Improved ε-uniform Accurate Schemes for Singularly Perturbed Elliptic Convection-Diffusion 3D Problems. The Richardson Extrapolation Technique *

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Abstract

In this paper we consider the Dirichlet problem for a singularly perturbed elliptic equation of convection-diffusion type on a rectangular parallelepiped. For such a problem, ε-uniformly convergent monotone difference schemes on *piecewise uniform* meshes are well known; their ε-uniform order of accuracy does not exceed 1. Based on solutions of finite difference schemes on *piecewise uniform embedded* meshes by using the Richardson extrapolation technique, we construct a numerical solution that converges ε-uniformly with the second (up to a logarithmic factor) order of accuracy. The given technique can be applied to the construction and justification of higher-order accurate numerical solutions for n-dimensional problems, where $n > 3$.

1. Introduction

Special numerical methods for singularly perturbed boundary value problems are at present developed sufficiently well (see, e.g., [1–4]). These methods, in contrast to methods for regular boundary value problems (see, e.g., [5, 6]), allow us to obtain discrete solutions that converge uniformly with respect to a perturbation parameter ε (i.e. solutions convergent ε-uniformly). In the case of singularly perturbed boundary value problems for reaction-diffusion equations, special finite difference schemes have the ε-uniform order of accuracy close to two. However, for convection-diffusion equations the ε-uniform order of accuracy is not higher than one (see, e.g., [1–4, 7–9] and also the bibliography therein). Note that the most part of known results concerns problems in one and two dimensions; many-dimensional problems are considered only in [1]. Since

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the efficiency of numerical methods is largely defined by their convergence rate, interest arises in the construction of special schemes for convection-diffusion problems whose order of convergence is higher than one. It is very attractive to use a technique which allows us to construct such high-order accurate methods for many-dimensional problems.

Effective methods for constructing approximate solutions with increased order of accuracy for regular boundary value problems are the Richardson and defect correction methods (see, e.g., [5, 10, 11] and also the bibliography therein). In the case of singularly perturbed boundary value problems, the defect correction method is applied in order to increase accuracy with respect to the time variable for nonstationary boundary value problems of reaction-diffusion in [12, 13] and convection-diffusion in [14, 15]. The improvement in accuracy of solutions with respect to the space and time variables for singularly perturbed nonstationary reaction-diffusion problems by using the Richardson method is considered in [16]; the improvement in accuracy of solutions for singularly perturbed elliptic convection-diffusion equations on a strip is studied in [17, 18].

Note that grid constructions on uniform meshes are essentially used in the construction and justification of the defect-correction and Richardson methods for regular boundary value problems. In the case of singularly perturbed equations, for sufficiently large classes of boundary value problems it is necessary to apply the condensing (in the boundary layer region) mesh technique for ε-uniform convergence (see, e.g., [1, 19]). Thus, in order to construct schemes of higher-order accuracy for boundary value problems, in particular, for singularly perturbed equations, the most attractive is a technique based on the defect-correction and Richardson methods, which uses nonuniform meshes.

In the present paper we consider a boundary value problem for a singularly perturbed elliptic convection-diffusion equation on a rectangular parallelepiped. For such a problem, the known special difference scheme from [1] on piecewise uniform meshes converges ε-uniformly, however, its ε-uniform order of accuracy does not exceed one. Based on solutions of finite difference schemes on piecewise uniform embedded meshes by using the Richardson extrapolation technique, we construct a numerical solution that converges ε-uniformly with the order of accuracy close to two. To justify the convergence of the numerical solution we use expansions of solutions of special difference schemes with respect to the effective step-size of the mesh domain. Coefficients in the expansions of solutions of the difference schemes (their regular and singular components) are defined from solutions of the corresponding singularly perturbed boundary value problems. The same technique can be applied to the construction of discrete solutions with improved order of accuracy (of the second order up to a logarithmic factor) for boundary value problems in more than three dimensions.

2. Problem formulation

1. On the rectangular parallelepiped $\overline{D}$, where

$$D = \{x : 0 < x_s < d_s, \quad s = 1, 2, 3\}, \tag{2.1}$$

we consider the first boundary value problem for the elliptic equation [1]

$$L_{(2.2)}\, u(x) = f(x), \quad x \in D, \quad u(x) = \varphi(x), \quad x \in \Gamma. \tag{2.2a}$$

Here $\Gamma = \overline{D} \setminus D$,

$$L_{(2.2)} = L^{(2)}_{(2.2)} + L^{(1)}_{(2.2)}, \quad L^{(2)}_{(2.2)} \equiv \varepsilon \sum_{s=1}^{3} a_s(x) \frac{\partial^2}{\partial x_s^2}, \quad L^{1}_{(2.2)} \equiv \sum_{s=1}^{3} b_s(x) \frac{\partial}{\partial x_s} - c(x),$$

$a_s(x)$, $b_s(x)$, $c(x)$, $f(x)$, $x \in \overline{D}$, $\varphi(x)$, $x \in \Gamma_j$ are sufficiently smooth functions, where Γ_j, $j = 1, \ldots, 6$ are faces of the set D, $\Gamma = \cup \Gamma_j$, $\Gamma_j = \overline{\Gamma}_j$, moreover

$$a_0 \le a_s(x) \le a^0, \quad b_0 \le b_s(x) \le b^0, \quad c(x) \ge 0, \tag{2.2b}$$
$$x \in \overline{D}, \; s = 1, 2, 3, \; a_0, \, b_0 > 0;$$

the singular perturbation parameter ε takes arbitrary values in the half-open interval $(0, 1]$.

We suppose that on the set Γ^*, i.e. on the edges of D, compatibility conditions are satisfied which ensure the sufficient smoothness of the solution of problem (2.2), (2.1) for each fixed value of the parameter.

We use the following notation. The faces Γ_j, Γ_{j+3}, $j = 1, 2, 3$ of the set D are orthogonal to the x_j-axis; Γ_1, Γ_2 and Γ_3 contain the vertex $(0, 0, 0)$. By Γ^- (by Γ^+), we denote that part of the boundary Γ across which the characteristics of the reduced equation passing through the points $x \in D$ leave (enter) the set D, $\Gamma^- = \bigcup_{j=1}^{3} \Gamma_j$.

When the parameter ε tends to zero, a boundary layer appears in a neighbourhood of the set Γ^-. This layer is regular in the neighbourhood of the smooth parts of Γ^- and is elliptic (corner) in the neighbourhood of the intersection of the faces Γ_j, $j = 1, 2, 3$.

For the boundary value problem under consideration it is required, by using the Richardson extrapolation technique, to construct a discrete solution that converges ε-uniformly with the order of accuracy higher than 1.

[1] Throughout the paper, the notation $L_{(j.k)}$ $(M_{(j.k)}, \; G_{h(j.k)})$ means that these operators (constants, grids) are introduced in formula $(j.k)$.

About the contents. Discrete constructions related to problem (2.2), (2.1) are considered in Sections 4, 5 and 6. *A priori* estimates for solutions of problem (2.2), (2.1) used in the constructions are discussed in Section 3. In Section 4 we examine a variant of the Richardson method on special embedded meshes where the transition points of piecewise uniform meshes remain fixed when these meshes are refined. Conditions imposed on the width of the finer part of the mesh (in the boundary layer region) that are necessary to construct approximate solutions with the increased ε-uniform order of accuracy are given in Section 5. Expansions of solutions to finite difference schemes and approximate solutions of the improved order of accuracy are considered in Section 6. When performing the discrete constructions and justifying convergence of the approximate solutions, the majorant function technique is used (see, e.g., [6, 20]).

3. *A priori* estimates of solutions and derivatives

1. In this section we discuss *a priori* estimates of solutions and derivatives for the boundary value problem (2.2), (2.1); the derivation of the estimates is similar to that in [1].

1.1. Using the majorant function technique (see, e.g., [20]), we find the estimate [2]

$$|u(x)| \le M, \quad x \in \overline{D}. \tag{3.1}$$

We represent the problem solution as the decomposition

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{3.2a}$$

where $U(x)$ and $V(x)$ are the regular and singular components of the solution. The function $U(x)$, $x \in \overline{D}$ is the restriction, onto $\overline{D}$, of the function $U^0(x)$, $x \in \overline{D}^0$. The function $U^0(x)$, $x \in \overline{D}^0$ is the solution of the boundary value problem

$$L^0 U^0(x) = f^0(x), \quad x \in D^0, \qquad U^0(x) = \varphi^0(x), \quad x \in \Gamma^0. \tag{3.3}$$

Here $\overline{D}^0$ is the VII-th octant (opposite to the first octant), which is an extension of D beyond the sides Γ_1, Γ_2 and Γ_3 (the intersecting planes forming this octant contains the sides Γ_4, Γ_5 and Γ_6); the data of problem (3.3) are smooth continuations of the data of problem (2.2), (2.1) that preserve properties (2.2b) on $\overline{D}^0$; $L^0 = L^{0(2)} + L^{0(1)}$. We assume that the functions $f^0(x)$ and $\varphi^0(x)$,

[2] Here and below M, M_i (or m) denote sufficiently large (or small) positive constants which do not depend on ε and on the discretization parameters.

$x \in \overline{D}^0$ are equal to zero outside an m_1-neighbourhood of the set $\overline{D}$. The singular component $V(x)$ is the solution of the problem

$$L_{(2.2)} V(x) = 0, \quad x \in D, \quad V(x) = \varphi(x) - U(x), \quad x \in \Gamma. \tag{3.4}$$

The regular component $U(x)$ is decomposed into the sum of functions

$$U(x) = U_0(x) + \varepsilon U_1(x) + \cdots + \varepsilon^n U_n(x) + v_U(x), \quad x \in \overline{D}, \tag{3.2b}$$

which corresponds to the following representation of the function $U^0(x)$:

$$U^0(x) = U_0^0(x) + \varepsilon U_1^0(x) + \cdots + \varepsilon^n U_n^0(x) + v_U^0(x), \quad x \in \overline{D}^0; \tag{3.5a}$$

$U(x) = U^0(x), \ldots, v_U(x) = v_U^0(x), \quad x \in \overline{D}$, where $U^0(x)$ is the solution of problem (3.3). In (3.5a) the components $U_0^0(x)$ and $U_i^0(x)$, $i = 1, \ldots, n$ are solutions of the problems

$$L_{(3.3)}^{0(1)} U_0^0(x) = f^0(x), \quad x \in \overline{D}^0 \setminus \Gamma^{0+}, \tag{3.5b}$$

$$U_0^0(x) = \varphi^0(x), \quad x \in \Gamma^{0+};$$

$$L_{(3.3)}^{0(1)} U_i^0(x) = -\varepsilon^{-1} L_{(3.3)}^{0(2)} U_{i-1}^0(x), \quad x \in \overline{D}^0 \setminus \Gamma^{0+},$$

$$U_i^0(x) = 0, \quad x \in \Gamma^{0+}, \quad i = 1, \ldots, n.$$

We consider that the data of problem (2.2), (2.1) (in addition to the compatibility conditions on the set Γ^*, which ensure the smoothness of the solution $u(x)$) satisfy supplemental conditions on the set $\Gamma^{*+} = \Gamma^* \cap \{\Gamma \setminus \Gamma^-\}$ that ensure the sufficient smoothness of the functions $U_0^0(x)$ and $U_i^0(x)$, $i = 1, \ldots, n$. It is not difficult to write out such conditions, for example, in that case when the boundary function $\varphi(x)$ together with its derivatives vanishes on the set Γ^*.

For simplicity, we suppose the following inclusions to be satisfied:

$$u \in C^{n+2+\alpha}(\overline{D}), \quad U_i \in C^{3n+2-2i+\alpha}(\overline{D}), \tag{3.6}$$

$$i = 0, 1, \ldots, n, \quad n \geq 1, \quad \alpha > 0.$$

In this case $U \in C^{n+2+\alpha}(\overline{D})$ is true; for $U(x)$ and $V(x)$ we obtain the estimates

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} U(x) \right| \leq M \left[1 + \varepsilon^{n+1-k}\right], \quad x \in \overline{D}, \quad k \leq K; \tag{3.7a}$$

$$|\, V(x) \,| \leq M \exp(-m\, \varepsilon^{-1}\, r(x, \Gamma^-)), \quad x \in \overline{D},$$

where $r(x, \Gamma^-)$ is the distance between the point x and the set Γ^-, m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{s,\overline{D}}[a_s^{-1}(x)b_s(x)]$; $K = n+2$ if the data of problem (2.2), (2.1) are sufficiently smooth.

1.2. We now present estimates for derivatives of the function $V(x)$. The function $V(x)$ is decomposed as

$$V(x) = \sum_{j=1}^{3} V_{(j)}(x) + \sum_{j=1}^{3} V_{(j,j+1)}(x) + V_{(1,2,3)}(x), \quad x \in \overline{D}. \tag{3.2c}$$

Here $V_{(j)}(x)$, $V_{(j,j+1)}(x)$ and $V_{(1,2,3)}(x)$ are respectively one-dimensional, two-dimensional and three-dimensional boundary layers; $V_{(j,j+1)}(x) = V_{(1,3)}(x)$ for $j = 3$.

The functions $V_{(j)}(x)$ and $V_{(j,j+1)}(x)$, $x \in \overline{D}$ are the restrictions on $\overline{D}$ of the functions

$$V^0_{(j)}(x), \quad x \in \overline{D}_{(j)}; \qquad V^0_{(j,j+1)}(x), \quad x \in \overline{D}_{(j,j+1)}. \tag{3.8}$$

Here $\overline{D}_{(j)}$ is a slab in $\overline{D}^0$ containing the faces Γ_j, Γ_4, Γ_5 and Γ_6, and $\overline{D}_{(j,j+1)}$ is a cylinder in $\overline{D}^0$ that is the intersection of the slabs $\overline{D}_{(j)}$ and $\overline{D}_{(j+1)}$.

The functions $V^0_{(j)}(x)$ are solutions of the boundary value problems

$$L^0_{(3.3)} V^0_{(j)}(x) = 0, \qquad x \in D_{(j)}, \tag{3.9a}$$

$$V^0_{(j)}(x) = \varphi^0_{(j)}(x), \quad x \in \Gamma_{(j)}, \quad j = 1, 2, 3. \tag{3.9b}$$

The function $\varphi^0_{(j)}(x) = \varphi^0(x) - U^0(x)$, $x \in \Gamma_{(j)}$ satisfies the condition

$$\varphi^0_{(j)}(x) = \begin{cases} \varphi(x) - U(x), & x \in \Gamma_j. \\ 0, & x \in \Gamma^+; \end{cases}$$

This function $\varphi^0_{(j)}(x)$ vanishes outside the m-neighbourhood of the set Γ_j. The functions $V^0_{(j,j+1)}(x)$, $x \in \overline{D}_{(j,j+1)}$ are solutions of the problems

$$L^0_{(3.3)} V^0_{(j,j+1)}(x) = 0, \qquad x \in D_{(j,j+1)},$$

$$V^0_{(j,j+1)}(x) = \varphi^0_{(j,j+1)}(x), \quad x \in \Gamma_{(j,j+1)}, \quad j = 1, 2, 3.$$

The function $\varphi^0_{(j,j+1)}(x)$, being smooth on the faces $D_{(j,j+1)}$, vanishes on the set $\Gamma^+(D_{(j,j+1)})$ and satisfies the condition

$$U(x) + \sum_{s=j,j+1} V_{(s)}(x) + \varphi^0_{(j,j+1)} = \varphi(x), \quad x \in \Gamma_s, \quad s = j, j+1.$$

For the components in the representation (3.2c) we have

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}V_{(j)}(x)\right| \le M[\varepsilon^{-s_1}+\varepsilon^{1-k}]\exp(-m\varepsilon^{-1}r(x,\Gamma_j)), \qquad (3.7\text{b})$$

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}V_{(j,j+1)}(x)\right| \le M[\varepsilon^{-s_2}+\varepsilon^{1-k}]\exp(-m\varepsilon^{-1}r(x,\Gamma_j\cup\Gamma_{j+1})),$$

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}V_{(1,2,3)}(x)\right| \le M\varepsilon^{-k}\exp(-m\varepsilon^{-1}r(x,\Gamma^-)),$$

$$x\in\overline{D},\quad j=1,2,3,\quad k\le K,\quad K=K(n)=n+2,$$

where $s_1 = s_1(k_1,k_2,k_3,j)$, $s_1 = k_i$ for $j=i$, $s_2 = s_2(k_1,k_2,k_3,j)$, $s_2 = k_i + k_{i+1}$ for $j = i$; otherwise $s_1(k_1,k_2,k_3,j)$ and $s_2(k_1,k_2,k_3,j)$ are equal to zero; $k_{i+1} = k_1$ for $i=3$, $\Gamma_{j+1}=\Gamma_1$ for $j=3$; $m = m_{(3.7\text{a})}$.

The following estimates are also valid:

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}u(x)\right| \;\le\; M\varepsilon^{-k},\quad x\in\overline{D},\quad k\le K. \qquad (3.10)$$

Theorem 3.1. *Let the condition* $a_s,\ b_s,\ c,\ f\ \in\ C^{3n+2+\alpha}(\overline{D}),\ \varphi\ \in C^{3n+2+\alpha}(\overline{D}),\ s=1,2,3,\ n\ge 1,\ \alpha>0$ *be valid for the data of the boundary value problem* (2.2), (2.1), *and let its solution and the components* $U_i(x)$ *in the representation* (3.2b) *satisfy condition* (3.6). *Then the solution of the boundary value problem and its components in the representation* (3.2) *satisfy estimates* (3.7) *and* (3.10), *where* $K=n+2$.

4. Monotone difference schemes

In this section we give an ε-uniformly convergent difference scheme for problem (2.2), (2.1). We will use solutions of this scheme in order to construct numerical solutions with the increased order of accuracy.

1. On the set $\overline{D}_{(2.1)}$ we introduce the mesh

$$\overline{D}_h = \overline{\omega}_1\times\overline{\omega}_2\times\overline{\omega}_3, \qquad (4.1)$$

where $\overline{\omega}_s$ is a mesh on $[0,d_s]$; in general, the mesh $\overline{\omega}_s$ is *non-uniform*. Suppose that $h_s^i = x_s^{i+1}-x_s^i$, $x_s^i,\ x_s^{i+1}\in\overline{\omega}_s$, $h_s=\max_i h_s^i$, $h=\max_s h_s$. We assume that the condition $h \le M N^{-1}$ holds, where $N = \min[N_1, N_2, N_3]$; $N_s + 1$ is the number of mesh points in $\overline{\omega}_s$.

On the mesh $\overline{D}_{h(4.1)}$, we approximate the boundary value problem (2.2), (2.1) by the difference scheme

$$\Lambda z(x) = f(x), \quad x \in D_h, \quad z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{4.2}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$,

$$\Lambda z(x) \equiv \left\{ \varepsilon \sum_{s=1}^{3} a_s(x) \delta_{\overline{xs}\widehat{xs}} + \sum_{s=1}^{3} b_s(x) \delta_{xs} - c(x) \right\} z(x),$$

$\delta_{xs} z(x)$ and $\delta_{\overline{xs}\widehat{xs}} z(x)$ are the first and second difference derivatives; for example, $\delta_{\overline{x1}\widehat{x1}} z(x) = 2\,(h_1^i + h_1^{i-1})^{-1} [\delta_{x1} z(x) - \delta_{\overline{x1}} z(x)]$, $x = (x_1^i, x_2, x_3)$.

The difference scheme (4.2), (4.1) is monotone [6] ε-uniformly on a mesh with an arbitrary nodal distribution.

In the case of uniform meshes

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2 \times \overline{\omega}_3, \tag{4.3}$$

where $\overline{\omega}_s$ are uniform meshes on $[0, d_s]$, $s = 1, 2, 3$, the convergence for solutions of the difference scheme, by taking account of the *a priori* estimates, is verified under the condition $h = o(\varepsilon)$ with the error estimate

$$|u(x) - z(x)| \le M\, N^{-1} (\varepsilon + N^{-1})^{-1}, \quad x \in \overline{D}_{h(4.3)}. \tag{4.4}$$

2. We now consider a scheme on *piecewise-uniform* meshes.

On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h^* = \overline{\omega}_1^* \times \overline{\omega}_2^* \times \overline{\omega}_3^*. \tag{4.5a}$$

Here $\overline{\omega}_s^*$ is a mesh with *piecewise-constant* step, $s = 1, 2, 3$. To construct the mesh $\overline{\omega}_s^*$ we divide the segment $[0, d_s]$ into two parts $[0, \sigma_s]$ and $[\sigma_s, d_s]$; σ_s is a parameter from the interval $(0, d_s)$. The step-sizes of the mesh are constant on each of the subintervals and equal to $h_s^{(1)} = 2\sigma_s N_s^{-1}$ on $[0, \sigma_s]$ and $h_s^{(2)} = 2(d_s - \sigma_s) N_s^{-1}$ on $[\sigma_s, d_s]$. Assume

$$\sigma_s = \sigma_s(\varepsilon, N_s, d_s; l, m) = \min\,[2^{-1} d_s,\ l m^{-1} \varepsilon \ln N_s], \quad s = 1, 2, 3, \tag{4.5b}$$

where $m = m_{(3.7)}$, $l > 0$ is a parameter of the mesh. The meshes $\overline{\omega}_s^*$, and hence the mesh

$$\overline{D}_h^* = \overline{D}_h^*(l) \tag{4.5c}$$

have been constructed.

To approximate problem (2.2), (2.1), we use scheme (4.2) on the mesh

$$\overline{D}_h^* = \overline{D}_{h(4.5)}^*(l = 1). \tag{4.6}$$

For solutions of the difference scheme (4.2), (4.6) we obtain the estimates

$$|u(x) - z(x)| \le M\, N^{-1} \ln N, \quad x \in \overline{D}_h^*; \tag{4.7}$$

$$|u(x) - z(x)| \le M\, N^{-1} \min\,[\varepsilon^{-1}, \ln N], \quad x \in \overline{D}_h^*. \tag{4.8}$$

Definition. Let the function $z(x)$, $x \in \overline{D}_h$ be a solution of some scheme, and let the following estimate hold for this function:

$$|u(x) - z(x)| \le M\, \mu(N^{-1}, \varepsilon), \quad x \in \overline{D}_h, \tag{4.9}$$

where $\mu(N^{-1}, \varepsilon) \to 0$ as $N \to \infty$ for fixed values of the parameter ε. We say that this estimate is *unimprovable* with respect to the values of N and ε if the estimate

$$|u(x) - z(x)| \le M\, \mu_0(N^{-1}, \varepsilon), \quad x \in \overline{D}_h,$$

in general, fails in that case when $\mu_0(N^{-1}, \varepsilon) = o(\mu(N^{-1}, \varepsilon))$ at least for some values of the parameter ε, $\varepsilon \in (0, 1]$.

Estimates (4.4), (4.8), i.e. ε-dependent estimates for errors of the discrete solutions, are unimprovable with respect to the values of N and ε, and the ε-uniform estimate (4.7) is unimprovable with respect to the values of N.

Theorem 4.1. *Let the solutions of the boundary value problem* (2.2), (2.1) *satisfy a priori estimates* (3.7), (3.10) *for* $K = 3$. *Then the solution of the difference scheme* (4.2), (4.6) *converges to the solution of the boundary value problem at a rate of* $\mathcal{O}(N^{-1} \ln N)$ *ε-uniformly as* $N \to \infty$. *For the discrete solutions the error estimates* (4.4), (4.7) *and* (4.8) *are valid; estimates* (4.4), (4.8) *and* (4.7) *are unimprovable with respect to the values of* N, ε *and* N *respectively.*

5. Necessary condition for improving the accuracy by the Richardson method

1. The Richardson (extrapolation) method to increase accuracy of mesh solutions for regular boundary value problems is fairly well developed in the case of difference schemes on uniform meshes (see, e.g.,[10]). This method uses an expansion of the discrete solution in power series with respect to the step-size of the mesh domain, where the coefficients of this expansion are independent of the value of the step-size. The linear combination (extrapolation) of the mesh

solutions on meshes with different step-sizes, applied in the method, allows us to increase accuracy of the discrete solution.

We mention the Richardson method that was applied in [10] to solve ordinary differential equations with discontinuous coefficients, where finite difference schemes on piecewise uniform meshes were used. The step-sizes of such meshes on those parts where the coefficients are smooth were commensurable.

We consider the Richardson method in the case of problem (2.2), (2.1).

At first, we introduce some definitions. We say that a mesh $\overline{D}_h^0$ is *ε-uniformly dense* on $\overline{D}$, if variations of the function $u(x)$ (i.e. the solution of the boundary value problem (2.2), (2.1)) in neighboring nodes of the mesh $\overline{D}_h^0$ tend to zero ε-uniformly as $N \to \infty$. Let a discrete solution $z^0(x)$ have been constructed on some ε-uniformly dense mesh $\overline{D}_h^0$ by using the Richardson method, and moreover

$$|u(x) - z^0(x)| \le M N^{-\gamma}, \quad x \in \overline{D}_h^0.$$

We say that the function $z^0(x)$, $x \in \overline{D}_h^0$ converges ε-uniformly with the order higher than one if $\gamma > 1$, and strictly higher than one if $\gamma \ge 1 + m_1$.

2. We now describe the Richardson method used to improve accuracy of discrete solutions on the basis of special difference schemes of the form (4.2), (4.5).

In the case of scheme (4.2), (4.5) the mesh region $\overline{D}_h^*$ and the discrete solution $z(x)$, $x \in \overline{D}_h^*$ are defined by the scheme parameters N_1, N_2, N_3 and by the perturbation parameter ε. Associated with this base scheme, it is required to construct "daughter" difference schemes, whose solutions have the same main terms of the expansion with respect to some effective "mesh step-size" as the solution of the base scheme. It is convenient to use the value N^{-1} as the effective mesh step-size. For the daughter difference schemes we introduce the meshes

$$\overline{D}_h^{*k} = \overline{\omega}_1^{*k} \times \overline{\omega}_2^{*k} \times \overline{\omega}_3^{*k}, \tag{5.1}$$

where $\overline{\omega}_s^{*k}$ are *piecewise-uniform* meshes whose step-size on the segments $[0, \sigma_s]$ and $[\sigma_s, d_s]$ is k times larger than the step-size in the basic mesh $\overline{\omega}_{s(4.5)}^*$; $\overline{D}_h^{*k} = \overline{D}_h^{*k}(l)$. In the daughter meshes $\overline{\omega}_s^{*k} = \overline{\omega}_s^{*k}(\sigma_s^0)$, $\sigma_s^0 = \sigma_{s(4.5)}(N_{s(4.5)})$; the value σ_s^0 is independent of k. We denote the solution of problem (4.2), (5.1) by $z^k(x)$, $x \in \overline{D}_h^{*k}$.

For the uniform meshes (4.3) the meshes $\overline{D}_h^k$ are defined in a similar way.

In the case of the base scheme (4.2), (4.6), we use the meshes

$$\overline{D}_h^{*k} = \overline{D}_{h(5.1)}^{*k}(l = 1). \tag{5.2}$$

3. Let us look over requirements that are necessary for using the Richardson method in the case of meshes (4.6), (5.2).

3.1. We consider a model example. Let on the semiaxis $\overline{D}$, where

$$D = (0, \infty), \tag{5.3}$$

it be required to solve the boundary value problem

$$L\,u(x) \equiv \left\{ \varepsilon \frac{d^2}{dx^2} + \frac{d}{dx} \right\} u(x) = 0, \quad x \in D, \quad u(x) = \varphi(x) = 1, \quad x \in \varGamma, \tag{5.4}$$

moreover, $u(x) \to 0$ for $x \to \infty$.

On the set (5.3), we introduce a *uniform* mesh

$$\overline{D}_h \tag{5.5}$$

with the step-size $h = N^{-1}$. To solve problem (5.4), (5.3) we use the difference scheme

$$\begin{aligned} \Lambda\, z(x) &\equiv \{\varepsilon \delta_{\overline{x}\widehat{x}} + \delta_x\}\; z(x) = 0, \quad x \in D_h, \\ z(x) &= \varphi(x), \quad x \in \varGamma; \quad z(x) \to 0 \;\; \text{for} \;\; x \to \infty. \end{aligned} \tag{5.6}$$

Let $z^{k_0}(x)$, $x \in \overline{D}{}_h^{k_0}$ be the solution of problem (5.6) on the mesh

$$\overline{D}{}_h^{k_0} \tag{5.7}$$

whose step-size equals $k_0 h$, where $k_0 > 1$. Let

$$\overline{D}{}_h^0 = \overline{D}_h \cap \overline{D}{}_h^{k_0}. \tag{5.8}$$

Using the functions $z(x)$, $x \in \overline{D}_h$ and $z^{k_0}(x)$, $x \in \overline{D}{}_h^{k_0}$, we construct the discrete solution

$$z^0(x) = \gamma z(x) + (1 - \gamma) z^{k_0}(x), \quad x \in \overline{D}{}_h^0.$$

It is necessary to find the coefficient γ for which the function $z^0(x)$, $x \in \overline{D}{}_h^0$ converges ε-uniformly. Note that the solution of the difference scheme (5.6), (5.5) (as well as scheme (5.6), (5.7)) does not converge ε-uniformly to the solution of problem (5.4), (5.3).

Considerations of explicit solutions to the boundary value problem (5.4), (5.3) and schemes (5.6), (5.5) and (5.6), (5.7) show that there does not exist a parameter γ for which the function $z^0(x)$, $x \in \overline{D}{}_h^0$ converges ε-uniformly as $N \to \infty$.

Thus, in the case of singularly perturbed boundary value problems (2.2), (2.1) and base schemes (4.2), (4.3) (schemes on uniform meshes) the Richardson method does not allow us to construct approximate solutions that converge ε-uniformly. Note that the mesh $\overline{D}^0_{h(5.8)}$ in the case of problem (5.4), (5.3) is not ε-uniformly dense on $\overline{D}$.

3.2. On the unit segment $\overline{D}$, where

$$D = (0,1), \tag{5.9}$$

we consider the problem

$$\begin{aligned} L_{(5.4)}\, u(x) &= 0, \qquad x \in D, \\ u(x) &= \varphi(x), \quad x \in \Gamma, \quad \varphi(0)=1, \quad \varphi(1)=0. \end{aligned} \tag{5.10}$$

On $\overline{D}_{(5.9)}$, we construct the mesh

$$\overline{D}^*_h = \overline{\omega}^*_h, \tag{5.11}$$

where $\overline{\omega}^*_1 = \overline{\omega}^*_{1(4.5)}(\sigma_1)$ for $\sigma_1 = \min[2^{-1},\, l\, m^{-1}\varepsilon \ln N]$, m is an arbitrary number from the interval $(0,1)$, $l=1$. We approximate problem (5.10), (5.9) on the mesh (5.11) by the following difference scheme:

$$\Lambda_{(5.6)}\, z(x) = 0, \quad x \in D^*_h, \quad z(x) = \varphi(x), \quad x \in \Gamma^*_h. \tag{5.12}$$

Let $z^{k_0}(x)$, $x \in \overline{D}^{*k_0}_h$ be the solution of problem (5.12) on the *piecewise-uniform* mesh $\overline{D}^{*k_0}_h$. Using the functions $z(x)$, $x \in \overline{D}^*_h$ and $z^{k_0}(x)$, $x \in \overline{D}^{*k_0}_h$, it is required to construct the discrete solution

$$z^0(x) = \gamma z(x) + (1-\gamma) z^{k_0}(x), \quad x \in \overline{D}^{*0}_h$$

that converges ε-uniformly with the order strictly higher than one. Here $\overline{D}^{*0}_h = \overline{D}^*_h \cap \overline{D}^{*k_0}_h$. Note that the mesh $\overline{D}^{*0}_h$ is ε-uniformly dense on $\overline{D}$ in the case of problem (5.10), (5.9).

The examination of explicit solutions to the boundary value and difference problems shows that the following estimate is attainable for their solutions (by virtue of a choice of $m_{(5.11)}$):

$$u(x),\, z(x),\, z^{k_0}(x) \ge m N^{-1-\alpha} \quad \textit{for} \quad x = \sigma_{1(5.11)} \tag{5.13}$$

for an arbitrarily small value of $\alpha > 0$. Considering the boundary value and discrete problems for $x \ge \sigma_{1(5.11)}$, we justify that there does not exist a function $z^0(x)$, $x \in \overline{D}^{*0}_h$ for which the following estimate is satisfied:

$$\left| u(x) - z^0(x) \right| \le M N^{-1-2\alpha}, \quad x \in \overline{D}^{*0}_h, \quad \alpha = \alpha_{(5.13)}.$$

Thus, in the case of singularly perturbed boundary value problems (2.2), (2.1) and base schemes (4.2), (4.6) the Richardson method does not allow us to construct approximate solutions that converge ε-uniformly with the order strictly higher than 1.

Theorem 5.1. *In the case of the base scheme* (4.2), (4.6) *and the daughter schemes* (4.2), (5.2) *the Richardson method does not allow us to construct discrete solutions with ε-uniform order of accuracy strictly higher than 1 on ε-uniformly dense meshes in $\overline{D}$.*

3.3. Studying schemes (4.2), (4.5) and (4.2), (5.1) under the condition

$$l \geq n, \quad n > 1, \tag{5.14}$$

we arrive to the following statement.

Theorem 5.2. *The condition* (5.14) *is necessary for achieving the order of ε-uniform convergence higher than n (on ε-uniformly dense meshes) for the Richardson method using the base scheme* (4.2), (4.5) *and the associated (daughter) schemes* (4.2), (5.1). *Under the condition $l = n$, the Richardson method does not allow us to construct discrete solutions with ε-uniform order of accuracy strictly higher than n on ε-uniformly dense meshes in $\overline{D}$.*

6. The Richardson method for problem (2.2), (2.1)

1. We will find a discrete solution of the boundary value problem (2.2), (2.1) with the order of accuracy close to two in the form

$$z^0(x) = \gamma z(x) + (1-\gamma) z^{k_0}(x), \quad x \in \overline{D}_h^{*0}, \tag{6.1}$$

where $z(x)$, $x \in \overline{D}_h^{*}$ and $z^{k_0}(x)$, $x \in \overline{D}_h^{*k_0}$ are solutions of the difference schemes (4.2), (4.5) and (4.2), (5.1) under the condition

$$l = 2. \tag{6.2}$$

The coefficient γ in (6.1) is defined by expansions (two first terms) of the functions $z(x)$ and $z^{k_0}(x)$ with respect to the value N^{-1}, where $N = N_{(4.5)}$. We construct these expansions of the functions assuming that the values σ_s, $s = 1, 2, 3$ for the meshes $\overline{D}_h^{*}$ and $\overline{D}_h^{*k_0}$ are the same, $\sigma_s = \sigma_{s(4.5)}(l = 2)$. Note that the leading first terms in the expansions of the functions $z(x)$ and $z^{k_0}(x)$ are the function $u(x)$, i.e. the solution of the boundary value problem (2.2), (2.1).

2. It is convenient to discuss the concept for the construction of expansions with a model example.

2.1. On the segment $\overline{D}$, where

$$D = (0, d_1), \tag{6.3}$$

we consider the boundary value problem

$$\begin{aligned} Lu(x) &\equiv \left\{\varepsilon \frac{d^2}{dx^2} + b(x)\frac{d}{dx} - c(x)\right\} u(x) = f(x), \quad x \in D, \qquad (6.4) \\ u(x) &= \varphi(x), \quad x \in \Gamma; \end{aligned}$$

the functions $b(x)$, $c(x)$, $f(x)$ are sufficiently smooth, moreover, $b(x) \geq b_0 > 0$, $c(x) \geq 0$, $x \in \overline{D}$.

On $\overline{D}$, we construct the *piecewise uniform* mesh

$$\overline{D}_h^* = \overline{\omega}_1^*, \quad l = 2, \tag{6.5}$$

where $\overline{\omega}_1^* = \overline{\omega}_{1(4.5)}^*(\sigma_1)$, $\sigma_1 = \sigma_{1(4.5)}\ (\varepsilon, N_1, d_2; l = 2, m)$, m is an arbitrary number from the interval $(0, m_0)$, $m_0 = \min\limits_{\overline{D}}[b(x)]$.

Problem (6.4), (6.3) is approximated by the difference scheme

$$\begin{aligned} \Lambda z(x) &\equiv \{\varepsilon \delta_{\overline{x}\widehat{x}} + b(x)\delta_x - c(x)\}\, z(x) = f(x), \quad x \in D_h, \qquad (6.6) \\ z(x) &= \varphi(x), \quad x \in \Gamma_h. \end{aligned}$$

We introduce the notation. Let $v(x)$, $x \in \overline{D}$ be a sufficiently smooth function. By $z_v(x)$, $x \in \overline{D}_h$, we denote a solution of the problem

$$\Lambda z(x) = Lv(x), \quad x \in D_h, \quad z(x) = v(x), \quad x \in \Gamma_h.$$

We decompose the solution of problem (6.4), (6.3) into the sum of its regular and singular components:

$$u(x) = U(x) + V(x), \quad x \in \overline{D}. \tag{6.7a}$$

In a similar way, the solution of the discrete problem (6.6), (6.5) can be represented in the form

$$z(x) = z_U(x) + z_V(x), \quad x \in \overline{D}_h^*. \tag{6.7b}$$

2.2. We now find an expansion of the function $z_V(x)$.

For the function $\omega_V(x) = z_V(x) - V(x)$, $x \in \overline{D}_h^*$, i.e. a component of the truncation error for the solution of problem (6.6), (6.5), we have the relation

$$\begin{aligned} &\Lambda\omega_V(x) = (\Lambda - L)V(x) = \\ &= \varepsilon\left(\delta_{\overline{x}\widehat{x}} - \frac{d^2}{dx^2}\right)V(x) + b(x)\left(\delta_x - \frac{d}{dx}\right)V(x), \quad x \in D_h^*, \end{aligned}$$

where

$$\varepsilon\left|\left(\delta_{x\overline{x}}-\frac{d^2}{dx^2}\right)V(x)\right|\le M\varepsilon^{-3}(h_1^{(1)})^2\exp(-m\varepsilon^{-1}x),\qquad x<\sigma_1;$$

$$|(\Lambda-L)V(x)|\le\begin{cases}MN^{-1}, & x=\sigma_1,\\ MN^{-2}(\varepsilon+N^{-1})^{-1}\exp(-m\varepsilon^{-1}(x-\sigma_1)), & x\ge\sigma_1+M_1h_1^{(2)};\end{cases}$$

$$\left|\left\{\left[\delta_x-\frac{d}{dx}\right]-2^{-1}h_1^{(1)}\frac{d^2}{dx^2}\right\}V(x)\right|\le M\varepsilon^{-3}(h_1^{(1)})^2\exp(-m\varepsilon^{-1}x),\ x<\sigma_1;$$

$$x\in D_h^*.$$

Thus, the largest discrepancy of $\omega_V(x)$ is caused by the particular truncation error

$$(\delta_x-\frac{d}{dx})V(x)\approx 2^{-1}h_1^{(1)}\frac{d^2}{dx^2}V(x)$$

on the subdomain $x<\sigma_1$; this component contributes to the total error with an error of order $\varepsilon^{-1}\,h_1^{(1)}$. The remainder part of the truncation error gives a contribution to $\omega_V(x)$ not exceeding the magnitude $M\,\varepsilon^{-2}(h_1^{(1)})^2\le M\,N^{-2}\,\ln^2 N$.

We introduce the function $V_1(x)$, $x\in\overline{D}$, i.e. the solution of the boundary value problem

$$L\,V_1(x)=-\sigma_1 b(x)\frac{d^2}{dx^2}V(x),\quad x\in D,\quad V_1(x)=0,\quad x\in\Gamma. \tag{6.8a}$$

The function $V_1(x)$ satisfies the estimate

$$|V_1(x)|\le M\ln N\exp(-m\varepsilon^{-1}x),\quad x\in\overline{D}, \tag{6.8b}$$

moreover,

$$|(\Lambda-L)V_1(x)|\le M\varepsilon^{-1}N^{-1}\ln^2 N\exp(-m\varepsilon^{-1}x),\quad x\in D_h^*,\quad x<\sigma_1.$$

Thus, for the function $\omega_V(x)-N^{-1}V_1(x)$ we obtain the estimate

$$\left|\Lambda[\omega_V(x)-N^{-1}V_1(x)]\right|\le M\varepsilon^{-1}N^{-2}\ln^2 N\exp(-m\varepsilon^{-1}x),\quad x\in D_h^*,\quad x<\sigma_1.$$

Taking account of the given relations, we find

$$\left|z_V(x)-(V(x)+N^{-1}V_1(x))\right|\le MN^{-2}\ln^2 N,\quad x\in\overline{D}_h^*.$$

Thus, we have the following expansion for the singular component $z_V(x)$:

$$z_V(x)=V(x)+N^{-1}V_1(x)+\rho_V(x),\quad x\in\overline{D}_h^*, \tag{6.9}$$

where $|\rho_V(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}^*_h$.

2.3. An expansion for the regular component $z_U(x)$ in the representation (6.7b) can be constructed in a similar way. We obtain the following expansion:

$$z_U(x) = U(x) + N^{-1} U_1(x) + \rho_U(x), \quad x \in \overline{D}^*_h, \tag{6.10a}$$

which can be verified directly. Here

$$U_1(x) = U_1^1(x) + U_1^2(x), \quad x \in \overline{D}, \tag{6.10b}$$

the functions $U_1^1(x)$ and $U_1^2(x)$ are solutions of the problems

$$L U_1^1(x) = -\sigma_1 b(x) \frac{d^2}{dx^2} U(x), \quad x \in D, \quad U_1^1(x) = 0, \quad x \in \Gamma; \tag{6.11a}$$

$$L U_1^2(x) = \left\{ \begin{array}{ll} -(d_1 - 2\sigma_1) b(x) \dfrac{d^2}{dx^2} U(x), & x < \sigma_1 \\ 0, & x > \sigma_1 \end{array} \right\}, \quad x \in D, \tag{6.11b}$$

$$U_1^2(x) = 0, \quad x \in \Gamma.$$

For the components $U_1^1(x)$, $U_1^2(x)$ and $\rho_U(x)$ we obtain the estimates

$$\left| U_1^1(x) \right| \le M, \quad \left| U_1^2(x) \right| \le M \sigma_1, \quad x \in \overline{D}, \tag{6.10c}$$

$$| \rho_U(x) | \le M N^{-2} \ln N, \quad x \in \overline{D}^*_h.$$

2.4. The expansions (6.9) and (6.10) imply the following expansion for the function $z(x)$, $x \in \overline{D}^*_h$, i.e. the solution of problem (6.6), (6.5):

$$z(x) = u(x) + N^{-1} u_1(x) + \rho_u(x), \quad x \in \overline{D}^*_h, \tag{6.12}$$

where

$$u_1(x) = U_1(x) + V_1(x), \; x \in \overline{D}, \quad \rho_u(x) = \rho_U(x) + \rho_V(x), \; x \in \overline{D}^*_h,$$

moreover,

$$|u_1(x)| \le M \ln N, \quad x \in \overline{D}, \quad |\rho_u(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}^*_h. \tag{6.13}$$

For the function $z^{k_0}(x)$, $x \in \overline{D}^{*k_0}_h$, we obtain the expansion

$$z^{k_0}(x) = u(x) + k_0 N^{-1} u_1(x) + \rho_u^{k_0}(x), \quad x \in \overline{D}^{*k_0}_h, \tag{6.14a}$$

where $u_1(x) = u_{1(6.12)}(x)$, and also

$$|\rho_u^{k_0}(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}^{*k_0}_h. \tag{6.14b}$$

It follows from expansions (6.12), (6.14a) and estimates (6.13), (6.14b) that the function

$$z^0(x) = \gamma z(x) + (1-\gamma) z^{k_0}(x), \quad x \in \overline{D}_h^{*0}, \tag{6.15}$$

where $z(x)$ and $z^{k_0}(x)$ are the solutions of problem (6.6) on the meshes $\overline{D}^*_{h(6.5)}$ and $\overline{D}_h^{*k_0}$, with

$$\gamma = \gamma^{k_0} = k_0(k_0-1)^{-1}, \tag{6.16}$$

satisfies the estimate

$$|u(x) - z^0(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}_h^{*0}. \tag{6.17}$$

Theorem 6.1. *Let the data of the boundary value problem* (6.4), (6.3) *satisfy the condition* $b,\ c,\ f \in C^{4+\alpha}(\overline{D}),\ \alpha > 0$. *Then the function* $z^0_{(6.15)}(x)$, $x \in \overline{D}_h^{*0}$, *i.e. the numerical approximation by the Richardson method based on solutions of the difference scheme* (6.6) *on the meshes* $\overline{D}^*_{h(6.5)}$ *and* $\overline{D}_h^{*k_0}$, *under the conditions* (6.2) *and* (6.16) *converges to the solution of the boundary value problem* (6.4), (6.3) *ε-uniformly at a rate of* $\mathcal{O}(N^{-2}\ln^2 N)$ *as* $N \to \infty$; *for the functions* $z(x)$ *and* $z^{k_0}(x)$ *the expansions* (6.12) *and* (6.14) *are valid, and estimate* (6.17) *holds for the function* $z^0(x)$.

3. We determine a discrete higher-order accurate solution of problem (2.2), (2.1) by the relation

$$z^0(x) = \gamma z(x) + (1-\gamma) z^{k_0}(x), \quad x \in \overline{D}_h^{*0}, \tag{6.18a}$$

where $z(x)$ is the solution of problem (4.2), (4.5), and $z^{k_0}(x)$ is the solution of problem (4.2) on the mesh $\overline{D}_h^{*k_0}$. Here

$$\overline{D}_h^{*0} = \overline{D}_h^* \cap \overline{D}_h^{*k_0}; \quad \overline{D}_h^{*k} = \overline{\omega}_1^{*k} \times \overline{\omega}_2^{*k} \times \overline{\omega}_3^{*k}, \quad \overline{D}_h^{*k} = \overline{D}_h^{*k}(l), \tag{6.18b}$$

where $\overline{\omega}_s^{*k} = \overline{\omega}_{s(5.1)}^{*k}(l), \quad s = 1, 2, 3;$

$$\gamma = \gamma^{k_0} = k_0(k_0-1)^{-1}, \quad l = 2; \tag{6.18c}$$

the value k_0 is, in general, arbitrary, $k_0 > 1$.

We suppose that the data of the boundary value problem (2.2), (2.1) and the components of its solution in the representation (3.2) are sufficiently smooth. Then the function $z^0_{(6.18)}(x)$ satisfies the estimate

$$|u(x) - z^0_{(6.18)}(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}_h^{*0}. \tag{6.19}$$

This estimate can be proved by using expansions for the functions $z(x)$, $x \in \overline{D}_h^*$ and $z^{k_0}(x)$, $x \in \overline{D}_h^{*k_0}$.

4. We now construct expansions for solutions of the difference schemes (4.2), (4.5) and (4.2), (5.1) in the case of condition (6.2).

The decomposition of the continual solution $u(x)$ of the boundary value problem (2.2), (2.1)

$$u(x) = U(x) + \sum_{j=1}^{3} V_{(j)}(x) + \sum_{j=1}^{3} V_{(j,j+1)}(x) + V_{(1,2,3)}(x), \quad x \in \overline{D} \tag{6.20}$$

(see, for example, representation (3.2)) corresponds to the discrete decomposition of the solution $z(x)$ of the finite difference scheme

$$z(x) = z_U(x) + \sum_{j=1}^{3} z_{V_{(j)}}(x) + \sum_{j=1}^{3} z_{V_{(j,j+1)}}(x) + z_{V_{(1,2,3)}}(x), \quad x \in \overline{D}_h^*; \tag{6.21}$$

here $j+1=1$ for $j=3$.

4.1. We consider the singular components from (6.21). The function $z_{V_{(1,2,3)}}(x)$ is represented as a sum of functions

$$z_{V_{(1,2,3)}}(x) = V_{(1,2,3)}(x) + N^{-1} \sum_{i=1}^{3} V_{(1,2,3)i}(x) + \rho_{V_{(1,2,3)}}(x), \quad x \in \overline{D}_h^*, \tag{6.22}$$

$N = N_{(4.5)}$, where $V_{(1,2,3)i}(x)$, $x \in \overline{D}$ are solutions of the boundary value problems

$$\begin{aligned} L_{(2.2)} V_{(1,2,3)i}(x) &= -\sigma_i \, N \, N_i^{-1} \, b_i(x) \, \frac{\partial^2}{\partial x_i^2} \, V_{(1,2,3)}(x), \quad x \in D, \\ V_{(1,2,3)i}(x) &= 0, \quad x \in \Gamma, \quad i = 1,2,3. \end{aligned}$$

The function $z_{V_{(j,j+1)}}(x)$ is decomposed into the sum of functions

$$z_{V_{(j,j+1)}}(x) = V_{(j,j+1)}(x) + N^{-1} \sum_{i=1}^{3} V_{(j,j+1)i}(x) + \rho_{V_{(j,j+1)}}(x), \quad x \in \overline{D}_h^*; \tag{6.23}$$

where the function $V_{(j,j+1)i}(x)$ for $i \neq j,\ j+1$ has the following representation:

$$V_{(j,j+1)i}(x) = \sum_{k=1}^{3} V_{(j,j+1)i}^{k}(x), \quad x \in \overline{D}, \quad i = 1,2,3.$$

The components from (6.23) are restrictions of the functions $V^0_{(j,j+1)}(x)$ and $V^0_{(j,j+1)i}(x)$, $x \in \overline{D}_{(j,j+1)(3.8)}$. These functions $V^0_{(j,j+1)i}(x)$ and their components $V^{k0}_{(j,j+1)i}(x)$ are solutions of the problems

$$L_{(2.2)} V^0_{(j,j+1)i}(x) = -\sigma_i \, N N_i^{-1} \, b_i(x) \frac{\partial^2}{\partial x_i^2} V^0_{(j,j+1)}(x), \quad x \in D_{(j,j+1)},$$

$$V^0_{(j,j+1)i}(x) = 0, \quad x \in \Gamma_{(j,j+1)}, \quad i = j, j+1;$$

$$L_{(2.2)} V^{10}_{(j,j+1)i}(x) = -(d_i - \sigma_i) \, N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} V^0_{(j,j+1)}(x), \quad x \in D_{(j,j+1)},$$

$$V^{10}_{(j,j+1)i}(x) = 0, \quad x \in \Gamma_{(j,j+1)};$$

$$L_i V^{20}_{(j,j+1)i}(x) = \left\{ \begin{array}{ll} (d_i - \sigma_i) N N_i^{-1} b_i(x) \dfrac{\partial^2}{\partial x_i^2} V^0_{(j,j+1)}(x), & x_i < \sigma_i \\ 0, & x_i > \sigma_i \end{array} \right\},$$

$$x \in \overline{D}_{(j,j+1)} \setminus \Gamma_{(j,j+1)i+3},$$

$$V^{20}_{(j,j+1)i}(x) = 0, \quad x \in \Gamma_{(j,j+1)i+3};$$

$$L_{(2.2)} V^{30}_{(j,j+1)i}(x) = -\left\{ L_{(2.2)} - L_i \right\} V^{20}_{(j,j+1)i}(x), \quad x \in D_{(j,j+1)},$$

$$V^{30}_{(j,j+1)i}(x) = 0, \quad x \in \Gamma_{(j,j+3)i+3}; \quad i \neq j, j+1, \quad i = 1, 2, 3.$$

Here $\quad V^0_{(j,j+1)}(x) = V^0_{(j,j+1)(3.8)}(x), \quad x \in \overline{D}_{(j,j+1)},$

$$L_i \equiv \varepsilon a_i(x) \frac{\partial^2}{\partial x_i^2} + b_i(x) \frac{\partial}{\partial x_i} - c_i(x), \quad c(x) = \sum_{i=1}^{3} c_i(x), \quad i = 1, 2, 3.$$

The function $z_{V_{(j)}}(x)$ has the expansion

$$z_{V_{(j)}}(x) = V_{(j)}(x) + N^{-1} \sum_{i=1}^{3} V_{(j)i}(x) + \rho_{V_{(j)}}(x), \quad x \in \overline{D}^*_h, \tag{6.24}$$

moreover, $V_{(j)i}(x)$ for $i \neq j$ has the representation

$$V_{(j)i}(x) = \sum_{k=1}^{3} V^k_{(j)i}(x), \quad x \in \overline{D}, \quad i = 1, 2, 3.$$

The components from (6.24) are restrictions on $\overline{D}$ of the corresponding components that are solutions of the problems

$$L_{(2.2)}V^0_{(j)i}(x) = -\sigma_i N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} V^0_{(j)}(x), \quad x \in D_{(j)},$$

$$V^0_{(j)i}(x) = 0, \quad x \in \Gamma_{(j)}, \quad i = j;$$

$$L_{(2.2)}V^{10}_{(j)i}(x) = -(d_i - \sigma_i) N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} V^0_{(j)}(x), \quad x \in D_{(j)},$$

$$V^{10}_{(j)i}(x) = 0, \quad x \in \Gamma_{(j)};$$

$$L_i V^{20}_{(j)i}(x) = \left\{ \begin{matrix} (d_i - \sigma_i) N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} V^0_{(j)}(x), & x_i < \sigma_i \\ 0, & x_i > \sigma_i \end{matrix} \right\}, \ x \in \overline{D}_{(j)} \setminus \Gamma_{(j)i+3},$$

$$V^{20}_{(j)i}(x) = 0, \quad x \in \Gamma_{(j)i+3};$$

$$L_{(2.2)}V^{30}_{(j)i}(x) = -\left\{ L_{(2.2)} - L_i \right\} V^{20}_{(j)i}(x), \quad x \in D_{(j)},$$

$$V^{30}_{(j)i}(x) = 0, \quad x \in \Gamma_{(j)}; \quad i \neq j, \quad i = 1, 2, 3,$$

where $V^0_{(j)}(x) = V^0_{(j)(3.8)}(x), \ x \in \overline{D}_{(j)}$.

4.2. The regular component $z_U(x)$ has the expansion

$$z_U(x) = U(x) + N^{-1} \sum_{i=1}^{3} U_i(x) + \rho_U(x), \quad x \in \overline{D}^*_h, \tag{6.25}$$

$$U_i(x) = \sum_{k=1}^{3} U_i^k(x), \quad x \in \overline{D}, \quad i = 1, 2, 3,$$

where $U_i^k(x)$, $x \in \overline{D}$ are restrictions of the functions $U_i^{k0}(x)$, $x \in \overline{D}^0_{(3.8)}$. The functions $U_i^{k0}(x)$ can be found by solving the problems

$$\begin{aligned} L_{(2.2)}U_i^{10}(x) &= -(d_i - \sigma_i) N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} U^0(x), \quad x \in D^0, \\ U_i^{10}(x) &= 0, \quad x \in \Gamma^0; \\ L_i U_i^{20}(x) &= \left\{ \begin{matrix} (d_i - 2\sigma_i) N N_i^{-1} b_i(x) \frac{\partial^2}{\partial x_i^2} U^0(x), & x_i < \sigma_i \\ 0, & x_i > \sigma_i \end{matrix} \right\}, \quad x \in \overline{D}^0 \setminus \Gamma_i^0, \\ U_i^{20}(x) &= 0, \quad x \in \Gamma_i^0; \\ L_{(2.2)}U_i^{30}(x) &= -\left\{ L_{(2.2)} - L_i \right\} U_i^{20}(x), \quad x \in D^0, \\ U_i^{30}(x) &= 0, \quad x \in \Gamma^0, \end{aligned}$$

where $U_i^0(x) = U_{i(3.5)}^0(x), \quad x \in \overline{D}^0$.

4.3. The expansions (6.22)–(6.25) for the components from the representation (6.21) imply the following expansion for the function $z(x)$, $x \in \overline{D}_h^*$:

$$z(x) = u(x) + N^{-1}[u_0(x) + u_1(x)] + \rho_u(x), \quad x \in \overline{D}_h^*, \tag{6.26}$$

where

$$u_0(x) = \sum_{i=1}^{3} \left[U_i^1(x) + V_{(i)i}(x) + V_{(i,i+1)i}(x) + V_{(i-1,i)i}(x) + V_{(1,2,3)i}(x)\right],$$

$$u_1(x) = \sum_{i=1}^{3} \left[U_i^2(x) + U_i^3(x) + V_{(i)i+1}(x) + V_{(i)i+2}(x) + V_{(i,i+1)i+2}(x)\right], \quad x \in \overline{D};$$

$$\rho_u(x) = \rho_U(x) + \sum_{j=1}^{3} \left[\rho_{V_{(j)}}(x) + \rho_{V_{(j,j+1)}}(x)\right] + \rho_{V_{(1,2,3)}}(x), \quad x \in \overline{D}_h^*.$$

For the functions $u_0(x)$, $u_1(x)$ and $\rho_u(x)$ we obtain the estimates

$$|u_0(x)| \le M \ln N, \quad |u_1(x)| \le M\varepsilon \ln N, \quad x \in \overline{D};$$
$$|\rho_u(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}_h^*.$$

The function $z^{k_0}(x)$, $x \in \overline{D}_h^{*k_0}$ has the expansion

$$z^{k_0}(x) = u(x) + N^{-1}[u_0(x) + u_1(x)] + \rho_u^{k_0}(x), \quad x \in \overline{D}_h^{*k_0}, \tag{6.27}$$

where $u_i(x) = u_{i(6.26)}(x)$, $i = 1, 2$, and also $|\,\rho_u^{k_0}(x)| \le MN^{-2} \ln^2 N$, $x \in \overline{D}_h^{*k_0}$.

Estimate (6.19) follows from expansions (6.26) and (6.27).

Theorem 6.2. *Let the solutions of the boundary value problem* (2.2), (2.1) *satisfy a priori estimates* (3.7) *for* $K = 7$. *Then the function* $z_{(6.18)}^0(x)$, $x \in \overline{D}_h^{*0}$, *i.e. the approximation by the Richardson method based on solutions of the difference scheme* (4.2) *on the meshes* $\overline{D}_{h(4.5)}^*$ *and* $\overline{D}_{h(6.18)}^{*k_0}$, *under condition* (6.18c) *converges to the solution of the boundary value problem* (2.2), (2.1) ε*-uniformly at a rate of* $\mathcal{O}(N^{-2} \ln^2 N)$ *as* $N \to \infty$; *for the functions* $z(x)$, $x \in \overline{D}_h^*$ *and* $z^{k_0}(x)$, $x \in \overline{D}_h^{*k_0}$ *the expansions* (6.26) *and* (6.27) *are valid, and estimate* (6.19) *holds for the function* $z_{(6.18)}^0(x)$, $x \in \overline{D}_h^{*0}$.

7. Remarks and generalizations

1. In the case of the condition

$$\varepsilon \le M N^{-1}, \tag{7.1}$$

expansions (6.26) and (6.27) are essentially simplified. For the functions $z(x)$ and $z^{k_0}(x)$ the following expansions are valid:

$$\begin{aligned} z\,(x) &= u(x) + N^{-1} u_0(x) + \rho_u(x), \qquad x \in \overline{D}{}^*_h, \\ z^{k_0}(x) &= u(x) + k_0 N^{-1} u_0(x) + \rho_u^{k_0}(x), \quad x \in \overline{D}{}^{*k_0}_h, \end{aligned}$$

where $u_0(x) = u_{0(6.26)}(x)$, moreover,

$$|\rho_u(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}{}^*_h, \qquad |\rho_u^{k_0}(x)| \le M N^{-2} \ln^2 N, \quad x \in \overline{D}{}^{*k_0}_h.$$

2. The given techniques for constructing and justifying ε-uniformly convergent difference schemes of increased accuracy is directly applicable to problem (2.2) on an unbounded tube domain $\overline{D}$, where

$$\overline{D} = D \cup \Gamma, \quad D = \{x : \ 0 < x_s < d_s, \ x_3 \in R, \quad s = 1, 2\}. \tag{7.2}$$

We denote the faces of D orthogonal to the x_j-axis by Γ_j and Γ_{j+2}, $j = 1, 2$; the faces Γ_1 and Γ_2 contain the axis $x_1 = x_2 = 0$; $\Gamma = \bigcup_{j=1}^4 \Gamma_j$; we set $\Gamma^- = \bigcup_{j=1,2} \Gamma_j$.

When the parameter ε tends to zero, a boundary layer appears in a neighbourhood of the set Γ^-. This layer is corner in the neighbourhood of the intersection of the faces Γ_1 and Γ_2.

To solve problem (2.2), (7.2), we use the monotone scheme (4.2), (4.1), where the mesh $\overline{\omega}_3$ in (4.1) is as follows

$$\overline{\omega}_3 \ \ \textit{is a mesh on the } x_3\textit{-axis}, \tag{7.3}$$

with $N_3 + 1$ being the maximal number of nodes in the mesh $\overline{\omega}_3$ on the unit interval. In the case of piecewise-uniform meshes condensing in the boundary layer, we consider the mesh $\overline{\omega}{}^*_3$ in (4.5) as

$$\overline{\omega}{}^*_3 \ \ \textit{is the uniform mesh}. \tag{7.4}$$

If the data of problem (2.2), (7.2) are sufficiently smooth, and under suitable compatibility conditions on the faces Γ^*, the solutions of the difference schemes (4.2), (4.1), (7.3) and (4.2), (4.5), (7.4) satisfy estimates (4.4) and (4.7), (4.8) respectively.

When constructing approximate solutions of increased accuracy, we use the Richardson method. The numerical solution is determined by the relation (see (6.18))

$$z^0(x) = \gamma\, z(x) + (1-\gamma)\, z^{k_0}(x), \quad x \in \overline{D}_h^{*0}, \tag{7.5}$$

where $z(x)$ is the solution of problem (4.2), (4.5), (7.4), and $z^{k_0}(x)$ is the solution of problem (4.2) on the mesh $\overline{D}^{*0}_{h(6.18b)}$; the coefficient γ satisfies condition (6.18c). Note that the step-size of the mesh $\overline{\omega}_3^{*k}$ in (6.18b) is k times larger than that of the mesh (7.4).

It is not difficult to find conditions imposed on the data of the boundary value problem (they are similar to the conditions given in Theorem 6.2) under which the following estimate (similar to estimate (6.19)) holds for the function $z^0_{(7.5)}(x)$:

$$|u(x) - z^0_{(7.5)}(x)| \le M\, N^{-2} \ln^2 N, \quad x \in \overline{D}_h^{*0}. \tag{7.6}$$

Estimate (7.6) can be justified similarly to estimate (6.19).

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