the formal difference scheme are studied in Sections 4 and 5, respectively. In Section 6 we develop constructive difference schemes that converge (in particular, $\overline{\varepsilon}$-uniformly) on prescribed (bounded) domains whose sizes are allowed to grow as the number of mesh points increases.

Note that the construction of $\overline{\varepsilon}$-uniformly convergent schemes for the boundary value problem under study was not previously considered even for the case of a bounded domain. The technique given in the paper can be applied to the development of parameter-uniform numerical methods, capable of actual computation, for other types of singularly perturbed problems in unbounded domains.

# 2. Problem Formulation. The aim of the research

**2.1.** In the quarter plane $\overline{D}$, where

$$\overline{D} = D \cup \mathit{\Gamma}, \quad D = \{x: \ x_s \in (0, \infty), \quad s = 1, 2\}, \tag{2.1}$$

we consider the Dirichlet problem for the singularly perturbed equation[1]

$$L_{(2.2)}\, u(x) = f(x), \quad x \in D, \quad u(x) = \varphi(x), \quad x \in \mathit{\Gamma}. \tag{2.2}$$

Here

$$L \equiv \varepsilon_1 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} + b_1(x) \frac{\partial}{\partial x_1} + \varepsilon_2\, b_2(x) \frac{\partial}{\partial x_2} - c(x),$$

the functions $a_s(x)$, $b_s(x)$, $c(x)$, $f(x)$ are assumed to be sufficiently smooth on $\overline{D}$, $s = 1, 2$, the function $\varphi(x)$ is sufficiently smooth on the sides $\mathit{\Gamma}_j$, $j = 1, 2$ and continuous on $\mathit{\Gamma}$; $\mathit{\Gamma} = \mathit{\Gamma}_1 \cup \mathit{\Gamma}_2$; $\mathit{\Gamma}_s = \overline{\mathit{\Gamma}}_s$, the side $\mathit{\Gamma}_s$ is orthogonal to the axis $x_s$, $s = 1, 2$. We assume that the following conditions are satisfied:[2]

$$\begin{gathered} a_0 \le a_s(x) \le a^0, \quad b_0 \le b_s(x) \le b^0, \quad c_0 \le c(x) \le c^0, \quad a_0,\, b_0,\, c_0 > 0; \\ |f(x)| \le M, \quad x \in \overline{D}; \quad |\varphi(x)| \le M, \quad x \in \mathit{\Gamma}. \end{gathered} \tag{2.3}$$

The parameters $\varepsilon_1$ and $\varepsilon_2$, i.e., the components of the vector-parameter $\overline{\varepsilon}$, take arbitrary values in the half-open interval $(0, 1]$ and the segment $[-1, 1]$ respectively.

[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that this operator (constant, grid) is introduced in formula $(j.k)$.

[2] Here and below $M$, $M_i$ (or $m$) denote sufficiently large (small) positive constants independent of the vector-parameter $\overline{\varepsilon}$ and the parameters of difference schemes.

By the solution of the boundary value problem, we mean its classical solution, i.e., a function $u \in C^2(D) \cap C(\overline{D})$ that is bounded on $\overline{D}$ and satisfies the differential equation on $D$ and the boundary condition on $\Gamma$. For simplicity, we suppose that the compatibility conditions ensuring the required smoothness of the solution for each fixed value of the vector-parameter $\overline{\varepsilon}$ are fulfilled on the set $\Gamma^c = \Gamma_1 \cap \Gamma_2$ of "corner points".

When the parameter $\varepsilon_1$ tends to zero, boundary layers arise in a neighborhood of the boundary $\Gamma$ (or its part). The nature of these layers and their properties in a neighborhood of the sets $\Gamma_1$, $\Gamma_2$, $\Gamma^c$ are determined by the vector-parameter $\overline{\varepsilon}$ (see considerations in Section 7). The presence of parabolic boundary layers in problem (2.2), (2.1) does not allow us to construct $\overline{\varepsilon}$-uniformly (i.e., uniformly with respect to $\varepsilon_1$ and $\varepsilon_2$) convergent numerical methods based on the fitted operator approach, and so we need a technique based on condensing meshes (see the description of the approaches to the construction of the special schemes in bounded domains for $\varepsilon_2 = 0$ or $|\varepsilon_2| \approx 1$ in [1, 3, 6, 10, 14]).

Problems of this type appear in the modelling of a heat transfer process for fluid flow over surfaces in the case of boundary layers controlled by blowing some fluid in or suction of some fluid out of a layer (see, e.g., [13], Chap. 14). In these problems the parameter $\varepsilon_1$ is determined by the relation $\varepsilon_1 = Re^{-1}$ or $\varepsilon_1 = Pe^{-1}$, where $Re$ and $Pe$ are the Reynolds and Peclet numbers, and the parameter $\varepsilon_2$ determines the intensity of "blowing/suction" on the streamlined surface.

**2.2.** Unlike problems in bounded domains (for singularly perturbed or regular equations), the construction of numerical methods in the case of unbounded domains is essentially complicated. The approximation of solutions of such problems on the domain of definition, as rule, requires to use disrete sets with an infinite number of mesh points (see Remark 5 to Theorem 7.1 in Section 7).

Thus, in the case of problems in unbounded domains it seems appropriate to use the following approach for the development of constructive numerical methods. Suppose that we are interested in finding a solution of problem (2.2), (2.1) on some prescribed bounded set $\overline{D}^0 \subset \overline{D}$. Let the set $\overline{D}^0$ be a rectangle defined by its lower left and upper right vertices $d^1 = (d_1^1,\, d_2^1)$ and $d^2 = (d_1^2,\, d_2^2)$, where $d^1$ is an arbitrary point of $\overline{D}$:

$$\overline{D}^0 = \overline{D}^0(d^1,\, d^2), \quad \overline{D}^0 = D^0 \cup \Gamma^0. \tag{2.4}$$

Thus, we have $\overline{D}^0 = [d_1^1,\, d_1^2] \times [d_2^1,\, d_2^2]$, $d^2 = d^1 + d^0$, $d^0 = (d_1^0,\, d_2^0)$; generally speaking, the value $d_s^0$, that is, the size of $D^0$ along the $x_s$-axis, may depend on the parameter $\overline{\varepsilon}$; let $d_0^0 = \max[d_1^0,\, d_2^0]$. It is required to construct a numerical method which allows us to approximate the solution of problem (2.2), (2.1) on the set $\overline{D}^0$. The accuracy of the discrete solution on $\overline{D}^0$ (just as the values $d_s^0$, $s = 1, 2$) can depend on the parameter $\overline{\varepsilon}$ and the values of $N_1$ and $N_2$, which define the number of mesh points used (in $x_1$ and $x_2$). It is desirable that the values $d_s^0$ are allowed to grow as $N_1$ and $N_2$ increase. When constructing an $\overline{\varepsilon}$-uniformly convergent method, we require that the size of the set $\overline{D}^0$ and the accuracy of the discrete solution (on $\overline{D}^0$) be independent of the parameter $\overline{\varepsilon}$ and be defined only by the values of $N_1$ and $N_2$.

Our aim is to construct $\overline{\varepsilon}$-uniformly convergent formal and constructive schemes for the boundary value problem (2.2) by using standard finite difference approximations. In the case of constructive difference schemes we are interested in finding the solution of problem (2.2), (2.1) on the bounded set $\overline{D}^0_{(2.4)}$.

## 3. Formal difference schemes

In the case of problem (2.2), (2.1) we consider formal difference schemes, viz. schemes on meshes with an infinite number of nodes.

**3.1.** On the set (2.1), we introduce the grid

$$\overline{D}_h^* = \overline{\omega}_1^* \times \overline{\omega}_2^*, \tag{3.1}$$

where $\overline{\omega}_s^*$ is a mesh on the semiaxis $x_s \geq 0$ with arbitrarily distributed mesh points. Let $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i$, $x_s^{i+1} \in \overline{\omega}_s^*$, $h_s = \max_i h_s^i$, $h = \max_s h^s$. Denote by $N_{*s} + 1$ the minimum number of nodes in the mesh $\overline{\omega}_s^*$ on a unit interval. Suppose that the condition $h \leq M N_*^{-1}$ is fulfilled, where $N_* = \min[N_{*1}, N_{*2}]$.

To solve problem (2.2), (2.1) we use the difference scheme

$$\Lambda z(x) = f(x), \quad x \in D_h^*, \quad z(x) = \varphi(x), \quad x \in \Gamma_h^*. \tag{3.2}$$

Here $D_h^* = D \cap \overline{D}_h^*$, $\Gamma_h^* = \Gamma \cap \overline{D}_h^*$;

$$\Lambda z(x) \equiv \left\{ \varepsilon_1 \sum_{s=1,2} a_s(x) \delta_{\overline{xs}\,\widehat{xs}} + b_1(x)\delta_{x1} + \varepsilon_2^+ b_2(x)\delta_{x2} + \varepsilon_2^- b_2(x)\delta_{\overline{x2}} - c(x) \right\} z(x), \quad x \in D_h^*;$$

$\delta_{\overline{xs}\,\widehat{xs}} z(x)$ and $\delta_{xs} z(x)$,$\delta_{\overline{xs}} z(x)$ are the second and first (forward and backward) difference derivatives on non-uniform grids, for example, $\delta_{\overline{x1}\,\widehat{x1}}\, z(x) = 2\big(h_1^{i-1} + h_1^i\big)^{-1}\big(\delta_{x1} - \delta_{\overline{x1}}\big)\, z(x)$, $x = (x_1^i, x_2) \in D_h^*$; $\varepsilon_2^+ = 2^{-1}(\varepsilon_2 + |\varepsilon_2|)$, $\varepsilon_2^- = 2^{-1}(\varepsilon_2 - |\varepsilon_2|)$.

Scheme (3.2), (3.1) is $\overline{\varepsilon}$-uniformly monotone [12].

Taking into account the estimates of Theorem 7.1 (see Section 7) and the maximum principle, we find the estimate

$$|u(x) - z(x)| \leq M\, \varepsilon_1^{-2}\, N_*^{-1}, \quad x \in \overline{D}_h^*. \tag{3.3}$$

In the case of a uniform mesh

$$\overline{D}_h^*, \tag{3.4}$$

we obtain the estimate

$$|u(x) - z(x)| \leq M\, (\varepsilon_1 + N_*^{-1})^{-1}\, N_*^{-1}, \quad x \in \overline{D}_h^*. \tag{3.5}$$

**Definition.** Let a discrete function $z(x)$, $x \in \overline{D}_h^*$, i.e., the solution of some difference scheme, satisfy the estimate

$$|u(x) - z(x)| \leq M\, \mu(N_*^{-1};\, \varepsilon_1), \quad x \in \overline{D}_h^*.$$

We say that this estimate is *unimprovable* with respect to $N_*$ and $\varepsilon_1$ if the estimate

$$|u(x) - z(x)| \leq M\, \mu_0(N_*^{-1};\, \varepsilon_1), \quad x \in \overline{D}_h^*,$$

is, in general, false in that case when $\mu_0(N_*^{-1};\varepsilon_1) = o(\mu(N_*^{-1};\varepsilon_1))$ at least for some values of $N_*$ and $\varepsilon_1$ such that $N_* \geq M$, $\varepsilon_1 \in (0, 1]$.

Considering solutions of model problems, we justify that estimate (3.5) is unimprovable with respect to $N_*$ and $\varepsilon_1$. The condition

$$N_*^{-1} = o(\varepsilon_1) \tag{3.6}$$

is necessary and sufficient for the convergence (for $N_* \to \infty$, $\varepsilon_1 \in (0,1]$) of solutions of the difference scheme (3.2), (3.4).

Under the condition $-1 \le \varepsilon_2 \le -\varepsilon_1^{1/2}$, we have the estimate

$$|u(x) - z(x)| \le M \sum_{s=1,2} (\varepsilon_s + N_{*s}^{-1})^{-1} N_{*s}^{-1}, \quad x \in \overline{D}_h^*. \tag{3.7}$$

**Theorem 3.1.** *For the difference scheme* (3.2), (3.4), *condition* (3.6) *is necessary for the convergence of discrete solutions to the solution of the boundary value problem* (2.2), (2.1) *and is also a sufficient condition if a priori estimate* (7.3) *holds, where* $K = 3$. *The discrete solutions satisfy estimates* (3.3), (3.5), (3.7); *estimate* (3.5) *is unimprovable with respect to* $N_*$ *and* $\varepsilon_1$.

**3.2.** To construct $\overline{\varepsilon}$-uniformly convergent schemes, we use meshes condensing in a neighbourhood of the boundary layers. The rule of mesh refinement is controlled by the nature of the arising boundary layers.

On the set $\overline{D}$ we introduce the mesh

$$\overline{D}_h^* = \overline{D}_h^{*S} = \overline{\omega}_1^{*S} \times \overline{\omega}_2^{*S}, \tag{3.8}$$

where $\overline{\omega}_s^{*S} = \overline{\omega}_s^{*S}(\sigma_s)$ is a piecewise uniform mesh on the semiaxis $x_s \ge 0$, $s = 1,2$. The stepsizes of the mesh $\overline{\omega}_s^{*S}$ are constant on the intervals $[0, \sigma_s]$ and $[\sigma_s, \infty)$ and are equal to $h_s^{(1)} = 2\sigma_s N_{*s}^{-1}$ and $h_s^{(2)} = 2(1 - \sigma_s) N_{*s}^{-1}$, respectively. The value $\sigma_1$ is chosen to satisfy the condition

$$\sigma_1 = \sigma_1(\varepsilon_1, N_{*1}) = \min\,[2^{-1},\ M_1\, \varepsilon_1\, \ln N_{*1}], \quad \textit{where} \ \ M_1 = m_{1(7.14)}^{-1},$$

The magnitude of $\sigma_2$ depends on the values of $\varepsilon_1$, $\varepsilon_2$ and $N_{*2}$ so that $\sigma_2 = \sigma_2(\varepsilon_1, \varepsilon_2, N_{*2})$:

$$\sigma_2 = \sigma_2(\varepsilon_1, \varepsilon_2, N_{*2}) = \begin{cases} \min\,[2^{-1},\ M_2\, \varepsilon_1^{1/2}\, \ln N_{*2}] & \text{for} \quad |\varepsilon_2| \le M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ M_3\, \varepsilon_1\, \varepsilon_2^{-1}\, \ln N_{*2}] & \text{for} \quad \varepsilon_2 > M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ M_4\, |\varepsilon_2|\, \ln N_{*2}] & \text{for} \quad \varepsilon_2 < -M^0\, \varepsilon_1^{1/2}. \end{cases}$$

Here $M^0$ is an arbitrary constant, $M_2 = m_{2(7.15)}^{-1}$, $m_{2(7.15)} = m_2(M^0)$, $M_3 = m_{2(7.19)}^{-1}$, $M_4 = m_{2(7.22)}^{-1}$, $m_{2(7.22)} = m_2(M^0)$.

Applying the technique from [10, 14] and taking into account *a-priori* estimates for the solutions of problem (2.2), (2.1), we find the estimate

$$|u(x) - z(x)| \le M\, N_{*1}^{-1} \min\,[\varepsilon_1^{-1}, \ln N_{*1}] + \tag{3.9}$$

$$+ M\, N_{*2}^{-1} \begin{cases} \min\,[\varepsilon_1^{-1/2}, \ln N_{*2}] & \text{for} \quad |\varepsilon_2| \le M^0\, \varepsilon_1^{1/2}, \\ \min\,[\varepsilon_1^{-1} \varepsilon_2, \ln N_{*2}] & \text{for} \quad \varepsilon_2 > M^0\, \varepsilon_1^{1/2}, \\ \min\,[\,|\varepsilon_2|^{-1}, \ln N_{*2}] & \text{for} \quad \varepsilon_2 < -M^0\, \varepsilon_1^{1/2}, \quad x \in \overline{D}_h^{*S}. \end{cases}$$

The following $\overline{\varepsilon}$-uniform estimate also holds:

$$|u(x) - z(x)| \le M\, N_*^{-1} \ln N_*, \quad x \in \overline{D}_h^{*S}. \tag{3.10}$$

Estimates (3.9) and (3.10) are unimprovable with respect to $N_{*1}, N_{*2}, \varepsilon_1, \varepsilon_2$ and $N_*$, respectively.

**Theorem 3.2.** *Let the components of the solution of the boundary value problem* (2.2), (2.1) *from representations* (7.5), (7.10) *and* (7.5), (7.24) *satisfy the a-priori estimates of Theorem* 7.1, *where* $K = 3$. *Then the solution of the difference scheme* (3.2), (3.8) *converges to the solution of the boundary value problem* (2.2), (2.1) $\overline{\varepsilon}$*-uniformly. For the discrete solutions estimates* (3.9) *and* (3.10) *hold, which are unimprovable with respect to* $N_{*1}$, $N_{*2}$, $\varepsilon_1$, $\varepsilon_2$ *and* $N_*$, *respectively.*

# 4. Domain of dependence for solutions of problem (2.2), (2.1)

In this section we consider how the finite variation of the data of problem (2.2), (2.1) on $\overline{D}$, but however far from its subset $\overline{D}^{\,0}$, influences the solution of the problem on this set $\overline{D}^{\,0}$.

**4.1.** Suppose that we are interested in finding the solution of problem (2.2), (2.1) on the set $\overline{D}^{\,0}_{(2.4)}$ in $\overline{D}$. Let the set $\overline{D}^{\,0}_{(2.4)}$ belong to the rectangle $\overline{D}^{[0]} \subset \overline{D}$ defined by the vertices $\widehat{d}^{\,1}$ and $\widehat{d}^{\,2}$, $\widehat{d}^{\,i} = (\widehat{d}^{\,i}_1, \widehat{d}^{\,i}_2)$, $i = 1, 2$:

$$\overline{D}^{[0]} = \overline{D}^{[0]}(\widehat{d}^{\,1}, \widehat{d}^{\,2}) \equiv \overline{D}^{\,0}_{(2.4)}(\widehat{d}^{\,1}, \widehat{d}^{\,2}), \quad \overline{D}^{[0]} = D^{[0]} \cup \Gamma^{[0]}. \tag{4.1a}$$

Here $\widehat{d}^{\,1} = d^1 - \eta^1$, $\widehat{d}^{\,2} = d^2 + \eta^2$, $d^i = d^i_{(2.4)}$, $\eta^i = (\eta^i_1, \eta^i_2)$; the values $(\eta^1_1, \eta^2_1) \equiv \overline{\eta}_{(1)}$ and $(\eta^1_2, \eta^2_2) \equiv \overline{\eta}_{(2)}$ determine a neighbourhood of the set $\overline{D}^{\,0}$ in the $x_1$– and $x_2$–directions. So, the set $\overline{D}^{[0]}$ contains the set $\overline{D}^{\,0}$ with its $\{\overline{\eta}_{(1)}, \overline{\eta}_{(2)}\}$-neighbourhood, $\overline{\eta}_{(i)} = (\eta_{(i)1}, \eta_{(i)2})$, $i = 1, 2$:

$$\overline{D}^{[0]} = \overline{D}^{[0]}(\overline{D}^{\,0};\ \overline{\eta}_{(1)}, \overline{\eta}_{(2)}). \tag{4.1b}$$

Let $u^{[0]}(x)$, $x \in \overline{D}^{[0]}$, be the solution of the problem

$$Lu^{[0]}(x) = f(x), \quad x \in D^{[0]}, \tag{4.2a}$$

$$u^{[0]}(x) = \varphi(x), \quad x \in \Gamma^{[0]} \cap \Gamma, \tag{4.2b}$$

$$u^{[0]}(x) = 0, \qquad x \in \Gamma^{[0]} \setminus \Gamma. \tag{4.2c}$$

Using the majorant function technique, we obtain the following estimate for the solution of problem (4.2), (4.1):

$$|u(x) - u^{[0]}(x)| \le M\ \left[\beta_1(\eta_{(1)1}) + \exp(-m^1_1 \eta_{(1)2}) + \beta_2(\eta_{(2)1})\right] + \tag{4.3}$$
$$+ M \begin{cases} \exp(-m^1_2\, \varepsilon_1^{-1/2}\, \eta_{(2)2}) & \text{for} \quad |\varepsilon_2| \le M_0\, \varepsilon_1^{1/2}, \\ \exp(-m^2_2\, \varepsilon_2^{-1}\, \eta_{(2)2}) & \text{for} \quad \varepsilon_2 > M_0\, \varepsilon_1^{1/2}, \\ \exp(-m^3_2\, \varepsilon_1^{-1}\, |\varepsilon_2|\, \eta_{(2)2}) & \text{for} \quad \varepsilon_2 < -M_0\, \varepsilon_1^{1/2}, \quad x \in \overline{D}^{\,0}. \end{cases}$$

Here

$$\beta_1(\eta_{(1)1}) = \begin{cases} \exp(-m^2_1\, \varepsilon_1^{-1}\, \eta_{(1)1}), & \Gamma^{[0]} \cap \Gamma_1 \ne \emptyset, \\ 0, & \Gamma^{[0]} \cap \Gamma_1 = \emptyset; \end{cases}$$

$$\beta_2(\eta_{(2)1}) = \begin{cases} \exp(-m^1_2\, \varepsilon_1^{-1/2}\, \eta_{(2)1}) & \text{for} \quad |\varepsilon_2| \le M_0\, \varepsilon_1^{1/2}, \\ \exp(-m^3_2\, \varepsilon_1^{-1}\, \varepsilon_2 \eta_{(2)1}) & \text{for} \quad \varepsilon_2 > M_0\, \varepsilon_1^{1/2}, \\ \exp(-m^2_2 |\, \varepsilon_2|^{-1}\, \eta_{(2)1}) & \text{for} \quad \varepsilon_2 < -M_0\, \varepsilon_1^{1/2}, \quad \Gamma^{[0]} \cap \Gamma_2 \ne \emptyset, \\ 0, & \qquad \Gamma^{[0]} \cap \Gamma_2 = \emptyset, \end{cases}$$

$M_0$ is an arbitrary constant, $m^1_1$ is an arbitrary number from the interval $(0, m^{10}_1)$, $m^{10}_1 = \min\{2^{-1/2} \min_{\overline{D}}{}^{1/2}[a_1^{-1}(x)\, c(x)],\ 2^{-1} \min_{\overline{D}}[b_1^{-1}(x)\, c(x)]\}$, $m^2_1 = m_{1(7.14)}$, $m^1_2 = m_{2(7.15)}(M_0)$, $m^3_2 = m_{2(7.19)}$, $m^2_2 = m_{2(7.22)}(M_0)$. Estimate (4.3) (up to constant factors multiplying $\eta_{(i)s}$, $i, s = 1, 2$) is unimprovable with respect to the values of $\overline{\eta}_{(1)}$, $\overline{\eta}_{(2)}$ and $\overline{\varepsilon}$.

**Theorem 4.1.** *For $\eta_{(i)s} \to \infty$, $i, s = 1, 2$, such that $\overline{D}^{[0]}(\overline{D}^0; \overline{\eta}_{(1)}, \overline{\eta}_{(2)}) \subset \overline{D}$, the solution of the boundary value problem* (4.2), (4.1) *converges, on $\overline{D}^0$, to the solution of the boundary value problem* (2.2), (2.1) *$\overline{\varepsilon}$-uniformly. On the set $\overline{D}^0$, estimate* (4.3) *holds for the solution of problem* (4.2), (4.1).

**Remark 1.** If the function $u^{[0]}(x)$, $x \in \overline{D}^{[0]}$, satisfies the condition $u^{[0]}(x) = u_{(2.2;2.1)}(x)$, $x \in \Gamma^{[0]} \setminus \Gamma$, instead of condition (4.2b), then $u^{[0]}(x) = u_{(2.2;2.1)}(x)$, $x \in \overline{D}^{[0]}$; here $u_{(2.2;2.1)}(x)$, $x \in \overline{D}$, is the solution of problem (2.2), (2.1). Thus, the solution of problem (4.2), (4.1) is the solution of the perturbed problem generated by a perturbation of "the data" of problem (2.2), (2.1), namely by a finite change in its solution (the function $u_{(2.2;2.1)}(x)$) outside the set

$$\widehat{D}^{[0]} = D^{[0]} \cup \left\{ \overline{D}^{[0]} \cap \Gamma \right\}. \tag{4.4}$$

It follows from estimate (4.3) that the perturbations of the solution caused by the above perturbation of the data are small on the set $\overline{D}^0$ when the distance $R^0$ between the sets $\overline{D}^0$ and $\overline{D} \setminus \widehat{D}^{[0]}_{(4.4)}$ is sufficiently large. The perturbation of the solution (the function $u^{[0]}(x) - u(x)$) decreases exponentially on the set $\overline{D}^0$ as $R^0$ increases. Thus, although the domain of dependence for the solution of problem (2.2), (2.1) on the set $\overline{D}^0$ is the whole set $\overline{D}$, it turns out however that the domain of "essential" dependence (when the perturbations of the solution are "essentially" different from zero) for the solution on the set $\overline{D}^0$ is bounded.

**4.2.** Let us estimate a neighbourhood of the domain of "essential" dependence for the solution of problem (2.2), (2.1) considered on the set $\overline{D}^0$.

First, we give a definition for the domain of essential dependence (or, shortly, the domain of dependence) of the solution of problem (2.2), (2.1) on the set $\overline{D}^0$.

**Definition.** Let $\overline{D}^\wedge$ be a subset of $\overline{D}$ which contains the set $\overline{D}^0$. We denote by $u^\wedge(x)$, $x \in \overline{D}^\wedge$ the solution of the perturbed problem

$$\begin{aligned} Lu^\wedge(x) &= f(x), \quad x \in D^\wedge, \\ u^\wedge(x) &= \varphi(x), \quad x \in \Gamma^\wedge \cap \Gamma, \quad u^\wedge(x) = 0, \quad x \in \Gamma^\wedge \setminus \Gamma. \end{aligned} \tag{4.5}$$

When the data of problem (2.2), (2.1) are disturbed, the solution of the perturbed problem is supposed to be zero on a part of the boundary $\Gamma^\wedge \setminus \Gamma$. Given a set $\overline{D}^0$ and a sufficiently small number $\beta > 0$, if there exists a set $\overline{D}^\wedge$ such that the function $u^\wedge(x)$, $x \in \overline{D}^\wedge$, considered on the set $\overline{D}^0$ satisfies the estimate

$$|u(x) - u^\wedge(x)| \le M\,\beta, \quad x \in \overline{D}^0,$$

we say that the set $\overline{D}^\wedge$ is the *domain of dependence* of the solution of problem (2.2), (2.1) on the set $\overline{D}^0$ with the perturbation threshold $\beta$ (or, in short, $\overline{D}^\wedge$ is the *domain of dependence for the set $\overline{D}^0$ with threshold $\beta$*); thus,

$$\overline{D}^\wedge = \overline{D}^\wedge(\overline{D}^0, \beta). \tag{4.6}$$

By $\eta^*_{(i)}$, $i = 1, 2$, we denote the vector-parameters $\overline{\eta}_{(i)}$ (see (4.1)) such that the set $\overline{D}^{[0]}_{(4.1)}$ is the domain of dependence $\overline{D}^\wedge(\overline{D}^0, \beta)$:

$$\overline{D}^{[0]}_{(4.1)}(\overline{\eta}_{(i)} = \eta^*_{(i)},\ i = 1, 2) = \overline{D}^\wedge(\overline{D}^0,\ \beta) \equiv \overline{D}^{[0]\wedge}_{(4.7)}. \tag{4.7}$$

By virtue of estimate (4.3), we can estimate the values $\eta^*_{(i)}$, $i = 1, 2$ (i.e., we can choose the parameters $\eta^*_{(1)}$, $\eta^*_{(2)}$ so that the following estimates hold):

$$\eta^*_{(1)1} \le \min\,[M_1^1\,\varepsilon_1\,\ln\beta^{-1},\, d_1^1] \le M_1\,\varepsilon_1\,\ln\beta^{-1}, \tag{4.8}$$

$$\eta^*_{(1)2} \le M_1^2\,\ln\beta^{-1},$$

$$\eta^*_{(2)1} \le \begin{cases} \min[M_2^1\varepsilon_1^{1/2}\ln\beta^{-1}, d_2^1] \le M_2^1\varepsilon_1^{1/2}\ln\beta^{-1} & \text{for } \;|\varepsilon_2| \le M_0\varepsilon_1^{1/2}, \\ \min[M_2^2\varepsilon_1\varepsilon_2^{-1}\ln\beta^{-1}, d_2^1] \le M_2^2\varepsilon_2\ln\beta^{-1} & \text{for } \;\varepsilon_2 > M_0\varepsilon_1^{1/2}, \\ \min[M_2^3|\varepsilon_2|\ln\beta^{-1}, d_2^1] \le M_2^3\varepsilon_1|\varepsilon_2|^{-1}\ln\beta^{-1} & \text{for } \;\varepsilon_2 < -M_0\varepsilon_1^{1/2}; \end{cases}$$

$$\eta^*_{(2)2} \le \begin{cases} M_2^1\varepsilon_1^{1/2}\ln\beta^{-1} & \text{for } \;|\varepsilon_2| \le M_0\varepsilon_1^{1/2}, \\ M_2^3\varepsilon_2\ln\beta^{-1} & \text{for } \;\varepsilon_2 > M_0\varepsilon_1^{1/2}, \\ M_2^2\varepsilon_1|\varepsilon_2|^{-1}\ln\beta^{-1} & \text{for } \;\varepsilon_2 < -M_0\varepsilon_1^{1/2}, \end{cases}$$

where $M_1^1 = (m^1_{1(4.3)})^{-1}$, $M_1^2 = (m^2_{1(4.3)})^{-1}$, $M_2^1 = (m^1_{2(4.3)})^{-1}$, $M_2^2 = (m^3_{2(4.3)})^{-1}$, $M_2^3 = (m^2_{2(4.3)})^{-1}$. The components $\eta^*_{(i)s}$, $i, s = 1, 2$, are highly weakly depending on $\beta$ and growing for $\beta \to 0$. From the unimprovability of estimate (4.3) it follows that estimates (4.8) for the component-parameters $\eta^*_{(i)s}$ defining the "sizes" of the set $\overline{D}^{[0]\wedge}_{(4.7)} \setminus D^0$, i.e., the neighbourhood of the set $\overline{D}^0$ from the domain of dependence, are unimprovable with respect to $\overline{\varepsilon}$ and $\ln\beta^{-1}$.

**Theorem 4.2.** *For the boundary value problem* (2.2), (2.1), *the parameters* $\eta^*_{(1)}$ *and* $\eta^*_{(2)}$, *which define the sizes of the neighbourhood of the set* $\overline{D}^0$ *from the domain of dependence* $\overline{D}^{[0]\wedge}_{(4.7)}$ *with the perturbation threshold* $\beta$, *are* $\overline{\varepsilon}$*-uniformly bounded for fixed values of* $\beta$ *and grow without bound for* $\beta \to 0$. *The parameters* $\eta^*_{(i)s}$ *satisfy estimate* (4.8), *which is unimprovable with respect to the values of* $\overline{\varepsilon}$ *and* $\ln\beta^{-1}$.

**Remark 2.** The estimate of the parameters $\eta^*_{(i)s}$ essentially depends on the vector-parameter $\overline{\varepsilon}$. It is convenient to consider domains of dependence such that the "sizes" of the set $\overline{D}^{[0]\wedge}_{(4.7)} \setminus D^0$ are controlled only by a single parameter. We denote by $\eta^*$ the parameter $\eta$ such that the set $\overline{D}^{[0]}_{(4.1)}$ under the condition $\eta_{(i)s(4.1\mathrm{b})} = \eta$, $i, s = 1, 2$, is the domain of dependence $\overline{D}^\wedge(\overline{D}^0, \beta)$:

$$\overline{D}^{[0]}_{(4.1)}(\eta_{(i)s} = \eta^*,\ i, s = 1, 2) = \overline{D}^\wedge(\overline{D}^0,\, \beta) = \overline{D}^\wedge_{(4.9)}(\eta^*). \tag{4.9}$$

For the parameter $\eta^*_{(4.9)}$ we obtain the $\overline{\varepsilon}$-uniform estimate

$$\eta^* \le M\,\ln\beta^{-1}, \tag{4.10}$$

where $M = \max\limits_{i,j}\,[M_1^i,\, M_2^j]$, $M_1^i = M^i_{1(4.8)}$, $M_2^j = M^j_{2(4.8)}$; the estimate is unimprovable with respect to $\ln\beta^{-1}$.

# 5. The domain of dependence for solutions of the formal difference scheme

Just as in the case of the differential problem (2.2), (2.1), it is of interest to consider domains of solution dependence in the case of the formal difference schemes introduced in Section 3.

**5.1.** Consider the discrete problem (3.2), (3.1). First, we introduce a definition for the domain of (essential) dependence.

**Definition.** Suppose that we are interested in the solution of the discrete problem on the set $\overline{D}^{\,0}_{(2.4)}$ under a specific perturbation of the data in problem (3.2), (3.1). Let

$$\overline{D}{}^{\wedge}_h = \overline{D}{}^{\wedge} \cap \overline{D}{}^{*}_h \tag{5.1a}$$

be a set containing $\overline{D}{}^{0}_h = \overline{D}{}^{0} \cap \overline{D}{}^{*}_h$; $\overline{D}{}^{\wedge} \neq \overline{D}$. We denote by $z^{\wedge}(x)$, $x \in \overline{D}{}^{\wedge}_h$ the solution of the perturbed problem

$$\begin{aligned} \Lambda z^{\wedge}(x) &= f(x), \quad x \in D^{\wedge}_h, \\ z^{\wedge}(x) &= \varphi(x), \quad x \in \Gamma^{\wedge}_h \cap \Gamma, \quad z^{\wedge}(x) = 0, \quad x \in \Gamma^{\wedge}_h \setminus \Gamma; \end{aligned} \tag{5.1b}$$

assume that the boundary $\Gamma^{\wedge}$ passes through the nodes of the mesh $\overline{D}{}^{*}_h$. Under the condition $\overline{D}{}^{\wedge} = \overline{D}$ the discrete problem is not perturbed. Given a set $\overline{D}{}^{0}$ and a value $\beta$ ($\beta > 0$ is sufficiently small), if there exists a set $\overline{D}{}^{\wedge}_h$ such that the function $z^{\wedge}(x)$, $x \in \overline{D}{}^{\wedge}_h$, considered on the set $\overline{D}{}^{0}$ satisfies the estimate $|z(x) - z^{\wedge}(x)| \le M\,\beta,\ x \in \overline{D}{}^{0}_h,$ we say that the set $\overline{D}{}^{\wedge}_h$ is the *domain of (essential) dependence* of the solution to problem (3.2), (3.1) on the set $\overline{D}{}^{0}_h$ with the perturbation threshold $\beta$, (or, in short, $\overline{D}{}^{\wedge}_h$ is the domain of dependence for the set $\overline{D}{}^{0}$ with threshold $\beta$); thus,

$$\overline{D}{}^{\wedge}_h = \overline{D}{}^{\wedge}_h\,(\overline{D}{}^{0},\, \beta). \tag{5.2}$$

For the discrete problems the domain of dependence for the set $\overline{D}{}^{0}$ is generally no smaller than that for the differential problem.

We denote by $\eta^{*h}_{(i)}$, $i = 1, 2$ the vector-parameters $\overline{\eta}_{(i)}$ such that the set

$$\overline{D}{}^{[0]}_h = \overline{D}{}^{[0]}_{(4.1)} \cap \overline{D}{}^{*}_h = \overline{D}{}^{[0]}_h\left(\overline{\eta}_{(1)},\, \overline{\eta}_{(2)}\right) \tag{5.3}$$

is the domain of dependence $\overline{D}{}^{\wedge}_h(\overline{D}{}^{0},\, \beta)$ of the discrete problem

$$\overline{D}{}^{[0]}_{h(5.3)}(\overline{\eta}_{(i)} = \eta^{*h}_{(i)},\ i = 1, 2) = \overline{D}{}^{\wedge}_h(\overline{D}{}^{0},\, \beta) = \overline{D}{}^{[0]\wedge}_{h(5.4)}(\eta^{*h}_{(i)},\ i = 1, 2). \tag{5.4}$$

By $\eta^{*h}$, we denote the parameter $\eta$ such that the set

$$\overline{D}{}^{[0]}_{(4.1)}\left(\overline{\eta}_{(i)s} = \eta,\ i, s = 1, 2\right) \cap \overline{D}{}^{*}_h = \overline{D}{}^{[0]}_h\,(\eta) \tag{5.5}$$

is the domain of dependence of the discrete problem

$$\overline{D}{}^{[0]}_{h(5.5)}\left(\eta = \eta^{*h}\right) = \overline{D}{}^{\wedge}_h\left(\overline{D}{}^{0},\, \beta\right) = \overline{D}{}^{[0]\wedge}_{h(5.6)}\left(\eta^{*h}\right). \tag{5.6}$$

The domain of dependence $\overline{D}{}^{[0]\wedge}_{(4.7)}$ for the differential problem (2.2), (2.1) is the domain of dependence for the discrete problem (3.2) in the case of mesh (3.4) provided that

$$\varepsilon_1^{-1}\, N_*^{-1} = \mathcal{O}(\beta), \tag{5.7}$$

and in the case of mesh (3.8) under the condition

$$N_*^{-1}\, \ln N_* = \mathcal{O}(\beta). \tag{5.8}$$

Thus, the parameters $\eta^{*h}_{(i)}$, $i = 1, 2$, in the case of mesh (3.4) (mesh (3.8)) under condition (5.7) (condition (5.8)) satisfy the (unimprovable) estimate (4.8), where $\eta^{*}_{(i)s}$ is $\eta^{*h}_{(i)s}$.

**Theorem 5.1.** *In the case of the difference scheme* (3.2) *on mesh* (3.4) *(mesh* (3.8)*) under condition* (5.7) *(condition* (5.8)*), the parameters* $\eta^{*h}_{(1)}$ *and* $\eta^{*h}_{(2)}$ *defining the domain of dependence* $\overline{D}^{[0]\wedge}_{h(5.4)}$ *of the discrete solution are* $\overline{\varepsilon}$*-uniformly bounded for fixed values of* $\beta$ *and grow without bound for* $\beta \to 0$*. The parameters* $\eta^{*h}_{(i)s}$ *satisfy estimate* (4.8)*, where* $\eta^{*}_{(i)s}$ *is* $\eta^{*h}_{(i)s}$*, which is unimprovable with respect to the values of* $\overline{\varepsilon}$ *and* $\ln \beta^{-1}$.

**5.2.** Note that the domain of dependence $\overline{D}^{\wedge}_{(4.9)}$ for the differential problem is the domain of dependence for the discrete problem (3.2) on meshes (3.1), (3.4) and (3.8) for any $\overline{\varepsilon}$ and $N_*$:

$$\overline{D}^{\wedge}_{h} = \overline{D}^{\wedge}_{(4.9)} \cap \overline{D}^{*}_{h} = \overline{D}^{\wedge}_{h}\left(\overline{D}^{0},\, \beta\right) = \overline{D}^{[0]\wedge}_{h(5.6)}\,(\eta^{*h} = \eta^{*}_{(4.9)}). \tag{5.9}$$

For the parameter $\eta^{*h}_{(5.6)}$, in the case of meshes (3.1), (3.4) and (3.8) we have the (unimprovable) estimate

$$\eta^{*h} \le M\, \ln \beta^{-1}, \quad M = M_{(4.10)}. \tag{5.10}$$

We denote by $N^{\wedge}_{*s} + 1$ the number of nodes in the mesh along the $x_s$-axis on the set $\overline{D}^{[0]\wedge}_{h(5.6)}$, $s = 1, 2$.

In the case of meshes (3.4) and (3.8), the following estimate holds for the values $N^{\wedge}_{*1}$ and $N^{\wedge}_{*2}$:

$$N^{\wedge}_{*s} \le M\, \left(d^{0}_{s} + \ln \beta^{-1}\right) N_{*s}, \quad s = 1, 2, \tag{5.11}$$

where $d^{0} = d^{0}_{(2.4)}$. But if the threshold $\beta$ satisfies the condition

$$m N_*^{-1} \le \beta \le M\, N_*^{-1}, \tag{5.12a}$$

then the following estimate is valid:

$$N^{\wedge}_{*s} \le M\, \left(d^{0}_{s} + \ln N_{*s}\right) N_{*s}, \quad s = 1, 2. \tag{5.12b}$$

Thus, in the case of the finite set $\overline{D}^{0}$ the number of mesh points along the $x_s$-axis on the set $\overline{D}^{[0]\wedge}_{h(5.6)}$ grows linearly, up to an logarithmic factor, as $N_{*s}$ increases.

**Theorem 5.2.** *In the case of the difference scheme* (3.2) *on meshes* (3.1), (3.4) *and* (3.8)*, the domains of dependence* $\overline{D}^{\wedge}_{(4.9)}$ *and* $\overline{D}^{[0]\wedge}_{h(5.6)}$ *of the solutions of the boundary value problem* (2.2), (2.1) *and of the difference scheme* (3.2) *obey the relation* (5.9)*; the parameter* $\eta^{*h}_{(5.6)}$ *satisfies the (unimprovable) estimate* (5.10)*. For the values* $N^{\wedge}_{*s}$*,* $s = 1, 2$*, which define the number of mesh points on the set* $\overline{D}^{[0]\wedge}_{(5.6)}$*, estimates* (5.11) *and* (5.12) *hold in the case of meshes* (3.4), (3.8).

# 6. Constructive difference schemes

In view of the fact that the sizes of the domain of dependence for solutions of the formal discrete problem (3.2), (3.8) depends weakly on the value $\beta$ (in the case of the finite set $\overline{D}^{0}$), the number of mesh points on the domain of dependence grows as $\beta \to 0$ at the rate $\mathcal{O}(N_{*1}\, N_{*2}\, \ln^2 \beta^{-1})$ (see estimate (5.11)). This nature of the dependence on $\beta$ allows us to construct sufficiently effective schemes.

**6.1.** Suppose that it is required to develop a constructive scheme which approximates the solution of problem (2.2), (2.1) on the set $\overline{D}^0_{(2.4)}$. For the set $\overline{D}^0$ we construct the domain of dependence

$$\overline{D}^\wedge = \overline{D}^\wedge_{(4.9)}(\eta^*), \tag{6.1a}$$

where $\eta^* = M_{(4.10)} \ln \beta^{-1}$, and $\beta > 0$ is a sufficiently small number chosen below. On the set $\overline{D}^\wedge_{(6.1)}$, we introduce the mesh with an arbitrary distribution of its nodes:

$$\overline{D}_h = \overline{D}^\wedge_h = \overline{\omega}_1 \times \; \overline{\omega}_2, \tag{6.1b}$$

$N_s + 1$ is the number of nodes in the mesh $\overline{\omega}_s$, $s = 1, 2$; let $N = \min[N_1, N_2]$. The condition $h_s \le M(d^0_s + \eta^*)N_s^{-1}$ is assumed to be satisfied, where $h_s$ is the maximal stepsize of the mesh $\overline{\omega}_s$. On the mesh $\overline{D}_h$, we build the scheme

$$\begin{aligned} \Lambda z(x) &= f(x), \quad x \in D_h, \\ z(x) &= \varphi(x), \quad x \in \mathit{\Gamma}_h \cap \mathit{\Gamma}, \quad z(x) = 0, \quad x \in \mathit{\Gamma}_h \setminus \mathit{\Gamma}. \end{aligned} \tag{6.2}$$

For the difference scheme (6.2), (6.1) we obtain the estimate

$$|u(x) - z(x)| \le M\, [\varepsilon_1^{-2}\, (d^0_0 + \ln \beta^{-1})\, N^{-1} + \beta], \quad x \in \overline{D}^0_h,$$

where $\overline{D}^0_h = \overline{D}^0 \cap \overline{D}_h$; $d^0_0 = d^0_{0(2.4)}$.

In order to complete the determination of the constructive difference scheme we should choose the parameter $\beta$. It is convenient to choose the parameter $\beta$ satisfying the condition

$$\beta = N^{-1}; \tag{6.1c}$$

in this case $\eta^* = M_{(4.10)} \ln N$. The constructive difference scheme (6.2), (6.1), i.e., the scheme with the finite number of mesh points, has been thus constructed.

**6.2.** For the solutions of the difference scheme (6.2), (6.1) we obtain the estimate

$$|u(x) - z(x)| \le M\, \varepsilon_1^{-2}\, (d^0_0 + \ln N)\, N^{-1}, \quad x \in \overline{D}^0_h. \tag{6.3}$$

On the mesh

$$\overline{D}_h \tag{6.4}$$

which is uniform with respect to both variables, $\overline{\omega}_s = \overline{\omega}^u_s$, $s = 1, 2$, we come to the estimate

$$|u(x) - z(x)| \le M(\varepsilon_1 + (d^0_0 + \ln N)N^{-1})^{-1}(d^0_0 + \ln N)N^{-1}, \quad x \in \overline{D}^0_h; \tag{6.5}$$

this estimate is unimprovable with respect to $N$, $d^0_0$ and $\varepsilon_1$.

Under the condition $-1 \le \varepsilon_2 \le -\varepsilon_1^{1/2}$ we obtain the estimate

$$|u(x) - z(x)| \le M \sum_{s=1,2} (\varepsilon_s + (d^0_0 + \ln N)N_s^{-1})^{-1}(d^0_0 + \ln N)N_s^{-1}, \; x \in \overline{D}^0_h. \tag{6.6}$$

In the case of meshes (6.1) and (6.4), when the following condition holds:

$$\overline{D}^\wedge_{(6.1)} \cap \mathit{\Gamma} \ne \emptyset, \tag{6.7}$$

we have the estimate

$$|u(x) - z(x)| \le M \sum_{s=1,2} N_s^{-1}(d_s^0 + \ln N_s), \quad x \in \overline{D}_h^{\,0}, \tag{6.8}$$

i.e., scheme (6.2) on meshes (6.1) and (6.4) converges on $\overline{D}_h^{\,0}$ $\overline{\varepsilon}$-uniformly.

We now present a mesh condensing in the boundary layer for which scheme (6.2) converges on the set $\overline{D}^{\,0}$ $\overline{\varepsilon}$-uniformly if condition (6.7) is violated.

Let the following condition be valid:

$$\overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_1 = \emptyset, \quad \overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_2 = \emptyset. \tag{6.9a}$$

In this case we apply the mesh

$$\overline{D}_h = \overline{\omega}_1^S \times \overline{\omega}_2^S. \tag{6.9b}$$

Here $\overline{\omega}_s^S = \overline{\omega}_s^S(\sigma_s)$, $s = 1, 2$ is a piecewise uniform mesh on the segment $[0, d_s]$, where $d_s = d_s(N) = d_s^2 + \eta^* = d_s^2 + M \ln N$, $d^2 = d^2_{(2.4)}$, $\eta^* = \eta^*_{(4.9)}$, $\sigma_s$ is a parameter depending on $\overline{\varepsilon}$ and $N_s$. The stepsizes of the mesh $\overline{\omega}_s^S$ are constant on the segments $[0, \sigma_s]$ and $[\sigma_s, d_s]$ and equal to $h_s^{(1)} = 2\sigma_s N_s^{-1}$ and $h_s^{(2)} = 2(d_s - \sigma_s) N_s^{-1}$, respectively. The value $\sigma_1$ is chosen to satisfy the condition

$$\sigma_1 = \sigma_1(\varepsilon_1, N_1) = \min\,[2^{-1},\ M_1\, \varepsilon_1\, \ln N_1],$$

where $M_1 = m^{-1}_{1(7.14)}$. The value $\sigma_2$ is defined by the relation

$$\sigma_2 = \sigma_2(\varepsilon_1,\, \varepsilon_2,\, N_2) = \begin{cases} \min\,[2^{-1},\ M_2 \quad \varepsilon_1^{1/2} \quad \ln N_2] & \text{for} \quad |\varepsilon_2| \le M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ M_3 \quad \varepsilon_1 \varepsilon_2^{-1} \quad \ln N_2] & \text{for} \quad \varepsilon_2 > M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ M_4 \quad |\varepsilon_2| \quad \ln N_2] & \text{for} \quad \varepsilon_2 < -M^0\, \varepsilon_1^{1/2}, \end{cases}$$

where $M^0$ and $M_i$, $i = 2, 3, 4$, are constants from (3.8).

In the case of the condition

$$\overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_1 = \emptyset, \ \ \overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_2 \ne \emptyset \tag{6.9c}$$

we use the mesh

$$\overline{D}_h = \overline{\omega}_1^S \times \overline{\omega}_2^u, \tag{6.9d}$$

where $\overline{\omega}_1^S = \overline{\omega}^S_{1(6.9\mathrm{b})}$, $\overline{\omega}_s^u$ is a uniform mesh on $[0, d_s]$. But if the following condition is satisfied:

$$\overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_1 \ne \emptyset, \ \ \overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma}_2 = \emptyset, \tag{6.9e}$$

then we apply the mesh

$$\overline{D}_h = \overline{\omega}_1^u \times \overline{\omega}_2^S, \tag{6.9f}$$

where $\overline{\omega}_2^S = \overline{\omega}^S_{2(6.9\mathrm{b})}$. Thus, the mesh $\overline{D}_{h(6.9)}$, which we use in the case of the condition

$$\overline{D}^{\wedge}_{(6.1)} \cap \mathit{\Gamma} = \emptyset, \tag{6.10}$$

has been constructed.

In the case of condition (6.10), the solution of the difference scheme (6.2) on mesh (6.9) converges on the set $\overline{D}_h^{\,0}$ $\overline{\varepsilon}$-uniformly with the following error estimate:

$$|u(x) - z(x)| \le M \sum_{s=1,2} N_s^{-1}(d_s^0 + \ln N_s), \quad x \in \overline{D}_h^{\,0}. \tag{6.11}$$

The estimate (6.11) is unimprovable with respect to the values of $N_1$, $N_2$, $d_1^0$ and $d_2^0$.

**Theorem 6.1.** *Let the condition of Theorem* 3.1 *be fulfilled. Then the solution of the difference scheme* (6.2) *on meshes* (6.1) *and* (6.4) *(on mesh* (6.9) *in the case of condition* (6.10)*) converges on the set* $\overline{D}_h^{\,0}$ *to the solution of the boundary value problem* (2.2), (2.1) *for fixed values of the parameter* $\overline{\varepsilon}$ *(*$\overline{\varepsilon}$*-uniformly). In the case of meshes* (6.1), (6.4), *the discrete solutions on the set* $\overline{D}_h^{\,0}$ *satisfy estimates* (6.3), (6.5) *and, under condition* (6.7)*, estimate* (6.8)*; in the case of mesh* (6.9)*, under condition* (6.10) *estimate* (6.11) *holds.*

**6.3.** Finally, we give a difference scheme which converges on $\overline{D}^{\,0}$ $\overline{\varepsilon}$-uniformly irrespective of the disposition of the set $\overline{D}^{\,0}$ on $\overline{D}$.

Given a set $\overline{D}^{\,0}$, we construct the set $\overset{\wedge}{\overline{D}}_{(6.1)}$. On this set, in the case of condition (6.7) we construct the mesh

$$\overline{D}_h = \overline{D}_{h(6.4)}, \tag{6.12a}$$

and in the case of condition (6.10) we construct the mesh

$$\overline{D}_h = \overline{D}_{h(6.9)}. \tag{6.12b}$$

The difference scheme (6.2), (6.12), for fixed values of $d_s^0$(2.4), converges on $\overline{D}^{\,0}$ $\overline{\varepsilon}$-uniformly:

$$|u(x) - z(x)| \leq M \sum_{s=1,2} N_s^{-1}(d_s^0 + \ln N_s), \quad x \in \overline{D}_h^{\,0}. \tag{6.13}$$

In the case of the condition

$$d_1^0,\, d_2^0 = \mathcal{O}(\ln N) \tag{6.14a}$$

we have the estimate

$$|u(x) - z(x)| \leq M\, N^{-1} \ln N, \quad x \in \overline{D}_h^{\,0}. \tag{6.14b}$$

According to (6.13), scheme (6.2), (6.12) converges $\overline{\varepsilon}$-uniformly under the condition

$$d_1^0,\, d_2^0 = o(N). \tag{6.15}$$

**Theorem 6.2.** *Let the condition of Theorem* 3.2 *be satisfied. Then the solution of the difference scheme* (6.2), (6.12) *converges on the set* $\overline{D}^{\,0}$ *to the solution of the boundary value problem* (2.2), (2.1) $\overline{\varepsilon}$*-uniformly; the scheme converges under condition* (6.15)*. The discrete solutions satisfy estimates* (6.13) *and* (6.14)*; estimate* (6.13) *is unimprovable with respect to the values of* $N_1$, $N_2$, $d_1^0$ *and* $d_2^0$.

# Appendices

# 7. A-priori estimates

**7.1.** In this section we give estimates of the solution and its derivatives used in the constructions; the technique from [5, 7, 8, 14, 15] is used to derive the estimates. Using the comparison theorems, we find that

$$|u(x)| \leq M, \quad x \in \overline{D}. \tag{7.1}$$

In the case of the condition

$$u \in C^{l+\alpha}(\overline{D}), \quad l \geq K, \quad \alpha \in (0,1) \tag{7.2}$$

the derivatives of solutions satisfy the estimate

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} \, u(x) \right| \leq M \, \varepsilon^{-k}, \quad x \in \overline{D}, \quad k \leq K, \tag{7.3}$$

where the value $K$ can be chosen sufficiently large depending on the smoothness of the problem data and suitable compatibility conditions on the set $\varGamma^c$ (see [9, 17]).

Let us present estimates obtained from asymptotic representations.

**7.2.** When analyzing solutions of the boundary value problem (2.2), (2.1), it is convenient to decompose the range of change of the vector-parameter $\overline{\varepsilon}$ into subdomains. Let the following condition be satisfied for the components $\varepsilon_1$ and $\varepsilon_2$:

$$|\varepsilon_2| \leq M_1 \, \varepsilon_1^{1/2}, \quad \varepsilon_1 = o(1), \tag{7.4}$$

where $M_1$ is an arbitrary number.

**7.2.1.** We represent the solution of the problem as a sum of functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{7.5}$$

where $U(x)$ and $V(x)$ are the regular and singular parts of the solution. The function $U(x)$, $x \in \overline{D}$, is the restriction to $\overline{D}$ of the function $U^0(x)$, $x \in D^0$, where $D^0$ is the whole plane. The function $U^0(x)$ is the bounded solution of the problem (a problem without boundary conditions)

$$L^0 U^0(x) = f^0(x), \quad x \in D^0. \tag{7.6}$$

Here $L^0$ and $f^0(x)$ are smooth extensions of the operator $L_{(2.2)}$ and the function $f(x)$ that preserve properties (2.3). For simplicity, we consider that the function $f^0(x)$ vanishes outside some neighbourhood of the set $\overline{D}$. The function $V(x)$ is the solution of the problem

$$\begin{aligned} L_{(2.2)} \, V(x) &= 0, & x \in D, \\ V(x) &= \varphi(x) - U(x) \equiv \varphi_V(x), & x \in \varGamma. \end{aligned}$$

We represent the function $U(x)$ as a sum of functions

$$U(x) = \sum_{k=0}^{n} \varepsilon_1^k U_k(x) + v_U^n(x) \equiv U^n(x) + v_U^n(x), \quad x \in \overline{D},$$

where $n \geq 0$, correspondingly to the representation of the function $U^0(x)$

$$U^0(x) = \sum_{k=0}^{n} \varepsilon_1^k U_k^0(x) + v_U^{0n}(x), \quad x \in D^0.$$

The functions $U_k^0(x)$, i.e., components in the expansion of the regular part of the solution to the problem, are the solutions of the following problems on $\overline{D}^{(0)} = \overline{D}^{(0)}(l^{(0)})$ for $l^{(0)} \to \infty$:

$$L_1 U_0^0(x) \equiv \left\{ b_1^0(x) \frac{\partial}{\partial x_1} + \varepsilon_2 b_2^0(x) \frac{\partial}{\partial x_2} - c^0(x) \right\} U_0^0(x) = f^0(x), \quad x \in D^{(0)}, \tag{7.7a}$$

$$U_0^0(x) = \varphi^0(x), \quad x \in \varGamma^{(0)}; \tag{7.7b}$$

$$L_1 U_k^0(x) \;=\; -\varepsilon_1 \sum_{s=1,2} a_s^0(x) \frac{\partial^2}{\partial x_s^2} U_{k-1}^0(x), \quad x \in D^{(0)}, \tag{7.8a}$$

$$U_k^0(x) \;=\; 0, \quad x \in \Gamma^{(0)}, \quad k > 0. \tag{7.8b}$$

Here $D^{(0)} = D^{(0)}(l^{(0)})$ is the half-plane $x_1 < l^{(0)}$, $\Gamma^{(0)} = \overline{D}^{(0)} \setminus D^{(0)}$; $\varphi^0(x)$, $x \in D^0$, is a sufficiently smooth function satisfying the condition $\varphi^0(x) = \varphi(x)$, $x \in \Gamma$ when $\varepsilon_2 < 0$. In the case of the condition $\varepsilon_2 \geq 0$ it will be convenient to take the set $\overline{D}^{(0)}$ as the set $D^{(0)}$ and to find the components $U_k^0(x)$ by considering only equations (7.7a) and (7.8a), i.e., to solve problems defined by differential equations without boundary conditions. For sufficiently large $n$ (for $n \geq K-1$), we obtain the following estimate for the function $U(x)$:

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} U(x) \right| \leq M, \quad x \in \overline{D}, \quad k \leq K. \tag{7.9}$$

**7.2.2.** The function $V(x)$ has the representation

$$V(x) = V_{(1)}(x) + V_{(2)}(x) + V_{(12)}(x), \quad x \in \overline{D}. \tag{7.10}$$

Here $V_{(1)}(x)$, $V_{(2)}(x)$ and $V_{(12)}(x)$ are the regular, parabolic and corner elliptic boundary layers, respectively. The functions $V_{(1)}(x)$ and $V_{(2)}(x)$, $x \in \overline{D}$, are the restrictions on $\overline{D}$ of the functions $V_{(1)}^0(x)$, $x \in \overline{D}_{(1)}$ and $V_{(2)}^0(x)$, $x \in \overline{D}_{(2)}$; $\overline{D}_{(j)}$ is a half-plane (containing $\overline{D}$), which has the set $\Gamma_j$ as a common part of the boundary with $\overline{D}$, $\Gamma_j = \Gamma \cap \Gamma_{(j)}$, $j = 1, 2$. The functions $V_{(j)}^0(x)$, $x \in \overline{D}_{(j)}$, are the solutions of the problems

$$L^0 V_{(j)}^0(x) = 0, \quad x \in D_{(j)} \quad V_{(j)}^0(x) = \varphi_{(j)}(x), \quad x \in \Gamma_{(j)}, \quad j = 1, 2; \tag{7.11}$$

the functions $V_{(j)}^0(x)$ decrease exponentially away from the boundary $\Gamma_{(j)}$. Here $\varphi_{(j)}(x)$, $x \in \Gamma_{(j)}$, are sufficiently smooth functions satisfying the condition $\varphi_{(j)}(x) = \varphi_V(x)$, $x \in \Gamma_j$, $j = 1, 2$.

The function $V_{(12)}(x)$ is the solution of the problem

$$L V_{(12)}(x) = 0, \quad x \in D, \quad V_{(12)}(x) = \varphi_V(x) - V_{(1)}(x) - V_{(2)}(x), \quad x \in \Gamma. \tag{7.12}$$

The main terms in the expansions of the functions $V_{(1)}^0(x)$, $x \in \overline{D}_{(1)}$ and $V_{(2)}^0(x)$, $x \in \overline{D}_{(2)}$, are the solutions of the following boundary value problems for the ordinary differential and parabolic equations:

$$\begin{aligned} &L_2 V_{(1)1}^0(x) \equiv \left\{ \varepsilon_1 a_1(x) \frac{d^2}{dx_1^2} + b_1(x) \frac{d}{dx_1} - c(x) \right\} V_{(1)1}^0(x) = 0, \quad x \in D_{(1)}, \\ &\quad V_{(1)1}^0(x) = \varphi_{(1)}(x), \quad x \in \Gamma_{(1)}; \end{aligned} \tag{7.13a}$$

$$\begin{aligned} &L_3 V_{(2)1}^0(x) \equiv \left\{ \varepsilon_1 a_2(x) \frac{\partial^2}{\partial x_2^2} + b_1(x) \frac{\partial}{\partial x_1} + \varepsilon_2 b_2(x) \frac{\partial}{\partial x_2} - c(x) \right\} V_{(2)1}^0(x) = 0, \quad x \in D_{(2)}, \\ &\quad V_{(2)1}^0(x) = \varphi_{(2)}(x), \quad x \in \Gamma_{(2)}; \end{aligned} \tag{7.13b}$$

the functions $V_{(j)1}^0(x)$ decrease exponentially away from $\Gamma_{(j)}$, $j = 1, 2$.

For the functions $V_{(j)}(x)$, $x \in \overline{D}$, we obtain the estimates

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} V_{(1)}(x)\right| \le M\varepsilon_1^{-k_1}\exp(-m_1\varepsilon_1^{-1}x_1), \tag{7.14}$$

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} V_{(2)}(x)\right| \le M\varepsilon_1^{-k_2/2}\exp(-m_2\varepsilon_1^{-1/2}x_2), \quad x \in \overline{D}, \quad k \le K, \tag{7.15a}$$

where $m_i$ is an arbitrary number in the interval $(0, m_i^0)$, $i = 1, 2$, $m_1^0 = \min\limits_{\overline{D}}[a_1^{-1}(x)b_1(x)]$, $m_2^0 = m_2^0(M_{1(7.4)}) = \min\{2^{-1}M_{1(7.4)}^{-1}\min\limits_{\overline{D}}[b_2^{-1}(x)\,c(x)],\ 2^{-1/2}\min\limits_{\overline{D}}{}^{1/2}\,[a_2^{-1}(x)\,c(x)]\}$.

Assuming that the following condition holds:

$$u \in C^{l+\alpha}(\overline{D}), \quad l = K + 4, \quad \alpha \in (0,1). \tag{7.16}$$

we find the estimate of the function $V_{(12)}(x)$

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} V_{(12)}(x)\right| \le \tag{7.15b}$$

$$\le M\,\varepsilon_1^{-k_1-k_2/2}\min[\exp(-m_1\,\varepsilon_1^{-1}\,x_1),\ \exp(-m_2\,\varepsilon_1^{-1/2}\,x_2)], \quad x \in \overline{D},\ k \le K,$$

where $m_i = m_{i(7.14)}$, $i = 1, 2$.

**7.3.** Let the following condition be satisfied for the components $\varepsilon_1$ and $\varepsilon_2$:

$$\varepsilon_2 \ge m_1\varepsilon_1^{1/2}, \quad \varepsilon_1 = o(1), \tag{7.17}$$

where $m_1$ is an arbitrary number.

The components in representations (7.5), (7.10) are the solutions of problems (7.6), (7.11), (7.12). In the case of condition (7.17), the component $V_{(2)}(x)$ in (7.10) is the regular boundary layer. The main term in the expansion of the function $V_{(2)}^0(x)$, $x \in \overline{D}_{(2)}$, can be found from the solution of the boundary value problem for the ordinary differential equation

$$L_4 V_{(2)1}^0(x) \equiv \left\{\varepsilon_1 a_2(x)\frac{d^2}{dx_2^2} + \varepsilon_2 b_2(x)\frac{d}{dx_2} - c(x)\right\} V_{(2)1}^0(x) = 0, \quad x \in D_{(2)},$$

$$V_{(2)1}^0(x) = \varphi_{(2)}(x), \quad x \in \Gamma_{(2)}. \tag{7.18}$$

The components $U(x)$ and $V_{(1)}(x)$ satisfy estimates (7.9) and (7.14). For the components $V_{(2)}(x)$ and $V_{(12)}(x)$ (in the case of condition (7.16)) we have the estimates

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} V_{(2)}(x)\right| \le M(\varepsilon_2\varepsilon_1^{-1})^{k_2}\exp(-m_2\varepsilon_2\varepsilon_1^{-1}x_2), \tag{7.19a}$$

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} V_{(12)}(x)\right| \le M\varepsilon_1^{-k}\varepsilon_2^{k_2}\min[\exp(-m_1\varepsilon_1^{-1}x_1), \exp(-m_2\varepsilon_2\varepsilon_1^{-1}x_2)],$$

$$x \in \overline{D}, \quad k \le K, \tag{7.19b}$$

where $m_1 = m_{1(7.14)}$, $m_2$ is an arbitrary number from $(0, m_2^0)$, $m_2^0 = \min\limits_{\overline{D}}\,[a_2^{-1}(x)b_2(x)]$.

**7.4.** But if the following condition is satisfied:

$$\varepsilon_2 \le -m_2 \varepsilon_1^{1/2}, \quad \varepsilon_1,\, |\varepsilon_2| = o(1), \tag{7.20}$$

then the component $V_{(2)}(x)$ (in (7.10)) is the hyperbolic boundary layer. The main term in the expansion of the function $V^0_{(2)}(x)$, $x \in \overline{D}_{(2)}$, is the solution of the (initial value) problem for the hyperbolic equation

$$\begin{aligned} L_5 V^0_{(2)1}(x) &\equiv \left\{ b_1(x)\frac{\partial}{\partial x_1} + \varepsilon_2 b_2(x)\frac{\partial}{\partial x_2} - c(x) \right\} V^0_{(2)1}(x) = 0, \quad x \in D_{(2)}, \\ V^0_{(2)1}(x) &= \varphi_{(2)}(x), \quad x \in \Gamma_{(2)}. \end{aligned} \tag{7.21}$$

For the components $V_{(2)}(x)$ and $V_{(12)}(x)$ we obtain the estimates

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V_{(2)}(x) \right| \le M |\varepsilon_2|^{-k_2} \exp(-m_2 |\varepsilon_2|^{-1} x_2), \tag{7.22a}$$

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V_{(12)}(x) \right| \le M \varepsilon_1^{-k_1} |\varepsilon_2|^{-k_2} \min[\exp(-m_1 \varepsilon^{-1} x_1), \exp(-m_2 |\varepsilon_2|^{-1} x_2)], \tag{7.22b}$$

$$x \in \overline{D}, \quad k \le K,$$

where $m_1 = m_{1(7.14)}$, $m_2$ is an arbitrary number in the interval $(0, m_2^0)$, $m_2^0 = \min\{2^{-1/2} m_{2(7.19)} \min_{\overline{D}} [a_2^{-1}(x) c(x)]^{1/2},\ 2^{-1} \min_{\overline{D}} [b_2^{-1}(x) c(x)]\} = m_2^0(m_{2(7.19)})$.

**7.5.** In the case of the condition

$$\varepsilon_1 = o(1), \quad \varepsilon_2 \le -m_1 \tag{7.23}$$

the function $U(x)$ in (7.5) is the restriction to $\overline{D}$ of the function $U_{(2)}(x)$, $x \in \overline{D}_{(2)}$, which is the solution of the boundary value problem on the half-plane $\overline{D}_{(2)}$

$$\begin{aligned} L^0 U_{(2)}(x) &= f^0(x), \quad x \in D_{(2)}, \\ U_{(2)}(x) &= \varphi^0(x), \quad x \in \Gamma_{(2)}, \end{aligned}$$

where $L^0 = L^0_{(7.6)}$, $f^0(x) = f^0_{(7.6)}(x)$, $\varphi^0(x)$, $x \in \Gamma_{(2)}$, is a smooth extension of the function $\varphi(x)$ from $\Gamma_2$ to $\Gamma_{(2)}$.

The function $V(x)$ from (7.5) has the representation

$$V(x) = V_{(1)}(x), \quad x \in \overline{D}, \tag{7.24}$$

where $V_{(1)}(x)$ is the regular boundary layer in a neighbourhood of the set $\Gamma_1$; no boundary layer arises in a neighbourhood of the set $\Gamma_2$. The function $V_{(1)}(x)$ is the solution of the problem

$$\begin{aligned} L V_{(1)}(x) &= 0, & x &\in D, \\ V_{(1)}(x) &= \varphi(x) - U(x), & x &\in \Gamma_1, \\ V_{(1)}(x) &= 0, & x &\in \Gamma_2. \end{aligned}$$

For the components $U(x)$ and $V_{(1)}(x)$ in representations (7.5) and (7.24), estimates (7.9) and (7.14) hold; estimate (7.14) occurs under condition (7.16).

**Theorem 7.1.** *Let* $a_s$,$b_s$,$c$,$f \in C^l(\overline{D})$, $s = 1,2$, $\varphi \in C(\Gamma)$, $\varphi \in C^{l+\alpha}(\Gamma_j)$, $j = 1,2$, $l \geq 3K-4$, $K \geq 3$, $\alpha \in (0,1)$. *Then, under the condition* (7.2)*, the function* $u(x)$*, i.e., the solution of problem* (2.2), (2.1)*, satisfy estimates* (7.1), (7.3)*. For the components* $U(x)$ *and* $V_{(1)}(x)$ *in representations* (7.5), (7.10) *(in representations* (7.5), (7.24)*) under the condition* $\varepsilon_1 = o(1)$ *and either* $\varepsilon_2 \geq 0$ *or* $|\varepsilon_2| = o(1)$, $\varepsilon_2 < 0$ *(under condition* (7.23)*), estimates* (7.9) *and* (7.14) *hold. For the components* $V_{(2)}(x)$, $V_{(12)}(x)$ *in the representation* (7.10)*, estimates* (7.15), (7.19) *and* (7.22) *are satisfied in the case of conditions* (7.4), (7.17) *and* (7.20)*, respectively. Estimates* (7.15b), (7.19b), (7.22b) *for the component* $V_{(1.2)}(x)$ *and estimate* (7.14) *for the component* $V_{(1)}(x)$ *from* (7.24) *are valid under the additional condition* (7.16).

**Remark 3.** The estimates for the components in representation (7.10) can be written in the compact form

$$\Big|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,U(x)\Big| \leq M, \quad \Big|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,V_{(1)}(x)\Big| \leq M\,\varepsilon_1^{-k_1}\,\exp(-m_1\,\varepsilon_1^{-1}\,x_1),$$

$$\Big|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,V_{(2)}(x)\Big| \leq M\,\lambda^{-k_2}\,\exp(-m_2\,\lambda^{-1}\,x_2),$$

$$\Big|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,V_{(12)}(x)\Big| \leq M\,\varepsilon_1^{-k_1}\,\lambda^{-k_2}\,\min\,[\exp(-m_1\,\varepsilon_1^{-1}\,x_1),\,\exp(-m_2\,\lambda^{-1}\,x_2)],$$

$$x \in \overline{D}, \quad k \leq K.$$

Here $\lambda = \lambda(\varepsilon_1,\,\varepsilon_2)$, $\lambda = \varepsilon_1\,(\varepsilon_1 + m^{(1)}\,\varepsilon_2^2)^{-1/2}$ for $\varepsilon_2 \geq 0$, $\lambda = (\varepsilon_1 + M^{(1)}\,\varepsilon_2^2)^{1/2}$ for $\varepsilon_2 < 0$, $m^{(1)} = 4^{-1}\,\inf\limits_{\overline{D}}\,[a_2^{-1}(x)\,b_2^2(x)\,c^{-1}(x)]$, $M^{(1)} = 4^{-1}\,\sup\limits_{\overline{D}}\,[a_2^{-1}(x)\,b_2^2(x)\,c^{-1}(x)]$, $m_i$ is an arbitrary number from $(0,\,m_i^0)$, $i = 1,2$, $m_1^0 = \inf\limits_{\overline{D}}\,[a_1^{-1}(x)\,b_1(x)]$, $m_2^0 = \inf\limits_{\overline{D}}\,[a_2^{-1/2}(x)\,c^{1/2}(x)]$.

**Remark 4.** The type of a boundary layer arising in a neighbourhood of the boundary $\Gamma_2$ depends on the relation between the parameters $\varepsilon_1$ and $\varepsilon_2$. This layer is regular (for $\varepsilon_1^{1/2} \ll \varepsilon_2 \leq 1$), parabolic (for $|\varepsilon_2| = \mathcal{O}(\varepsilon_1^{1/2})$) or hyperbolic (for $\varepsilon_2 < 0$, $\varepsilon_1^{1/2} \ll |\varepsilon_2| \ll 1$), or no layer appears (for $\varepsilon_2 < 0$, $|\varepsilon_2| \approx 1$). The main terms of asymptotic expansions for the singular component in a neighbourhood of the boundary $\Gamma_2$ are the solutions of problems (7.13b), (7.18) and (7.21). In a neighbourhood of the set $\Gamma_1$, but outside the nearest neighbourhood of $\Gamma^c$, the layer is regular (defined only by the parameter $\varepsilon_1$), and in a neighbourhood of the set $\Gamma^c$ the layer is elliptic (for $\varepsilon_1 = o(1)$ and either $\varepsilon_2 \geq 0$ or $\varepsilon_2 < 0$, $|\varepsilon_2| = o(1)$), or the strong layer does not appear (for $\varepsilon_1 = o(1)$, $\varepsilon_2 < 0$, $|\varepsilon_2| \approx 1$; the arising weak layer does not contain the first components of the expansion).

**Remark 5.** Let $\mathcal{U}$ be a set of solutions to the boundary value problem (2.2), (2.1), which are defined by the admissible class of the problem data, i.e., by the functions $f(x)$, $x \in \overline{D}$ and $\varphi(x)$, $x \in \Gamma$. Let the $\varepsilon$-net (see, e.g., [2]) be generated by a set of interpolants constructed on triangulations based on discrete sets ("meshes") on $\overline{D}$. In that case when the elements of $\mathcal{U}$ are approximated in the maximum norm, the $\varepsilon$-entropy $H_\varepsilon(\mathcal{U})$ (see [2]) of the set $\mathcal{U}$ is infinite.

# 8. Generalizations and remarks

**8.1.** In that case when condition (7.16) is violated and/or the smoothness of the data is not sufficiently high (for example, if $a_s$, $b_s$, $c$, $f \in C^\alpha(\overline{D})$, $\varphi \in C^{2+\alpha}C(\Gamma_j)$, $\varphi \in C(\Gamma)$, $\alpha \in (0,1)$,

$s, j = 1, 2$), the technique from [14, 16] allows us to establish the $\overline{\varepsilon}$-uniform convergence of the formal difference scheme (3.2), (3.8) at the rate $\mathcal{O}(N^{-\nu})$, where the convergence order $\nu = \nu(\alpha)$ is, in general, small. The use of the property that the domains of dependence of solutions for the differential problem (2.2), (2.1) and the discrete problem (3.2), (3.1) are bounded allows us to establish the $\overline{\varepsilon}$-uniform convergence on $\overline{D}^0$ of solutions of the difference scheme (6.2), (6.12) at the rate $\mathcal{O}(N^{-\nu} \ln N)$.

**8.2.** In that case when the condition $c(x) \geq c_0 > 0$, $x \in \overline{D}$ is violated, the domain of essential dependence of the solution of the boundary value problem (and the discrete problem) is not in general bounded if the $\varepsilon$-entropy $H_\varepsilon(\mathcal{U})$ of the set $\mathcal{U}$ is infinite. For such problems the technique considered here for the design of constructive difference schemes is directly inapplicable. But if the right hand-side of the equation and the boundary function decrease for $x_1 \to \infty$ in the case of the condition $c(x) \equiv 0$, $x \in \overline{D}$ (the $\varepsilon$-entropy $H_\varepsilon(U)$ of the set $U$ remains generally infinite), the technique developed in the paper makes it possible to construct $\overline{\varepsilon}$-uniformly convergent constructive difference schemes.

**8.3.** The exposed technique allows us to establish the property of the $\varepsilon$-uniform boundedness for domains of essential dependence of solutions to differential and formal difference problems also for other types of singularly perturbed problems on unbounded domains. Having this property and the sufficiently weak dependence of the sizes of such domains on the value $\beta$, one can develop constructive numerical methods that converge $\varepsilon$-uniformly.

# References


1. N.S. Bakhvalov, *On the optimization of methods for boundary-value problems with boundary layers*, Zh. Vychisl. Mat. Mat. Fiz., **9** (1969), No. 4, pp. 841–859, in Russian.

2. N.S. Bakhvalov, *Numerical Methods*, Moscow, Nauka, 1973, in Russian.

3. E.P. Doolan, J.J.H. Miller, and W.H.A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

4. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman and Hall/CRC, Boca Raton, 2000.

5. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

6. A.M. Il'in, *Differencing scheme for a differential equation with a small parameter affecting the highest derivative*, Math. Notes, **6** (1969), No. 2, pp. 596–602.

7. A.M. Il'in, A.S. Kalashnikov, and O.A. Oleinik. *Linear second-order equations of parabolic type*, Uspekhi Mat. Nauk, **17** (1962), No. 3, pp. 3–146, in Russian.

8. O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967, in Russian; Translations of Mathematical Monographs, **23**, American Mathematical Society, Providence, RI, 1968.

9. O.A. Ladyzhenskaya and N.N. Ural'tseva. *Linear and Quasilinear Equations of Elliptic Type*, Nauka, Moscow, 1973, in Russian; English transl. of 1st edn.: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York and London, 1968.

10. J.J.H. Miller, E. O'Riordan, and G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

11. H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion and Flow Problems*, Springer-Verlag, Berlin, 1996.

12. A.A. Samarskii, *Theory of Difference Schemes*, 3rd edn., Nauka, Moscow, 1989, in Russian; English transl.: *The Theory of Difference Schemes*, Marcel Dekker, Inc., New York, 2001.

13. H. Schlichting, *Boundary Layer Theory*, 7th edn., McGraw Hill, New York, 1979.

14. G.I. Shishkin, *Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations*, Ural Branch of Russian Acad. Sci., Ekaterinburg, 1992, in Russian.

15. G.I. Shishkin, *Grid approximations with an improved rate of convergence for singularly perturbed elliptic equations in domains with characteristic boundaries*, Sib. J. Numer. Math., **5** (2002), No. 1, pp. 71–92, in Russian.

16. G.I. Shishkin, *Approximation of singularly perturbed parabolic reaction-diffusion equations with nonsmooth data*, Computational Methods in Applied Mathematics, **1** (2001), No. 3, pp. 298–315.

17. E.A. Volkov, *On differential properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle*, Proc. Steklov Inst. Math., Moscow, **77** (1965), pp. 89–112, in Russian.