Domain Decomposition Method for a Singularly Perturbed Quasilinear Parabolic Convection-Diffusion Equation

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An initial boundary value problem of convection-diffusion type for a singularly perturbed quasilinear parabolic equation is considered on an interval. For this problem we construct ε-uniformly convergent difference schemes (nonlinear iteration-free schemes and their iterative variants) based on the domain decomposition method, which allow us to implement sequential and parallel computations on decomposition subdomains. Such schemes are obtained by domain decomposition applied to an ε-uniformly convergent nonlinear base scheme, which is a classic difference approximation of the differential problem on piecewise uniform meshes condensing in a boundary layer. The decomposition schemes constructed in this paper converge ε-uniformly at the rate of $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$, where N and N_0 denote respectively the number of mesh intervals in the space and time discretizations.

Key words: quasilinear parabolic PDEs, convection-diffusion, difference schemes, ε-uniform convergence, parallel algorithms, Schwartz-like domain decomposition methods.

1 Introduction

Special ε-uniformly convergent difference schemes for sufficiently wide classes of linear singularly perturbed elliptic and parabolic equations were constructed, see e.g., [1]–[6]. Numerical methods based on the domain decomposition technique were constructed and investigated for a number of linear boundary value and initial boundary value problems. Such methods allows us to apply sequential and parallel computations on the decomposition subdomains and, moreover, the convergence rate of these methods does not depend on the value of the parameter ε (see, e.g., [6]–[8]). Note that the use of parallel computations gives a possibility to accelerate the process of the numerical solution of the problem. In [7] the conditions are determined under which the parallelization of a difference scheme leads to the acceleration of the numerical solution of the singularly perturbed parabolic equation, and also the ε-uniform accuracy for the decomposition scheme is not less than the accuracy of the base scheme, i.e., a scheme subjected to the decomposition.

Attempts of developing ε-uniformly convergent numerical methods for nonlinear singularly perturbed equations have a fragmentary character (see, e.g., [9]–[12]). Thus, the development of ε-uniformly convergent numerical domain decomposition methods for nonlinear singularly perturbed equations is indeed the actual problem.

In this paper special ε-uniformly convergent finite difference schemes of the domain decomposition method are developed for an initial boundary value problem for a quasilinear singularly perturbed parabolic convection-diffusion equation on an interval. Nonlinear domain decomposition schemes and their iterative variants are constructed on the basis of a nonlinear difference scheme that is a classic grid approximation of the problem on piecewise uniform meshes. The constructed decomposition schemes allow us to use both sequential and parallel computations on subdomains. The rate of ε-uniform convergence for such schemes is $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$, i.e., the same as for schemes for linear equations; here N and N_0 denote the number of mesh intervals in the space and time discretizations respectively.

2 Problem formulation

In the domain

$$G = D \times (0, T], \quad D = (0, d) \tag{2.1}$$

with boundary $S = \overline{G} \setminus G$ we consider the following initial boundary value problem for a singularly perturbed parabolic equation with Dirichlet boundary conditions [1]

$$L_{(2.2)} u(x,t) \equiv \left\{ \varepsilon a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) =$$

$$= F(x,t,\, u(x,t)), \quad (x,t) \in G, \tag{2.2a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2b}$$

For $S = S_0 \cup S^L$, we distinguish the initial boundary $S_0 = \{(x,t) :\ x \in [0,1],\ t = 0\}$ and the lateral boundary $S^L = \{(x,t) :\ x = 0 \text{ or } x = d,\ 0 < t \le T\}$. In (2.2) $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $(x,t) \in \overline{G}$, $F(x,t,u)$, $(x,t,u) \in \overline{H}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions which satisfy [2]

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad p_0 \le p(x,t) \le p^0, \tag{2.2c}$$

$$0 \le c(x,t) \le c^0, \quad (x,t) \in \overline{G}, \quad a_0,\, b_0,\, p_0 > 0;$$

$$|\varphi(x,t)| \le M, \quad (x,t) \in S; \quad |F(x,t,\, u)| \le M, \quad (x,t,u) \in \overline{H};$$

where $\overline{H} = \overline{G} \times R$. The real parameter ε may take any positive value

$$\varepsilon \in (0,1]. \tag{2.2d}$$

For simplicity, the following condition is assumed to be fulfilled:

$$\frac{\partial}{\partial u} F(x,t,u) + c(x,t) \ge 0, \quad \frac{\partial^2}{\partial u^2} F(x,t,u) \le M, \quad (x,t,u) \in \overline{H}. \tag{2.3}$$

[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in equation $(j.k)$.

[2] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters.

The solution of problem (2.2), (2.1) exists and is unique for all values of $\varepsilon \in (0,1]$.

Let the lateral boundary S^L be represented as $S^L = S_1 \cup S_2$, where S_1 and S_2 are respectively the left and right sides of the boundary to the domain G. When the parameter ε tends to zero, a regular boundary layer appears in a neighborhood of the boundary S_1. This layer is described by an ordinary differential equation.

Our aim is to construct ε-uniformly convergent finite difference schemes of the domain decomposition method for problem (2.2), (2.1). In the case of iterative schemes we require that the number of iterations should be also independent of ε.

3 The difference scheme

To solve problem (2.2), (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ is the (possibly) non-uniform grid of nodal points x^i on $[0,1]$, $\overline{\omega}_0$ is a uniform grid on the interval $[0,T]$; N and N_0 are the numbers of intervals in the grids $\overline{\omega}$ and $\overline{\omega}_0$ respectively. We define $\tau = T/N_0$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \le M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

For problem (2.2), (2.1) we use the difference scheme [13]

$$\Lambda_{(3.2)} z(x,t) = F(x,t,z(x,t)), \qquad (x,t) \in G_h, \tag{3.2a}$$

$$z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h, \tag{3.2b}$$

where

$$\begin{aligned}
\Lambda_{(3.2)}\, z(x,t) &\equiv \{\varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}\}\; z(x,t), \\
\delta_{\overline{x}\widehat{x}}\, z(x^i,t) &= 2(h^{i-1} + h^i)^{-1} \left[\delta_x\, z(x^i,t) - \delta_{\overline{x}}\, z(x^i,t)\right], \\
\delta_x\, z(x^i,t) &= (h^i)^{-1} \left(z(x^{i+1},t) - z(x^i,t)\right), \\
\delta_{\overline{x}}\, z(x^i,t) &= (h^{i-1})^{-1} \left(z(x^i,t) - z(x^{i-1},t)\right), \\
\delta_{\overline{t}}\, z(x^i,t) &= \tau^{-1} \left(z(x^i,t) - z(x^i,t-\tau)\right),
\end{aligned}$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, and the difference $\delta_{\overline{x}\widehat{x}} z(x,t)$ is an approximation of the derivative $(\partial^2/\partial x^2) u(x,t)$ on a non-uniform mesh.

The difference scheme (3.2), (3.1) is monotone [13]. It is convenient to introduce a nonlinear operator associated with equation (3.2a):

$$\Lambda_{(3.3)}(z(x,t)) \equiv \Lambda_{(3.2)} z(x,t) - F(x,t,z(x,t)), \quad (x,t) \in G_h. \tag{3.3}$$

The following theorem is valid.

Theorem 3.1 *Let the conditions*

$$\begin{aligned}
\Lambda_{(3.3)}(z^1(x,t)) \;&\le\; \Lambda_{(3.3)}(z^2(x,t)), \quad (x,t) \in \overline{G}_h, \\
z^1(x,t) \;&\ge\; z^2(x,t), \qquad (x,t) \in S_h
\end{aligned}$$

be satisfied by the functions $z^i(x,t)$, $(x,t) \in \overline{G}_h$, $i = 1,2$. *Then* $z^1(x,t) \ge z^2(x,t)$, $(x,t) \in \overline{G}_h$.

Using the comparison theorem and taking into account *a-priori* estimates for the derivatives (see Theorem 8.1 in Section 8), we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε:

$$| \, u(x,t) - z(x,t) \, | \leq M \, \left[(\varepsilon^{-2} + N^{-1})^{-1} N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h. \tag{3.4}$$

On the mesh

$$\overline{G}_h = \overline{\omega}^u \times \overline{\omega}_0, \tag{3.5}$$

which is uniform with respect to x and t, we have the estimate

$$| \, u(x,t) - z(x,t) \, | \leq M \, \left[(\varepsilon^{-1} + N^{-1})^{-1} N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h. \tag{3.6}$$

The estimate (3.4), (3.6) are established according to the classical convergence proof for monotone difference schemes [6], [13].

Theorem 3.2 *Assume that estimate* (8.2), *where* $n = 0$, *holds for the solution of problem* (2.2). *Then, for a fixed value of the parameter* ε, *the solution of scheme* (3.2) *on meshes* (3.1) *and* (3.5) *converges to the solution of* (2.2) *with error bounds given by* (3.4) *and* (3.6), *respectively.*

4 The ε-uniformly convergent method

In this section we discuss an ε-uniformly convergent method for (2.2) by taking a special mesh condensed in a neighborhood of the boundary layer. The distribution of the nodes is derived from *a priori* estimates of the solution and its derivatives. We follow the approach described in [2], [4], [6], [14], i.e., we take

$$\overline{G}_h^* = \overline{\omega}^*(\sigma) \times \overline{\omega}_0 \;, \tag{4.1}$$

where $\overline{\omega}_0$ is the uniform mesh with step-size $\tau = T N_0^{-1}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter σ. We take $\sigma = \sigma(\varepsilon, N) = \min[\, 2^{-1}, \, m \, \varepsilon \ln N \,]$, where $m = m_{(8.7)}^{-1}$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[\,0,1\,]$ is divided in two parts $[\,0,\sigma\,]$, $[\,\sigma,1\,]$. In each part we use a uniform grid, with $N/2$ subintervals in each interval $[\,0,\sigma\,]$ and $[\,\sigma,1\,]$.

Using the maximum principle and taking into account *a-priori* estimates, similarly to [6], [15] we find

$$| \, u(x,t) - z(x,t) \, | \leq M \, \left[N^{-1} \ln N + N_0^{-1} \right], \quad (x,t) \in \overline{G}^*_{h(4.1)}. \tag{4.2}$$

Thus, the difference scheme (3.2), (4.1) converges ε-uniformly.

Theorem 4.1 *If the solution of problem* (2.2) *satisfies the hypotheses of Theorem* 8.1 *(see Section* 8*), where* $n = 0$, *then the solution of* (3.2), (4.1) *converges* ε*-uniformly to the solution of* (2.2) *and the estimate* (4.2) *is valid.*

5 Schwartz method for parabolic equations

In this section we introduce Schwartz' domain decomposition method for the boundary value problem (2.2), and for the solutions obtained we give the necessary and sufficient conditions for ε-uniform convergence.

5.1. We first describe Schwartz' classical method for problem (2.2). Let the set of open subdomains

$$D^k, \quad k = 1, \dots, \quad K \tag{5.1a}$$

with boundaries Γ^k, $\Gamma^k = \Gamma(D^k) = \overline{D}^k \setminus D^k$, cover the domain D: $D = \bigcup_{k=1}^{K} D^k$, and let

$$G^k = D^k \times (0, T], \quad k = 1, \dots, K. \tag{5.1b}$$

We denote by $D^{[k]}$ the union of the subdomains $D^1, \dots, D^K$ which does not include the set D^k:

$$D^{[k]} = \bigcup_{i=1,\, i\neq k}^{K} D^i\,. \tag{5.1c}$$

We denote the minimal overlap of the sets D^k and $D^{[k]}$ by δ^k, and by δ the smallest value of δ^k, i.e.,

$$\begin{gathered}\min_{k,\,x^1,\,x^2} \ \rho(x^1, x^2) = \delta, \\ x^1 \in \overline{D}^k, \quad x^2 \in \overline{D}^{[k]}, \quad x^1,\ x^2 \notin \{\ D^k \cap D^{[k]}\ \}, \quad k = 1, \dots, K,\end{gathered} \tag{5.2}$$

where $\rho(x^1, x^2)$ is the distance between the points x^1, $x^2 \in \overline{D}$. In general, the value δ may depend on the parameter ε.

Let

$$u^0(x, t), \quad (x, t) \in \overline{G} \tag{5.3a}$$

be a given arbitrary function satisfying the condition (2.2b). We are to find the sequence of the functions $u^r(x, t)$, $(x, t) \in \overline{G}$, $r = 1, 2, \dots$. Let the function $u^r(x, t)$ be known. The function $u^{r+1}(x, t)$ can be determined in the next way. First we find the functions $u^{r+\frac{k}{K}}(x, t)$, that is, the solutions of the following problems

$$\begin{aligned} L_{(5.4)}(u^{r+\frac{k}{K}}(x, t)) &= 0, \qquad (x, t) \in G^k, \\ u^{r+\frac{k}{K}}(x, t) &= u^{r+\frac{k-1}{K}}(x, t), \qquad (x, t) \in \overline{G} \setminus G^k, \quad k = 1, \dots, K. \end{aligned} \tag{5.3b}$$

The required function is defined by the relation

$$u^{r+1}(x, t) = u^{r+\frac{K}{K}}(x, t), \quad r = 0, 1, 2, \dots. \tag{5.3c}$$

In the case of the boundary value problem (2.2) the operator $L_{(5.4)}$ in (5.3b) is defined

$$L_{(5.4)}(u(x, t)) \equiv L_{(2.2)}u(x, t) - F(x, t,\, u(x, t)), \quad (x, t) \in G. \tag{5.4}$$

Each function $u^{r+\frac{k}{K}}(x, t)$, $(x, t) \in \overline{G}$, is the solution of the Dirichlet problem on the set $\overline{G}^k$ and coincides with the function $u^{r+\frac{k-1}{K}}(x, t)$ on the set $\overline{G} \setminus G^k$. This process is a natural classical Schwartz 'alternating' method.

In principle, we could give the conditions under which process (5.3), (5.4), (5.1) converges to the solution of the boundary value problem (2.2) as $r \to \infty$, where r is the number of iterations. However, in this paper we are interested in a non-iterative solver based on the modified Schwartz method.

5.2. We now describe the modified Schwartz method. Let

$$\overline{\omega}_0 \tag{5.5a}$$

be a uniform grid, just like $\overline{\omega}_{0(3.1)}$, on [0,T] with stepsize τ. By $G(t_1)$ we denote the strip

$$G(t_1) = \{\,(x,t):\ (x,t) \in G,\ t_1 < t \le t_1 + \tau\,\},\quad t_1, t_1+\tau \in \overline{\omega}_0.$$

Let $S(t_1) = \overline{G}(t_1) \setminus G(t_1)$ be the boundary of $G(t_1)$ and let $v(x,t) = v(x,t;t_1)$ be defined on $S(t_1)$. We denote an extension of the function $v(x,t)$ onto the whole set $\overline{G}(t_1)$ by $\overline{v}(x,t;t_1)$. The function $\overline{v}(x,t;t_1)$ is assumed to satisfy the Lipschitz condition with respect to t. We subdivide the strip $G(t_1)$ into sections $G^k(t_1) = G^k \cap G(t_1)$, $S^k(t_1) = \overline{G}^k(t_1) \setminus G^k(t_1)$.

Suppose that the function $u(x,t)$, $(x,t) \in \overline{G}$, for $t^n \in \overline{\omega}_0$, $t \le t^n < T$, $n = 0,1,\dots,N_0-1$, has already been constructed. Now we construct the function $u(x,t)$ for $t \le t^{n+1}$, i.e., we find the function $u(x,t)$ on the strip $G(t^n)$. This is done in the following way. First we find the functions $u^{\frac{k}{K}}(x,t)$ on the sections $\overline{G}^k(t^n)$ for $k = 1,\dots,K$, solving the boundary value problems

$$\left.\begin{array}{ll} L_{(5.4)}(u^{\frac{k}{K}}(x,t)) = 0, & (x,t) \in G^k(t^n) \\ u^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} \overline{u}(x,t;t^n), & k=1 \\ u^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array}\right\}, & (x,t) \in S^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}^k(t^n), \tag{5.5b}$$
$$k = 1,\dots,K;\quad t^n \in \overline{\omega}_0,\quad n \le N_0 - 1.$$

Here having $u^{\frac{k}{K}}(x,t)$ on $\overline{G}^k(t^n)$, we extend these functions for each value k onto the whole strip $\overline{G}(t^n)$ in the next way

$$u^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} u^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}^k(t^n) \\ \left.\begin{array}{ll} \overline{u}(x,t;t^n), & k=1 \\ u^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array}\right\}, & (x,t) \in \overline{G}(t^n) \setminus \overline{G}^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}(t^n), \tag{5.5c}$$
$$k = 1,\dots,K,\quad t^n \in \overline{\omega}_0.$$

Having $u^{\frac{k}{K}}(x,t)$, for $k = K$ we define the function $u(x,t)$ on the whole strip $\overline{G}(t^n)$ by

$$u(x,t) = u^{\frac{K}{K}}(x,t),\quad (x,t) \in \overline{G}(t^n),\quad t^n \in \overline{\omega}_0. \tag{5.5d}$$

Thereby we have the function $u(x,t)$ on the domain $\overline{G}$ for $t \in [0, t^{n+1}]$.

In the relations (5.5b), (5.5c) the function $\overline{u}(x,t;t^n)$ is constructed on the base of the function $\overline{v}(x,t;t^n)$

$$\overline{u}(x,t;t^n) = \overline{v}(x,t;t^n),\quad (x,t) \in \overline{G}(t^n). \tag{5.5e}$$

Using $v(x,t;t^n)$ which is defined on the boundary $S(t^n)$ in (5.5g), we find the function

$$\overline{v}(x,t;t^n),\qquad (x,t) \in \overline{G}(t^n), \tag{5.5f}$$

$$\begin{array}{lrcll} \text{supposing} & \overline{v}(x,t;t^n) & = & v\,(x,t\,;\,t^n) & \text{for}\quad (x,t) \in S(t^n) \\ \text{and} & \overline{v}(x,t;t^n) & = & v\,(x,t^n;\,t^n) & \text{for}\quad (x,t) \in G(t^n). \end{array}$$

Here

$$v(x,t;t^n) = \begin{cases} \varphi(x,t), & (x,t) \in S(t^n), & t^n = t^0 = 0, \\ \left.\begin{array}{lll} \varphi(x,t), & (x,t) \in S(t^n) \cap S, & t \ge t^n \\ u(x,t), & (x,t) \in S(t^n) \setminus S, & t = t^n \end{array}\right\}, & & t^n > 0, \quad (x,t) \in S(t^n), \end{cases} \tag{5.5g}$$

$$n = 0, 1, \ldots, N_0 - 1.$$

Thus the function $\overline{u}(x,t;t^n)$ on $\overline{G}(t^n)$ have been constructed.

The function $u^{\frac{k}{K}}(x,t)$ on each strip $\overline{G}(t^n)$ is the solution of the Dirichlet problem on the section $\overline{G}^k(t^n)$, whereas on the set $\overline{G}(t^n) \setminus G^k(t^n)$ it coincides with the function $\overline{u}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$ for $k = 1$, and with the function $u^{\frac{k-1}{K}}(x,t)$, $(x,t) \in \overline{G}(t^n)$ for $k \ge 2$. Thus we find the function $u(x,t)$, $(x,t) \in \overline{G}$, the solution of process (5.5), (5.4), (5.1), which we call the modified Schwartz method.

Note that the process (5.5), (5.4), (5.1), "the modified Schwartz method" is not an iterative process in the strict sense. The boundary value problems in (5.5), (5.4), (5.1) are solved only once at those points of $\overline{G}$ which do not belong to the intersection of the subdomains. The boundary value problem is solved twice only on the intersection of the subdomains.

In the continuous domain decomposition method (5.5), (5.4), (5.1) the intermediate problems on the subsets $\overline{D}^k_{(5.1)}$, $k = 1, \ldots, K$ are solved sequentially.

Using the comparison theorems [16], [17], we come to the estimate

$$|u(x,t) - u_{(5.5)}(x,t)| \le Q(\varepsilon, \delta) N_0^{-1}, \qquad (x,t) \in \overline{G},$$

where $u_{(5.5)}(x,t)$ is the solution of the process (5.5), (5.4), (5.1), $\delta = \delta_{(5.2)}(\varepsilon)$, i.e., the function $u_{(5.5)}(x,t)$ converges, as $N_0 \to \infty$, to the solution of boundary value problem (2.2) for each fixed value of the parameter ε. Note that the function $u_{(5.5)}(x,t)$ for $\delta = 0$ does not converge to the solution of boundary value problem (2.2) as $N_0 \to \infty$. Under the condition

$$\delta = \delta_{(5.2)}(\varepsilon) > 0, \quad \varepsilon \in (0,1], \qquad \inf_{\varepsilon \in (0,1]} [\, \varepsilon^{-1} \delta_{(5.2)}(\varepsilon) \,] > 0 \tag{5.6}$$

which is equivalent to the condition $\delta = \delta_{(5.2)}(\varepsilon) \ge m_{(5.6)}\varepsilon$, $\varepsilon \in (0,1]$, the function $u_{(5.5)}(x,t)$ converges ε-uniformly as $N_0 \to \infty$:

$$|u(x,t) - u_{(5.5)}(x,t)| \le M N_0^{-1}, \quad (x,t) \in \overline{G}. \tag{5.7}$$

If condition (5.6) is violated and the value δ satisfies the condition

$$\delta = \delta_{(5.2)}(\varepsilon) > 0, \quad \varepsilon \in (0,1], \qquad \inf_{\varepsilon \in (0,1]} [\varepsilon^{-1} \; \delta_{(5.2)}(\varepsilon)] = 0, \tag{5.8}$$

the function $u_{(5.5)}(x,t)$ does not converge ε-uniformly.

5.3. Here we describe the continuous variant of the modified Schwartz method that allows parallel computations on $P \ge 1$ processors.

Let D^k, $k = 1, \ldots, K$ be the subdomains from (5.1a) and let each D^k be partitioned in P disjoint (possibly empty) parts

$$D^k = \bigcup_{p=1}^{P} D^k_p, \quad k = 1, \ldots, K, \quad \overline{D}^k_i \cap \overline{D}^k_j = 0, \quad i \ne j. \tag{5.9a}$$

Here we assume that non-empty D_p^k do overlap, but generally D^k don't. We set

$$G_p^k = D_p^k \times (0,T], \quad p=1,\ldots,P, \quad k=1,\ldots,K. \tag{5.9b}$$

We find the function $u(x,t)$ by solving the problems (5.10) similar to (5.5), but now on the set $\overline{G}_p^{\,k}(t^n)$ instead of $\overline{G}^{\,k}(t^n)$:

$$L_{(5.4)}(u_p^{\frac{k}{K}}(x,t) = 0, \quad (x,t) \in G_p^k(t^n), \tag{5.10a}$$

$$u_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{u}(x,t;t^n), & k=1 \\ u^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S_p^k(t^n), \quad p=1,\ldots,P$$

$$\text{for} \quad (x,t) \in \overline{G}_p^{\,k}(t^n), \quad k=1,\ldots,K, \quad t^n \in \overline{\omega}_0, \quad n \leq N_0 - 1;$$

$$u^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} u_p^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_p^k(t^n),\ p=1,\ldots,P \\ \left. \begin{array}{ll} \overline{u}(x,t;t^n), & k=1 \\ u^{\frac{k-1}{K}}(x,t), & k\geq 2 \end{array} \right\}, & (x,t) \in \overline{G}(t^n) \setminus \bigcup_{p=1}^{P} \overline{G}_p^{\,k}(t^n) \end{array} \right\} \tag{5.10b}$$

$$\text{for} \quad (x,t) \in \overline{G}(t^n), \quad k=1,\ldots,K, \quad t^n \in \overline{\omega}_0.$$

$$u(x,t) = u^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}(t^n), \quad t^n \in \overline{\omega}_0. \tag{5.10c}$$

The function $\overline{u}(x,t;t^n) = \overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$, $t^n \in \overline{\omega}_0$. The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$ is determined as in (5.5f).

Stepwise, for $n = 1, 2, \cdots$, we find the function $u_{(5.10)}(x,t)$, $(x,t) \in \overline{G}$, i.e., the solution of process (5.10), (5.9), which we call the modified continuous Schwartz method for P "processors".

The scheme (5.10) on the decomposition (5.9) can be written in the "operator" form

$$Q(u(x,t);\, \omega_0,\, F(\cdot), \varphi(\cdot), \psi(\cdot)) = 0, \quad (x,t) \in G. \tag{5.10d}$$

Here the function $\psi(x,t;t^n)$, $(x,t) \in G(t^n)$ defines the prolonged function $\overline{u}(x,t;t^n)$:

$$\overline{u}(x,t;t^n) = \left\{ \begin{array}{ll} v(x,t;t^n), & (x,t) \in S(t^n) \\ v(x,t^n;t^n) + \psi(x,t;t^n), & (x,t) \in G(t^n) \end{array} \right\}, \ (x,t) \in \overline{G}(t^n), \tag{5.10e}$$

so that in the case of the conditions (5.5e), (5.5f), simply, $\psi(x,t;t^n) \equiv 0$. The problem (5.10), (5.9) for $P = 1$ is identical with problem (5.5), (5.1).

In the continuous domain decomposition method (5.10), (5.9) the intermediate problems on the subsets $\overline{D}_{p(5.9)}^{k}$, $p = 1,\ldots,P$, $k = 1,\ldots,K$ can be solved independent of each other, for all $p = 1,\ldots,P$. For solutions of Schwartz method (5.10), (5.9) we have the estimate

$$|u(x,t) - u_{(5.10)}(x,t)| \leq M\, N_0^{-1}, \quad (x,t) \in \overline{G}_h. \tag{5.11}$$

The following theorem which similar to the theorem in [7] is valid.

Theorem 5.1 *The condition* (5.6) *is necessary and sufficient for the ε-uniform convergence (as $N_0 \to \infty$) of the solution of problems* (5.5), (5.1) *and* (5.10), (5.9) *to the solution of the boundary value problem* (2.2), (2.1). *In that case when conditions of Theorem* 4.1 *and also condition* (5.6) *are fulfilled, the solutions of the continual Schwartz-like methods satisfy estimates* (5.7), (5.11).

6 Difference schemes based on the Schwartz method

6.1. Here we construct a difference scheme based on the process (5.5), (5.1) and give the necessary and sufficient conditions for ε-uniform convergence of this scheme. We introduce the rectangular grids on each of the sets $\overline{G}^k$ and $\overline{G}_p^k$:

$$\overline{G}_h^k = \overline{G}^k \bigcap \overline{G}_{h(3.1)}, \quad \overline{G}_{ph}^k = \overline{G}_p^k \bigcap \overline{G}_{h(3.1)}, \tag{6.1}$$

or

$$\overline{G}_h^{k*} = \overline{G}^k \bigcap \overline{G}^*_{h(4.1)}, \quad \overline{G}_{ph}^{k*} = \overline{G}_p^k \bigcap \overline{G}^*_{h(4.1)}. \tag{6.2}$$

where $\overline{G}_{ph}^k = \overline{G}_{p,h}^k$. We assume that the boundaries of $\overline{G}^k$ and $\overline{G}_p^k$ pass through the nodes of the grid $\overline{G}_h$ and $\overline{G}_h^*$ respectively.

Now we introduce the discrete function $v(x,t) = v(x,t;t_1)$ defined on the boundary of the discrete strip $S_h(t_1) = S(t_1) \cap \overline{G}_h, \quad t_1 \in \overline{\omega}_0$. By $\overline{v}(x,t;t_1)$ we denote the extension of this function $v(x,t)$ to the discrete set $\overline{G}_h(t_1) = \overline{G}(t_1) \cap \overline{G}_h$. The function $\overline{v}(x,t;t_1)$ is considered to satisfy the Lipschitz condition with respect to t. The "strip" $\overline{G}_h(t_1)$ consists of only two time levels

$$\overline{G}_h(t_1) = \{\overline{\omega} \times [t = t_1]\} \bigcup \{\overline{\omega} \times [t = t_1 + \tau]\},$$

where $\overline{\omega}$ was introduced in (3.1).

Now we find the discrete solutions $z^{\frac{k}{K}}(x,t)$ by a procedure similar to (5.5). That is, assuming that $z(x,t)$, $t \le t^n$, has been computed, we solve the following problems on the strip $\overline{G}_h(t^n)$

$$\left.\begin{array}{ll} \Lambda_{(6.3)}(z^{\frac{k}{K}}(x,t)) = 0, & (x,t) \in G_h^k(t^n) \\ z^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} \overline{z}(x,t;t^n), & k=1 \\ z^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array}\right\}, & (x,t) \in S_h^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}_h^k(t^n), \tag{6.3a}$$
$$k = 1, \dots, K, \quad t^n \in \overline{\omega}_0, \quad n \le N_0 - 1;$$

$$z^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} z^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_h^k(t^n) \\ \left.\begin{array}{ll} \overline{z}(x,t;t^n), & k=1 \\ z^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array}\right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \overline{G}_h^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}_h(t^n), \tag{6.3b}$$
$$k = 1, \dots, K, \quad t^n \in \overline{\omega}_0.$$

The required function $z(x,t)$ on the strip $\overline{G}_h(t^n)$ is defined by the relation

$$z(x,t) = z^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{6.3c}$$

In the relations (6.3a), (6.3b)

$$\overline{z}(x,t;t^n) = \overline{v}(x,t;t^n), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{6.3d}$$

The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$ is found, using $v(x,t;t^n)$, $(x,t) \in S_h(t^n)$, as

$$\overline{v}(x,t;t^n) = \left\{\begin{array}{ll} v(x,t\,;t^n), & (x,t) \in S_h(t^n) \\ v(x,t^n;t^n), & (x,t) \in G_h(t^n) \end{array}\right\}, \quad (x,t) \in \overline{G}_h(t^n), \tag{6.3e}$$

where

$$v(x,t;t^n) = \left\{ \begin{array}{lll} \varphi(x,t), & (x,t)\in S_h(t^n), & t^n = t^0 = 0 \\ \left.\begin{array}{ll} \varphi(x,t), & (x,t)\in S_h(t^n)\cap S_h, \quad t\ge t^n \\ z(x,t), & (x,t)\in S_h(t^n)\setminus S_h, \quad t = t^n \end{array}\right\}, & & t^n > 0 \end{array} \right\}, \tag{6.3f}$$

$$(x,t)\in S_h(t^n), \quad n = 0,1,\ldots,N_0-1.$$

On each strip $\overline{G}_h(t^n)$ the function $z^{\frac{k}{K}}(x,t)$ is the solution of the discrete Dirichlet problem on the set $\overline{G}_h^k(t^n)$. On the remaining part $\overline{G}_h(t^n)\setminus G_h^k(t^n)$, for $k=1$ it coincides with the function $\overline{z}(x,t;t^n)$, $(x,t)\in\overline{G}_h(t^n)$ and for $k\ge 2$ with the function $z^{\frac{k-1}{K}}(x,t)$, $(x,t)\in\overline{G}_h(t^n)$. We define the operator $\Lambda_{(6.3)}$ by the relation

$$\Lambda_{(6.3)}(z(x,t)) = \Lambda_{(3.3)}(z(x,t)), \quad (x,t)\in G_h. \tag{6.4}$$

It is required to find the function $z_{(6.3)}(x,t)$, $(x,t)\in\overline{G}_h$, i.e., the solution of difference scheme (6.3) either on the mesh (4.1) or on the mesh (3.1) . The difference scheme (6.3) can be written symbolically in the operator form

$$Q_{(6.3)}(z(x,t);\Lambda_{(3.3)},F(\cdot,z(\cdot)),\varphi(\cdot),\psi(\cdot)) = 0, \quad (x,t)\in\overline{G}_h. \tag{6.3g}$$

Similarly to (5.10e), here the function $\psi(x,t;t^n)$, $(x,t)\in G_h(t^n)$ defines the function $\overline{z}(x,t;t^n)$:

$$\overline{z}(x,t;t^n) = \left\{ \begin{array}{ll} v(x,t;t^n), & (x,t)\in S_h(t^n) \\ v(x,t^n;t^n)+\psi(x,t;t^n), & (x,t)\in G_h(t^n) \end{array} \right\}, \quad (x,t)\in\overline{G}_h(t^n). \tag{6.3h}$$

In the above case of the conditions (6.3d), (6.3e) we have $\psi(x,t;t^n)\equiv 0$.

In the discrete domain decomposition method (6.3), the intermediate problems on the subsets $\overline{D}_h^k = \overline{D}^k_{(5.1)}\bigcap\overline{D}_h$ are solved sequentially. Thus, to solve boundary value problem (2.2), here we used the difference scheme (6.3), (3.1), which is the discrete analog of (5.5), (5.1). In the following section we extend this to the "parallel" case (5.10).

6.2. To describe the difference scheme which approximates process (5.10), (5.9) with P parallel processors, we assume that $z(x,t)$ is known for $t\le t^n$. Then we solve the problems

$$\Lambda_{(6.3)}(z_p^{\frac{k}{K}}(x,t)) = 0, \quad (x,t)\in G^k_{p\,h}(t^n), \tag{6.5a}$$

$$z_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}(x,t;t^n), & k=1 \\ z^{\frac{k-1}{K}}(x,t), & k\ge 2 \end{array} \right\}, \quad (x,t)\in S^k_{p\,h}(t^n), \quad p=1,\ldots,P$$

$$\text{for} \quad (x,t)\in\overline{G}^k_{p\,h}(t^n), \quad k=1,\ldots,K, \quad t^n\in\overline{\omega}_0, \quad n\le N_0-1;$$

$$z^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z_p^{\frac{k}{K}}(x,t), & (x,t)\in\overline{G}^k_{ph}(t^n), \quad p=1,\ldots,P \\ \left.\begin{array}{ll} \overline{z}(x,t;t^n), & k=1 \\ z^{\frac{k-1}{K}}(x,t), & k\ge 2 \end{array}\right\}, & (x,t)\in\overline{G}(t^n)\setminus\bigcup_{p=1}^{P}\overline{G}_p^k(t^n) \end{array} \right\}$$

$$\text{for} \quad (x,t)\in\overline{G}_h(t^n), \quad k=1,\ldots,K, \quad t^n\in\overline{\omega}_0.$$

We define the function $z_{(6.5)}(x,t)$ on the strip $\overline{G}_h(t^n)$ by the relation

$$z_{(6.5)}(x,t) = z^{\frac{K}{K}}(x,t), \quad (x,t)\in\overline{G}_h(t^n), \quad t^n\in\overline{\omega}_0. \tag{6.5b}$$

In (6.5a) $\overline{z}(x,t;t^n) = \overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$. The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$ is found, using $v(x,t;t^n)$, $(x,t) \in S_h(t^n)$, which is determined by (6.3e). Thus, the function $z_{(6.5)}(x,t)$, $(x,t) \in \overline{G}_h$, i.e., the solution of the difference scheme (6.5), (3.1), is found.

The difference scheme (6.5) can be written in the operator form

$$Q_{(6.5)}(z(x,t);\Lambda_{(3.3)},F(\cdot,z(\cdot)),\varphi(\cdot),\psi(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \tag{6.5c}$$

with $\psi(x,t;t^n) \equiv 0$.

In the discrete domain decomposition method (6.5), (3.1) the intermediate problems on the subsets $\overline{D}^k_{ph} = \overline{D}^k_{p(5.9)} \bigcap \overline{D}_h$ are solved independent of each other ("in parallel") for all $p = 1,\ldots,P$. For $P = 1$ the difference scheme (6.5), (3.1) transforms into (6.3), (3.1).

Under condition (5.6), using a standard technique of the comparison theorems, we get the estimate

$$\mid z_{(3.2)}(x,t) - z_{(6.5)}(x,t) \mid \leq M N_0^{-1}, \quad (x,t) \in \overline{G}_h, \tag{6.6}$$

where $z_{(3.2)}(x,t)$ and $z_{(6.5)}(x,t)$ are the solutions of the difference schemes (3.2), (3.1) and (6.5), (3.1), respectively.

6.3. A technique similar to the one explained in [2], [3] gives us errors bounds for the discrete solutions which are obtained by the difference schemes described above. Under condition (5.6), using the difference schemes (6.3), (3.1) and (6.3), (4.1) (schemes (6.5), (3.1) and (6.5), (4.1)), we obtain the following error estimates for the solution of the boundary value problem (2.2)

$$|u(x,t) - z^d(x,t)| \leq M \ \left[(\varepsilon + N^{-1})^{-1}\, N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{h(3.5)}, \tag{6.7a}$$

$$|u(x,t) - z^d(x,t)| \leq M \ \left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}^*_{h(4.1)}. \tag{6.7b}$$

Here $z^d(x,t)$, $(x,t) \in \overline{G}_h$ is the solution of the domain decomposition scheme (schemes (6.3) and (6.5) in the case of the sequential and parallel methods respectively). The above formulation allows us to summarize briefly a result similar to [7] as follows:

Theorem 6.1 *Let the hypotheses of Theorem* 4.1 *hold for the data of the boundary value problem* (2.2) *and its solution. Then, under condition* (5.6) *and for* N, $N_0 \to \infty$, *the solutions of the difference schemes* (6.3), (6.4) *and* (6.5), (6.4) *on the mesh* (4.1) *(mesh* (3.1)*) converge to the solution of* (2.2) ε*-uniformly (for a fixed value of* ε*). The estimates* (6.6), (6.7) *hold for the solutions of these difference schemes.*

7 Iterative schemes based on approximations of nonlinear schemes

The difference scheme (3.2), (3.1) and also domain decomposition schemes (6.3), (6.4), (3.1) and (6.5), (6.4), (3.1) are nonlinear. We now give some variants of difference schemes that allow us to find approximations to solutions of nonlinear schemes and solutions of the differential problem (2.2), (2.1).

On mesh (3.1) we consider the difference scheme

$$\Lambda_{(7.1)}(z(x,t)) \equiv \Lambda_{(3.2)}z(x,t) - F(x,t,\check{z}(x,t)) = 0, \quad (x,t) \in G_h, \tag{7.1}$$
$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here $\breve{z}(x,t) = z(x, t-\tau)$, $(x,t) \in G_h$. In the "linear" difference scheme (7.1), (3.1) numerical solutions at each time level are the solutions of linear equations.

The difference scheme (7.1), (3.1) is ε-uniformly monotone in the case of the condition

$$N_0 \geq T \max_{\overline{H}} \left[p^{-1}(x,t) \frac{\partial}{\partial u} F(x,t,\, u) \right]. \tag{7.2}$$

Theorem 7.1 *Let the condition* (7.2) *hold and the functions* $z^i(x,t)$, $(x,t) \in \overline{G}_h$, $i = 1, 2$ *satisfy the inequalities*

$$\begin{aligned} \Lambda_{(7.1)}(z^1(x,t)) \;&\leq\; \Lambda_{(7.1)}(z^2(x,t)), \quad && (x,t) \in G_h, \\ z^1(x,t) \;&\geq\; z^2(x,t), && (x,t) \in S_h. \end{aligned}$$

Then $z^1(x,t) \geq z^2(x,t)$, $(x,t) \in \overline{G}_h$.

In the nonlinear scheme (3.2), (3.1) it is possible to compute its solution at the time level $t \in \omega_0$ using the method of successive approximations, where the values of $z(x,t)$ in the function $F(x,t,\, z(x,t))$ are taken from the previous iteration

$$\begin{aligned} &\Lambda_{(7.3)}(z^{(s)}(x,t)) \equiv \left\{ \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - \tau^{-1}\, p(x,t) \right\} z^{(s)}(x,t) + \\ &\quad + \tau^{-1}\, p(x,t)\, \breve{z}(x,t) - F(x,t,\, z^{(s-1)}(x,t)) = 0, \quad (x,t) \in G_h, \\ &\quad z^{(s)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h \cap S^L, \\ &\quad z\,(x,t) = \varphi(x,t), \quad (x,t) \in S_h \cap S_0; \\ &\quad z^{(0)}(x,t) = \breve{z}(x,t), \quad (x,t) \in \overline{G}_h, \quad t > 0; \quad s = 1, 2, \ldots, s_0. \end{aligned} \tag{7.3}$$

We take the value of z from the known level $t - \tau$ as the initial guess $z^{(0)}(x,t)$; assume $z(x,t) = z^{(s_0)}(x,t)$.

It is possible to realize the computational process with fixed numbers of iterations s_0. For $s_0 = 1$ in (7.3) this scheme coincides with a linear variant of scheme (7.1).

For $s_0 \to \infty$ the solution of problem (7.3) converges to the solution of problem (3.2). However, errors in the numerical solution can increase, in general, as s_0 grows. In the case of the condition

$$\frac{\partial}{\partial u} F(x,t,\, u) \leq 0, \quad (x,t,u) \in \overline{H} \tag{7.4}$$

scheme (7.3), (3.1) is monotone. The following theorem is valid.

Theorem 7.2 *Let the conditions* (7.2), (7.4) *hold and the functions* $z^{(s)i}(x,t)$, $(x,t) \in \overline{G}_h$, $s = 1, 2, \ldots, s_0$, $i = 1, 2$ *satisfy the inequalities*

$$\begin{aligned} \Lambda_{(7.3)}(z^{(s)1}(x,t)) \;&\leq\; \Lambda_{(7.3)}(z^{(s)2}(x,t)), \quad && (x,t) \in G_h, \\ z^{(s)1}(x,t) \;&\geq\; z^{(s)2}(x,t), && (x,t) \in S_h, \quad s = 1, 2, \ldots, s_0. \end{aligned}$$

Then $z^{(s)1}(x,t) \geq z^{(s)2}(x,t)$, $(x,t) \in \overline{G}_h$, $s = 1, 2, \ldots, s_0$.

When we decompose the difference scheme (7.1), (3.1) as above, we obtain the scheme for sequential computations (sequential scheme)

$$Q_{(6.3)}\Big(\, z(x,t);\Lambda_{(7.1)}, F(\cdot,\check z(\cdot)),\varphi(\cdot),\psi(\cdot)=0\,\Big)=0,\quad (x,t)\in\overline{G}_h \tag{7.5}$$

and the scheme for parallel computations (parallel scheme)

$$Q_{(6.5)}\Big(\, z(x,t);\Lambda_{(7.1)}, F(\cdot,\check z(\cdot)),\varphi(\cdot),\psi(\cdot)=0\,\Big)=0,\quad (x,t)\in\overline{G}_h \tag{7.6}$$

When we decompose the difference scheme (7.3), (3.1), we obtain the (sequential and parallel) schemes

$$Q_{(6.3)}\big(z^{(s)}(x,t);\Lambda_{(7.3)}, F(\cdot,z^{(s-1)}(\cdot)),\varphi(\cdot),\psi^{(s-1)}(\cdot)\big)=0, \tag{7.7}$$
$$(x,t)\in\overline{G}_h,\quad s=1,2,\ldots,s_0;$$

$$Q_{(6.5)}\big(z^{(s)}(x,t);\Lambda_{(7.3)}, F(\cdot,z^{(s-1)}(\cdot)),\varphi(\cdot),\psi^{(s-1)}(\cdot)\big)=0, \tag{7.8}$$
$$(x,t)\in\overline{G}_h,\quad s=1,2,\ldots,s_0;$$

Here $\psi^{(s-1)}(x,t;\,t^n)\equiv 0$ for $s=1$, $\psi^{(s-1)}(x,t;\,t^n)=z^{(s-1)}(x,t^n)$ for $s=2,3,\ldots,s_0$; the function $z^{(s)}(x,t)$, $(x,t)\in\overline{G}_h(t^n)$, $t^n\in\overline{\omega}_0$ is the solution of the decomposition scheme on the s-th iteration. We assume that $z(x,t)=z^{(s_0)}(x,t)$, $(x,t)\in\overline{G}_h$.

Using the majorant function technique, in the case of decomposition schemes (7.5)–(7.8) on meshes (3.5) and (4.1) we obtain the estimates

$$|u(x,t)-z^d(x,t)|\le M\,\left[(\varepsilon+N^{-1})^{-1}\,N^{-1}+N_0^{-1}\right],\quad (x,t)\in\overline{G}_{h(3.5)}, \tag{7.9a}$$
$$|u(x,t)-z^d(x,t)|\le M\,\left[N^{-1}\ln N+N_0^{-1}\right],\quad (x,t)\in\overline{G}^*_{h(4.1)}. \tag{7.9b}$$

Here the error constants for the sequential and parallel schemes are, in general, different but independent of the value s_0 (in the case of schemes (7.7), (7.8)). The convergence rate for sequential and parallel schemes (7.5)–(7.8), and also for nonlinear schemes (6.3), (6.5) is the same as that for scheme (3.2) subjected to the decomposition.

Theorem 7.3 *Let the hypotheses of Theorem* 4.1 *be fulfilled. Then, under condition* (5.6) *and additional condition* (7.2) *(condition* (7.4)*) the solutions of the difference schemes* (7.5) *and* (7.6) *(schemes* (7.7) *and* (7.8)*) on the mesh* (4.1) *converge to the solution of problem* (2.2), (2.1) *ε-uniformly. The estimates* (7.9) *are valid for the solutions of these difference schemes.*

Remark 1. The nonlinear scheme (3.2) can be linearized by the Newton method [13]

$$\Lambda_{(7.10)}(z^{(s)}(x,t))\equiv\left\{\varepsilon a(x,t)\delta_{\overline{x}\widehat{x}}+b(x,t)\delta_x-c(x,t)-\tau^{-1}p(x,t)\right\}z^{(s)}(x,t)+$$
$$+\tau^{-1}\,p(x,t)\,\check z(x,t)-\frac{\partial}{\partial u}\,F(x,t,\,z^{(s-1)}(x,t))\,z^{(s)}(x,t)- \tag{7.10}$$
$$-F(x,t,\,z^{(s-1)}(x,t))-\frac{\partial}{\partial u}\,F(x,t,\,z^{(s-1)}(x,t))\,z^{(s-1)}(x,t)=0,\quad (x,t)\in G_h,$$
$$z^{(0)}(x,t)=\varphi(x,t),\quad (x,t)\in S_h\cap S^L,$$
$$z\ (x,t)=\varphi(x,t),\quad (x,t)\in S_h\cap S_0;\quad s=1,2,\ldots,s_0.$$

Here $z^{(0)}(x,t)=\check z(x,t)$, $(x,t)\in\overline{G}_h$, $t>0$; assume $z(x,t)=z^{(s_0)}(x,t)$. Using scheme (7.10) as a base scheme, one can construct sequential and parallel decomposition schemes similar to schemes (7.7) and (7.8), which converge ε-uniformly.

8 Estimates of the solution and its derivatives

Here we rely on the a-priori estimates for the solution of problem (2.2) and its derivatives as derived for elliptic and parabolic equations in [6], [14].

We denote by $C^{(\alpha)}(\overline{G}) = C^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number [17]. We suppose that the functions $F(x,t,\,u(x,t))$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity, we assume that at the corner points $S_0 \cap \overline{S}^L$ the following conditions hold

$$\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad k+2k_0 \le [\ \alpha\ ] + 2n, \tag{8.1}$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,F(x,t,\,0) = 0, \quad k+2k_0 \le [\ \alpha\ ] + 2n - 2,$$

where $[\ \alpha\ ]$ is the integer part of a number α, $\alpha > 0$, $n \ge 0$ is an integer number. We also suppose that $[\ \alpha\ ] + 2n \ge 2$.

Using interior a-priori estimates and estimates up to the boundary for the regular function $\widetilde{u}(\xi,t)$, [17], where $\widetilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}u(x,t)\right| \le M\,\varepsilon^{-k}, \quad k+2k_0 \le 2n+4, \quad n \ge 0. \tag{8.2}$$

This estimate holds, for example, for

$$u \in C^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0, \tag{8.3}$$

where ν is some small number.

For example, (8.3) is guaranteed for the solution of (2.2) if the coefficients satisfy a, b, c, $p \in C^{(\alpha+2n-2)}(\overline{G})$, $F \in C^{(\alpha+2n)}(\overline{H})$, $\varphi \in C^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \ge 0$ and condition (8.1) is fulfilled.

In fact we need a more accurate estimate than (8.2). Therefore, we represent the solution of the boundary value problem (2.2) in the form of the sum

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G}, \tag{8.4}$$

where $U(x,t)$ represents the regular part, and $W(x,t)$ the singular part. The function $U(x,t)$ is the smooth solution of equation (2.2a) satisfying condition (2.2b) on $S_2 \cup S_0$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (2.2a) smoothly extended to the domain $\overline{G}^*$ ($\overline{G}^*$ is a sufficiently large neighbourhood of $\overline{G}$). On the domains $\overline{G}$, $\overline{H}$ and $S_2 \cup S_0$, respectively, the coefficients, the right-hand side $F(x,t,\,u)$ and the boundary function $\varphi(x,t)$ of the extended problem are the same as for (2.2). Then the function $U(x,t)$ is the restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in C^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of a boundary value problem for the parabolic equation

$$L_{(2.2)}W(x,t) = F(x,t,\,u(x,t)) - F(x,t,\,U(x,t)), \quad (x,t) \in G, \tag{8.5}$$

$$W(x,t) = u(x,t) - U(x,t), \quad (x,t) \in S.$$

If (8.3) is true then $W \in C^{(4+2n+\nu)}(\overline{G})$. We suppose that $a, b, c, p \in C^{(\alpha+6n)}(\overline{G})$, $F \in C^{(\alpha+6n)}(\overline{H})$, $\varphi \in C^{(\alpha+6n)}(\overline{G})$, $\alpha > 6$, $n \geq 0$. Now, for the functions $U(x,t)$ and $W(x,t)$ we derive the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} U(x,t) \right| \leq M \, [1 + \varepsilon^{2n+2-k}], \tag{8.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} W(x,t) \right| \leq M \, \varepsilon^{-k} \, \exp(-m_{(8.7)} \, \varepsilon^{-1} \, x) \,), \tag{8.7}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \leq 2n + 4,$$

where $m_{(8.7)}$ is an arbitrary number from the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} \, [a^{-1}(x,t) \, b(x,t)]$. For example, the similar estimates are deduced in [8] for the case when $F(x, t, \, u(x,t)) = f(x,t)$.

Theorem 8.1 *Assume in equation* (2.2) *that* $a, \, b, \, c, \, p \in C^{(\alpha+6n)}(\overline{G})$, $F \in C^{(\alpha+6n)}(\overline{H})$, $\varphi \in C^{(\alpha+6n)}(\overline{G})$, $\alpha > 6$, $n \geq 0$ *and let condition* (8.3) *be fulfilled. Then, for the solution* $u(x,t)$ *of problem* (2.2) *and for its components in representation* (8.4), *it follows that* $u, U, W \in C^{(4+2n)}(\overline{G})$ *and that the estimates* (8.2), (8.6), (8.7) *hold.*

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