Approximation of Solutions and Derivatives for Singularly Perturbed Elliptic Convection-Diffusion Equations

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Abstract

In this paper we consider mesh approximations of a boundary value problem for singularly perturbed elliptic equations of convection-diffusion type on a strip. To approximate the equations, we use classical finite difference approximations on piecewise-uniform meshes condensing in a neighbourhood of the boundary layer. The approximation errors of solutions and derivatives are analysed in the ρ-metric. In this metric the error of a solution is defined by an absolute error, while the error of its derivative $(\partial/\partial x_1)u(x)$, i.e. the derivative in the direction across of the boundary layer, is defined by the relative error in that part of the domain where the derivative is large, and by the absolute error in the remainder part of the domain. It is shown that in the class of meshes, whose stepsize in the boundary layer does not decrease with moving away from the outflow boundary, there are no meshes on which the scheme converges ε-uniformly in the ρ-metric. We establish conditions, imposed on the distribution of the nodes of piecewise uniform meshes, under which the scheme converges in the ρ-metric ε-uniformly up to a logarithmic factor.

1. Introduction

Solutions of partial differential equations with a small parameter multiplying the highest derivatives have a limited smoothness that gives rise to difficulties of their numerical solving [1] – [3]. This leads to a need for the development of special mesh methods, the accuracy of which depends weakly on the perturbation parameter ε, and, in particular, methods that converge ε-uniformly. Difficulties also arise in that case when it is required to find not only the solution, but also its derivatives, for example, in the computation of diffusion flows. Thus, it is actual to construct such numerical methods that allow us to approximate both the solution of the problem and its derivatives with errors weakly depending on the perturbation parameter ε.

For several boundary value problems ε-uniformly convergent finite difference schemes have been constructed (see, e.g., [4]–[6] for the description of approaches to the construction of such schemes). In some publications the approximation of both solutions and derivatives was considered for ε-uniform numerical methods (see, for example, [7, 8]). However, the construction of special numerical methods for the approximation of solutions and derivatives in that case

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when the approximation errors depend weakly on the parameter ε has not yet been considered practically.

In the present paper mesh approximations of a boundary value problem on a strip for singularly perturbed elliptic convection-diffusion equations are considered; we use classical approximations of the equation on piecewise-uniform meshes condensing in a neighbourhood of the boundary layer. The derivative of the solution in the normal direction (towards the outflow boundary) $(\partial/\partial x_1)u(x)$ grows without bound (in a neighbourhood of the boundary layer) as $\varepsilon \to 0$; the normalized derivative $\varepsilon\,(\partial/\partial x_1)u(x)$ is ε-uniformly bounded. However, the normalized derivative outside of the boundary layer tends to zero when $\varepsilon \to 0$, and, as a result, the information about its behavior fails.

In this paper we consider the approximation errors for solutions and derivatives in the ρ-metric: the error of a solution $u(x)$ is defined by the absolute error; the error of the derivative $(\partial/\partial x_1)u(x)$ is defined by the relative error in that part of the domain where the derivative is large, and by the absolute error in the remainder part of the domain. We analyze the convergence of classical schemes (on arbitrary meshes, e.g., uniform meshes) and of ε-uniformly convergent schemes on meshes from [4]. Then we establish conditions imposed on the distribution of the nodes of these piecewise uniform meshes, under which the scheme converges in the ρ-metric ε-uniformly up to a logarithmic factor.

About contents of the paper. Problem formulation and the goal of research are given in Section 2. Finite difference schemes for this problem are considered in Sections 4, 7 and 8; schemes for a model one-dimensional problem are considered in Sections 3, 5 and 6. The convergence analysis in the ρ-metric is given in Sections 3, 4 for classical schemes, and in Sections 5, 7 for schemes on piecewise-uniform meshes (which ensure the ε-uniform convergence of the solutions in C). Schemes convergent in the ρ-metric ε-uniformly up to a logarithmic factor are considered in Sections 6, 8. *A-priori* estimates used in the constructions and proofs are given in Section 9.

2.   Problem Formulation

1.   On the domain $\overline{D}$ with boundary Γ, where

$$D = \{x:\ \ x_1 \in (0, d),\ \ x_2 \in R\}, \tag{2.1}$$

we consider a boundary value problem for the singularly perturbed elliptic equation with convective terms

$$L_{(2.2)}u(x) \equiv \left\{ \varepsilon \sum_{s=1,2} a_s(x)\frac{\partial^2}{\partial x_s{}^2} + \sum_{s=1,2} b_s(x)\frac{\partial}{\partial x_s} - c(x) \right\} u(x) = f(x), \quad x \in D,$$
$$\tag{2.2}$$
$$u(x) = \varphi(x), \quad x \in \Gamma.$$

Here $a_s(x),\, b_s(x),\, c(x),\, f(x),\ x \in \overline{D},\ \varphi(x),\, x \in \Gamma$ are sufficiently smooth functions, moreover[1] $a_0 \le a_s \le a^0,\ b_0 \le b_1(x) \le b^0,\ |b_2(x)| \le b^0,\ 0 \le c(x) \le c^0,\ x \in \overline{D},\ a_0, b_0 > 0;\ |f(x)| \le M,$

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[1] Here and below we denote by M, M_i (or m) sufficiently large (or small) positive constants which are independent of the parameter value ε or the difference operators. The notation $L_{(j.k)}$ $(m_{(j.k)},\ D_{h(j.k)})$ means that this operator (constant, grid) is first introduced in the formula $(j.k)$.

$x \in \overline{D}$, $|\varphi(x)| \leq M$, $x \in \Gamma$; $\varepsilon \in (0,1]$ is the perturbation parameter. For simplicity, we suppose

$$\text{either } b_2(x) \geq 0 \quad \text{or } b_2(x) \leq 0, \quad x \in \overline{D}.$$

When ε tends to zero, a boundary layer appears in a neighbourhood of the boundary Γ_1. Here $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 are the left and right boundaries.

To solve this problem, we use classical finite difference schemes on rectangular meshes [14].

2. We are interested in the approximation of not only the solution of problem (2.2), (2.1), but also of its partial derivatives.

Note that the solution of problem (2.2), (2.1) and its derivatives with respect to x_2 are ε-uniformly bounded. However, the derivatives in x_1 grow without bound as $\varepsilon \to 0$ and remains bounded ε-uniformly outside a neighbourhood of the boundary layer. To estimate the derivatives $(\partial^{k_1}/\partial x_1^{k_1})u(x)$ it would be possible to use the products $\varepsilon^{k_1} \dfrac{\partial^{k_1}}{\partial x_1^{k_1}} u(x) \equiv \left(\dfrac{\partial^{k_1}}{\partial x_1^{k_1}}\right)^{*} u(x)$, i.e. the normalized derivatives which are ε-uniformly bounded on the whole of $\overline{D}$. For example, for the approximation of the solution and its first derivatives one can use the norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^1}^{*}$, where

$$\|u\|_{C^1}^{*} = \|u\| + \left\|\frac{\partial}{\partial x_2} u\right\| + \left\|\left(\frac{\partial}{\partial x_1}\right)^{*} u\right\|, \quad \left(\frac{\partial}{\partial x_1}\right)^{*} u(x) \equiv \varepsilon \left(\frac{\partial}{\partial x_1}\right) u(x). \tag{2.3}$$

Here $\|\cdot\|$ is the C-norm. But the norm $\|\cdot\|_{C^1}$ is not bounded ε-uniformly. This leads to an inconvenience to construct and analyze ε-uniformly convergent numerical methods. The norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^1}^{*}$ are inadequate norms; in these norms the information about the behavior of the derivative $(\partial/\partial x_1)u(x)$ outside of the boundary layer fails. For small values of the parameter ε the magnitude of the derivative outside of the boundary layer does not influence on the value of $\|\cdot\|_{C^1}^{*}$; the derivative outside of the layer becomes "invisible".

3. In the case of the approximation of the solution and its derivatives based on the maximum norm it seems to be appropriate to approximate the solution in this norm on the whole of $\overline{D}$, but the derivatives only on that part of the domain $\overline{D}$ where the derivatives are not too large. In that part of the domain where the derivatives are large, the approximation is defined by the relative error in the maximum norm.

Let it be required to approximate the solution $u(x)$, $x \in \overline{D}$ and its derivatives up to order n, $n \leq 2$. It is convenient to introduce a distance between the solution $u(x)$, $x \in \overline{D}$ of problem (2.2), (2.1) and some sufficiently smooth function $v(x)$, $x \in \overline{D}$ in the following way. Let β^n, $n = 1, 2$ be an arbitrary number from the interval $(0, \infty)$. Let $\overline{D}_\beta^n$ denote the set on which the following condition holds:

$$\left|\frac{\partial^n}{\partial x_1^n} u(x)\right| \geq \beta^n, \quad x \in \overline{D}_\beta^n, \tag{2.4a}$$

moreover, the condition

$$\max_{\overline{D}_\beta^{[n]}} \left|\frac{\partial^n}{\partial x_1^n} u(x)\right| \leq \beta^n \tag{2.4b}$$

is satisfied on the set $\overline{D}_\beta^{[n]} = \overline{D} \setminus D_\beta^n$.

By $\overline{D}_\beta^{n-}$ we denote the set $\overline{D}_\beta^n$ which does not include its right boundary. For the function $v(x)$, $x \in \overline{D}$, $v \in C^n(\overline{D})$ we define the magnitude ("seminorm")

$$\rho_u^{(n)}(v) = \inf_{\beta^n} \max \left\{ \left\| \frac{\partial^n}{\partial x_1^n}(u - v) \right\|_{\overline{D}_\beta^{[n]}}, \quad \left\| \left(\frac{\partial^n}{\partial x_1^n} u \right)^{-1} \frac{\partial^n}{\partial x_1^n}(u - v) \right\|_{\overline{D}_\beta^{n-}} \right\}.$$

The magnitude $\rho_u^k(v)$, $k = 1, 2$, i.e. the distance between the functions $u(x)$ and $v(x)$, $x \in \overline{D}$ and between their derivatives up to order k, is defined by

$$\rho_u^k(v) \equiv \|u - v\| + \sum_{k_1=1}^{k} \rho_u^{(k_1)}(v) + \sum_{k_2=1}^{k} \left\| \frac{\partial^{k_2}}{\partial x_2^{k_2}}(u - v) \right\|, \quad k = 1, 2. \tag{2.5}$$

We introduce the distance between the solution $u(x)$, $x \in \overline{D}$ and a mesh function $z(x)$, $x \in \overline{D}_h$, which is defined on a rectangular mesh $\overline{D}_h$ on $\overline{D}$. For this, we construct some interpolants from the values of the function $z(x)$, $x \in \overline{D}_h$ and put them in correspondence to the function $u(x)$, $x \in \overline{D}_h$ (and its derivatives up to order k, $k \le 2$). We will use these interpolants to estimate the approximation errors.

Let $\overline{z}(x)$, $x \in \overline{D}$ denote the interpolant (bilinear in x_1 and x_2) which is constructed from the values of $z(x)$, $x \in \overline{D}_h$; $\overline{z}(x)$, $x \in \overline{D}$ is a continuous function which has piecewise continuous derivatives $\frac{\partial}{\partial x_1}\overline{z}(x)$ and $\frac{\partial}{\partial x_2}\overline{z}(x)$. These derivatives are discontinuous on the lines $x_1 = x_1^i$ and $x_2 = x_2^j$ respectively; $(x_1^i, x_2^j) \in \overline{D}_h$. On the lines of discontinuity $x_1 = x_1^i$ and $x_2 = x_2^j$ we complete a definition of the derivatives $\frac{\partial}{\partial x_1}\overline{z}(x)$ and $\frac{\partial}{\partial x_2}\overline{z}(x)$ up to continuity for $x_1 = x_1^i + 0$ and $x_2 = x_2^j + 0$ respectively.

Further we construct the functions $\overline{z}^1(x)$ and $\overline{z}^2(x)$, $x \in \overline{D}$, which approximate the second difference derivatives w.r.t. x_1 and x_2. Let $\overline{D}_h = \overline{\omega}_1 \times \omega_2$, $\overline{\omega}_1$ and ω_2 be meshes on the interval $[0, d]$ and on the axis x_2. We introduce the meshes $\overline{\omega}_1^*$ and ω_2^*, whose nodes x_1^* and x_2^* are defined by the relations $x_1^{*i+1/2} = 2^{-1}(x_1^i + x_1^{i+1})$, $x_2^{*j+1/2} = 2^{-1}(x_2^j + x_2^{j+1})$, $x_1^i, x_1^{i+1} \in \overline{\omega}_1$, x_2^j, $x_2^{j+1} \in \omega_2$.

We define the functions $\overline{z}^1(x)$ and $\overline{z}^2(x)$ on the lines $x_1 = x_1^{*i+1/2}$ and $x_2 = x_2^{*j+1/2}$ by

$$\overline{z}^1(x) = \delta_{x1}\left(\frac{\partial}{\partial x_1}\overline{z}(x) \right), \quad x \in \overline{\omega}_1^* \times R; \quad \overline{z}^2(x) = \delta_{x2}\left(\frac{\partial}{\partial x_2}\overline{z}(x) \right), \quad x \in [0, d] \times \omega_2^*.$$

We extend the functions $\overline{z}^1(x)$ and $\overline{z}^2(x)$ linearly between the lines $x_1 = x_1^{*i+1/2}$, $x_1^{*i+3/2}$ and $x_2 = x_2^{*j+1/2}$, $x_2^{*j+3/2}$, ...; the function $\overline{z}^1(x)$ is linearly extended up to the boundary Γ if $x_1^i = 0$ or $x_1^{i+1} = d$. The functions $\overline{z}^1(x)$ and $\overline{z}^2(x)$, $x \in \overline{D}$ are "extensions" (onto $\overline{D}$) of the second difference derivatives of the function $z(x)$, $x \in \overline{D}_h$ with respect to x_1 and x_2.

We introduce the "function"

$$\tilde{z}(x), \quad x \in \overline{D} \tag{2.6}$$

and its "derivatives". This function coincides with $\overline{z}(x)$ on $\overline{D}$. Its first derivatives $\frac{\partial}{\partial x_1}\tilde{z}$ and

$\frac{\partial}{\partial x_2}\widetilde{z}$ are the same as the first derivatives of the function $\overline{z}(x)$, i.e. $\frac{\partial}{\partial x_1}\overline{z}(x)$ and $\frac{\partial}{\partial x_2}\overline{z}(x)$. The second "derivatives" $\frac{\partial^2}{\partial x_1^2}\widetilde{z}$ and $\frac{\partial^2}{\partial x_2^2}\widetilde{z}$ are the functions $\overline{z}^1(x)$ and $\overline{z}^2(x)$ respectively. Note that, in the case of sufficiently smooth functions $v(x)$, $x \in \overline{D}$, the function $\widetilde{z}(x)$, $x \in \overline{D}$ and its "partial" derivatives up to the second order, where $z(x) = v(x)$, $x \in \overline{D}$, approximate the function $v(x)$ and its derivatives in the maximum norm.

We define the distance between the functions $u(x)$ and $z(x)$ by the relation

$$\rho_u^{kh}(z) \equiv \|u - \widetilde{z}\| + \sum_{k_1=1}^{k} \rho_u^{(k_1)h}(z) + \sum_{k_2=1}^{k} \left\|\frac{\partial^{k_2}}{\partial x_2^{k_2}}(u - \widetilde{z})\right\|, \quad k = 1, 2, \tag{2.7}$$

where

$$\rho_u^{(n)h}(z) = \inf_{\beta^n} \max \left\{ \left\|\frac{\partial^n}{\partial x_1^n}(u - \widetilde{z})\right\|_{\overline{D}_\beta^{[n]}}, \ \left\|\left(\frac{\partial^n}{\partial x_1^n}u\right)^{-1}\frac{\partial^n}{\partial x_1^n}(u - \widetilde{z})\right\|_{\overline{D}_\beta^{n-}} \right\}, \quad n = 1, 2.$$

In that case when the function $u(x)$, $x \in \overline{D}$ has ε-uniformly bounded derivatives $(\partial^n/\partial x_1^n)u(x)$, $n \leq k$, the distance $\rho_{u(2.7)}^{kh}(z)$ between $u(x)$, $x \in \overline{D}$ and $z(x)$, $x \in \overline{D}_h$ is equivalent to the "usual" distance between $u(x)$, $x \in \overline{D}$ and $\widetilde{z}(x)$, $x \in \overline{D}$

$$\|u - \widetilde{z}\|_{C_*^k} = \sum_{k_2=0}^{k} \left\|\frac{\partial^{k_2}}{\partial x_2^{k_2}}(u - \widetilde{z})\right\| + \sum_{k_1=1}^{k} \left\|\frac{\partial^{k_1}}{\partial x_1^{k_1}}(u - \widetilde{z})\right\|, \quad k = 1, 2; \tag{2.8}$$

$\|u - \widetilde{z}\|_{C_*^k} = \|u - \widetilde{z}\|$ for $k = 1$. Generally speaking, $\rho_u^{kh}(z) \leq M \|u - \widetilde{z}\|_{C^k}$.

Note that for the function $V_1(x) = x_1 \exp(-\varepsilon^{-1}x_1)$, $x \in \overline{D}$, which is the leading term in the asymptotic expansion (in powers of ε) of the singular part of the solution $u(x)$ (for suitable parameters of the boundary value problem), we have the estimates

$$\|V_1\|_{C_{(2.3)}^1}^* \leq 2\varepsilon; \qquad 1 \leq \|V_1\|_{C^1}, \ \rho_{u(2.5)}^1(V_1) \leq 2.$$

Thus, the norm $\|\cdot\|_{C^1}$ is an *inadequate* metric for small values of the parameter ε even in the case of the singular components (the first terms of their asymptotic expansions in ε).

4. In the case of classical mesh approximations to problem (2.2), (2.1), the discrete solutions on uniform meshes converge in the $\rho_u^{1h}(\cdot)$ metric only under the condition

$$N_1^{-1} \ll \varepsilon \ln^{-1}(\varepsilon^{-1} + 1), \tag{2.9}$$

where N_1 is the number of mesh points on the interval $[0, d]$ (see, e.g., the statements of Lemma 3.1, Theorem 4.2). The use of piecewise-uniform meshes from [4, 5, 9], on which the discrete solutions converge ε-uniformly in the norm $\|\cdot\|$, allows us rather to weaken condition (2.9) in the case of the $\rho_u^{1h}(\cdot)$ metric (see, for example, the remark to Theorem 7.2). In connection with the marked behavior of the mesh solutions it is of interest to construct schemes that converge in the $\rho_u^{1h}(\cdot)$ metric under a weaker condition than condition (2.9).

We say that the solution $z(x)$, $x \in \overline{D}_h$ of the difference scheme converges in the $\rho_u^{1h}(\cdot)$ metric (for $N \to \infty$, where the value N indicates the number of mesh elements in $\overline{D}_h$ on each of the axes) *almost ε-uniformly* with the convergence defect $\mathcal{O}(\varepsilon^{-\nu})$, if for any arbitrarily small number $\nu > 0$ there exists a function $\mu(\eta)$ such that the following estimate holds:

$$\rho_u^{1h}(z) \leq M \mu (\varepsilon^{-\nu} N^{-1}), \tag{2.10}$$

where $\mu(\eta) \to 0$ uniformly in ε as $\eta \to 0$. If $\nu = 0$ the scheme converges ε-uniformly. In a similar way almost ε-uniform convergence can be defined in the $\rho_u^{2h}(\cdot)$ metric and in the norms $\| \cdot \|$, $\| \cdot \|_{C_*^k}$.

Our goal is to construct special meshes, on which classical difference approximations to problem (2.2), (2.1) converge in the $\rho_u^{kh}(\cdot)$ metric almost ε-uniformly and, in particular, ε-uniformly.

3. A classical finite difference scheme. The model problem

1. It is suitable to consider a number of constructions with a model example for an ordinary singularly perturbed differential equation. On the interval $\overline{D}$ on the axis x, where

$$D = (0, d) \tag{3.1}$$

we consider the boundary value problem

$$Lu(x) \equiv \left\{ \varepsilon a \, \frac{d^2}{dx^2} + b \, \frac{d}{dx} \right\} u(x) = f(x), \quad x \in D, \tag{3.2}$$

$$u(x) = \varphi(x), \quad x \in \Gamma.$$

Here $\Gamma = \overline{D} \setminus D$, $f(x)$, $x \in \Gamma$ is a sufficiently smooth function, and also $a, b > 0$. We approximate problem (3.2), (3.1) by a classical finite difference scheme [14]. On the interval $\overline{D}$ we introduce the mesh

$$\overline{D}_h = \overline{\omega} \tag{3.3}$$

with any distribution of the mesh points satisfying the condition $h \leq M N^{-1}$; $h = \max_i h^i$, $h_i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}$, $N + 1$ is the number of nodes in the mesh $\overline{\omega}$. For problem (3.2), (3.1) we use the finite difference scheme

$$\Lambda z(x) \equiv \left\{ \varepsilon a \, \delta_{\widehat{xx}} + b \, \delta_x \right\} z(x) = f(x), \quad x \in D_h, \quad z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{3.4}$$

The solutions of scheme (3.4), (3.3) satisfy the error estimate

$$| \, u(x) - z(x) \, | \leq M \, (\varepsilon^2 + N^{-1})^{-1} N^{-1}, \quad x \in \overline{D}_h. \tag{3.5}$$

In the case of the uniform mesh

$$\overline{D}_h^u \tag{3.6}$$

we have

$$| \, u(x) - z(x) \, | \; \leq \; M \, (\varepsilon + N^{-1})^{-1} N^{-1}, \quad x \in \overline{D}_h^u. \tag{3.7}$$

For difference derivatives of the mesh solutions we have

$$\left| \frac{d}{dx} u(x) - \delta_x \, z(x) \right| \; \leq \; M \, \varepsilon^{-1} \, (\varepsilon + N^{-1})^{-2} N^{-1}, \quad x \in \overline{D}_{h(3.3)}^{-}; \tag{3.8a}$$

$$\left| \frac{d^2}{dx^2} u(x) - \delta_{\widehat{\overline{x} x}} \, z(x) \right| \; \leq \; M \, \varepsilon^{-2} \, (\varepsilon + N^{-1})^{-2} N^{-1}, \quad x \in D_{h(3.3)}; \tag{3.8b}$$

$$\left| \frac{d}{dx} u(x) - \delta_x \, z(x) \right| \; \leq \; M \, \varepsilon^{-1} \, (\varepsilon + N^{-1})^{-1} N^{-1}, \quad x \in \overline{D}_{h(3.6)}^{u\,-}; \tag{3.9a}$$

$$\left| \frac{d^2}{dx^2} u(x) - \delta_{\widehat{\overline{x} x}} \, z(x) \right| \; \leq \; M \, (\varepsilon + N^{-1})^{-3} N^{-1}, \quad x \in D_{h(3.6)}^u. \tag{3.9b}$$

Here $\overline{D}_h^{-} = \omega \cup \{x = 0\}$. Estimates (3.9a), (3.9b) are obtained with taking into account the explicit form of the singular parts and the leading terms of the asymptotics (in ε) for the regular parts of solutions to problems (3.2), (3.1) and (3.4), (3.3).

Definition. Let $z(x)$, $x \in \overline{D}_h$ be a solution of a difference scheme. An estimate of the following form

$$|u(x) - z(x)| \; \leq \; M \, \varepsilon^{-\nu_1} (\varepsilon + N^{-1})^{-\nu_2} N^{-\nu_3} \equiv \mu_0(N, \varepsilon), \quad x \in \overline{D}_h, \quad \nu \geq 0$$

is said to be unimprovable with respect to the values of N, ε if the estimate

$$| \, u(x) - z(x) \, | \; \leq \; M \, \mu_1(N, \varepsilon), \quad x \in \overline{D}_h, \quad \mu_1(N, \varepsilon) = o(\mu_0(N, \varepsilon)),$$

generally speaking, fails for some values of N and ε, $N \geq N_0$, $\varepsilon \in (0, 1]$.

The consideration of the model problems with sufficiently simple data shows that estimate (3.9) is unimprovable w.r.t. the values of N, ε, and estimate (3.7) is unimprovable however under the (unimprovable) condition

$$N^{-1} = \mathcal{O}(\varepsilon). \tag{3.10}$$

For the "function" $\widetilde{z}(x)$, $x \in \overline{D}$, which is defined similarly to $\widetilde{z}_{(2.6)}(x)$, we have the following estimate in the case of scheme (3.4), (3.6):

$$\left\| \frac{d^k}{dx^k} (u - \widetilde{z}) \right\| \; \leq \; M \, \varepsilon^{-k} \, (\varepsilon + N^{-1})^{-1} N^{-1}, \quad k = 0, 1, 2; \tag{3.11}$$

this estimate is unimprovable w.r.t. the values of N, ε.

Thus, the function $\widetilde{z}(x)$ and its derivatives $(d^k/dx^k)\widetilde{z}(x)$ converge in the $\| \cdot \|$ norm if the following unimprovable condition holds:

$$N^{-1} = o(\varepsilon^{1+k}), \quad k = 0, 1, 2. \tag{3.12}$$

Lemma 3.1. *In the case of difference scheme* (3.4), (3.6) *the condition* (3.12) *is necessary and sufficient for the convergence of the derivatives* $(d^k/dx^k)\tilde{z}(x)$, $x \in \overline{D}$, $k = 0, 1, 2$ *in the* $\| \cdot \|$ *norm. Estimates* (3.5)–(3.9), (3.11) *hold for the mesh solutions and for the function* $\tilde{z}(x)$, $x \in \overline{D}$; *estimates* (3.9), (3.11), *and also the estimate* (3.7), *if the condition* $N^{-1} = \mathcal{O}(\varepsilon)$ *is valid, are unimprovable with respect to the values of* N, ε.

2. In the case of the $\rho_u^{1h}(\cdot)$-metric, we consider the behavior of the function

$$u_0^h(x) = u(x), \qquad x \in \overline{D}_h, \tag{3.13}$$

which is *a projection* of the solution on the mesh $\overline{D}_h$ of type (3.3). The distance between the functions $u(x)$ and $z(x)$, $x \in \overline{D}_{h(3.3)}$ is defined by

$$\rho_u^{kh}(z) \equiv \|u - \tilde{z}\| + \sum_{n=1}^k \rho_u^{(n)h}(z), \quad k = 1, 2, \tag{3.14}$$

where

$$\rho_u^{(n)h}(z) = \inf_{\beta} \max \left\{ \left\| \frac{d^n}{dx^n}(v - \tilde{z}) \right\|_{\overline{D}_\beta^{[n]}}, \ \left\| \left(\frac{d^n}{dx^n} u \right)^{-1} \frac{d^n}{dx^n}(v - \tilde{z}) \right\|_{\overline{D}_\beta^{n-}} \right\}, \quad n = 1, 2.$$

Let us evaluate the proximity of the functions $u(x)$, $x \in \overline{D}$ and $u_0^h(x)$, $x \in \overline{D}_h$ in the $\rho_u^{1h}(\cdot)$-metric. The solution of problem (3.2), (3.1) can be decomposed into the sum of two functions which are the regular and singular components

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \quad U(x) = U^0(x), \quad x \in \overline{D}, \tag{3.15}$$

where $U^0(x)$, $x \in \overline{D}^0$ and $V(x)$, $x \in \overline{D}$ are the solutions of the problems

$$L\, U^0(x) = f^0(x), \quad x \in D^0, \qquad U^0(x) = \varphi(x), \qquad x \in \Gamma^0;$$
$$L\, V(x) = 0, \qquad x \in D, \qquad V(x) = \varphi(x) - U(x), \quad x \in \Gamma.$$

Here $D^0 = (-\infty, d)$, $f^0(x)$, $x \in \overline{D}^0$ is a sufficiently smooth function, moreover, $f^0(x) = f(x)$, $x \in \overline{D}$, $f(x)$ vanishes outside an m-neighbourhood of the set $\overline{D}$; the function $U^0(x)$ is bounded on $x \in \overline{D}^0$. It is convenient to represent the function $u_0^h(x)$, $x \in \overline{D}_h$ as the sum

$$u_0^h(x) = U_0^h(x) + V_0^h(x), \quad x \in \overline{D}_h, \quad U_0^h(x) = U(x), \quad V_0^h(x) = V(x), \quad x \in \overline{D}_h.$$

For the component $U_0^h(x)$, $x \in \overline{D}_h$ we have the estimate

$$\left\| \frac{d}{dx}(U - \tilde{U}_0^h) \right\| \leq M\, N^{-1}. \tag{3.16}$$

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For the component $V_0^h(x)$, $x \in \overline{D}_h$ we have the estimates

$$\left\| \left(\frac{d}{dx} V \right)^{-1} \frac{d}{dx} \left(V - \tilde{V}_0^h \right) \right\| \leq \tag{3.17a}$$

$$\leq \begin{cases} M \, \varepsilon^{-1} \, h^i & for \quad h^i \leq M_1 \, \varepsilon, \\ M \, (\varepsilon^{-1} \, h^i)^{-1} \exp(m_1 \varepsilon^{-1} h^i) & for \quad h^i \geq M_2 \, \varepsilon; \end{cases}$$

$$\left\| \frac{d}{dx} (V - \tilde{V}_0^h) \right\| \leq M \, \varepsilon^{-1} \, (\varepsilon + h^i)^{-1} \, h^i \, \exp(-m_1 \varepsilon^{-1} x^i) \tag{3.17b}$$

$$for \quad x \in \overline{D}, \quad r(x, \Gamma_1) \leq m, \quad x \in [x^i, x^{i+1}];$$

$$\left\| \frac{d}{dx} (V - \tilde{V}_0^h) \right\| \leq M \, N^{-1} \quad for \quad x \in \overline{D}, \quad r(x, \Gamma_1) > m. \tag{3.17c}$$

Here $r(x, \Gamma_1)$ is the distance between the point x and the set Γ_1, $x^i, x^{i+1} \in \overline{D}_h$, $m_1 = a^{-1} b$, $M_2 \leq M_1$. Estimates (3.16), (3.17a), (3.17b) are unimprovable w.r.t. the values of N, h^i, ε.

It follows from estimates (3.16), (3.17) that the value $\rho_u^{1h}(u_0^h)$ is not bounded ε-uniformly. In the case of meshes (3.3) (meshes with an arbitrary distribution of the nodes) the condition

$$N^{-1} = \mathcal{O}\left(\varepsilon\right) \tag{3.18}$$

is necessary and sufficient for $\rho_u^{1h}(u_0^h)$ to be ε-uniformly bounded. Under condition (3.18) we have

$$\rho_u^{1h}(u_0^h) \leq M \, \varepsilon^{-1} \, N^{-1}; \tag{3.19}$$

this estimate is unimprovable w.r.t. the values of N, ε; the convergence defect for the projection $u_0^h(x)$, $x \in \overline{D}_h$ in the $\rho_u^{1h}(\cdot)$ metric is $\mathcal{O}\left(\varepsilon^{-1}\right)$ for meshes (3.3), i.e. the defect is the same as that for scheme (3.4), (3.6) in the norm $\| \cdot \|$.

Lemma 3.2. *Condition* (3.18) *in the case of meshes* (3.3) *is necessary and sufficient for the projection* $u_{0(3.13)}^h(x)$, $x \in \overline{D}_h$ *to be ε-uniformly bounded in the $\rho_u^{1h}(\cdot)$ metric. Under condition* (3.18), *the function $u_0^h(x)$ satisfies the estimate* (3.19) *being unimprovable with respect to the values of N, ε; the function $u_0^h(x)$, $x \in \overline{D}_h$ converges to $u(x)$, $x \in \overline{D}$ in the $\rho_u^{1h}(\cdot)$ metric with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$.*

3. The consideration of the solutions of the boundary value and discrete problems in the case of mesh (3.6) for $h = \varepsilon$ shows that $\rho_u^{1h}(z)$ grows without bound as $N \to \infty$; thus, the condition (3.18) is not sufficient for the boundedness of $\rho_u^{1h}(z)$.

For the discrete solutions on meshes (3.6), we have the rough estimate

$$\rho_u^{1h}(z) \leq M \, N^{-1}(\varepsilon + N^{-1})^{-2} \quad for \quad \overline{D}_h = \overline{D}_{h(3.6)}^u. \tag{3.20}$$

Thus, in contrast to the magnitude $\left\| \frac{d}{dx}(u - \tilde{z}) \right\|$ which grows without bound for $N^{-1} \gg \varepsilon^2$ and $\varepsilon \to 0$, the function $z(x)$ in the $\rho_u^{1h}(\cdot)$ metric (i.e. the distance between $u(x)$ and $z(x)$ in

the $\rho_u^{1h}(\cdot)$ metric) is ε-uniformly bounded for fixed values of N, however, this function is not bounded N-uniformly.

Taking account of estimates (3.16), (3.17), we establish that the condition

$$N^{-1} = \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)\right) \tag{3.21}$$

is necessary and sufficient for the solutions of scheme (3.4), (3.6) to be N- and (N, ε)-uniformly bounded in the $\rho_u^{1h}(\cdot)$ metric. Under condition (3.21) we have

$$\rho_u^{1h}(z) \ \leq \ M \, N^{-1} \, \varepsilon^{-1} \, \ln(\varepsilon^{-1} + 1); \tag{3.22}$$

this estimate is unimprovable w.r.t. the values of N, ε. Thus, the convergence defect of scheme (3.4), (3.6) in the $\rho_u^{1h}(\cdot)$ metric is $\mathcal{O}\left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1)\right)$.

Lemma 3.3. *Condition* (3.21) *is necessary and sufficient for the solutions of finite difference scheme* (3.4), (3.6) *to be bounded in the* $\rho_u^{1h}(\cdot)$ *metric. The mesh solutions satisfy estimate* (3.20), *and also estimate* (3.22) *unimprovable w.r.t. the values of N, ε holds if condition* (3.21) *is true. The convergence defect of this scheme is* $\mathcal{O}\left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1)\right)$.

Remark. In the $\rho_u^{2h}(\cdot)$ metric, the solution of finite difference scheme (3.4), (3.6) is (N, ε)-uniformly bounded if condition (3.21) is satisfied. Moreover, under this condition the estimate for the $\rho_u^{2h}(\cdot)$ metric is the same as that for the $\rho_u^{1h}(\cdot)$ metric (see (3.22)):

$$\rho_u^{2h}(z) \ \leq \ M \, N^{-1} \, \varepsilon^{-1} \, \ln(\varepsilon^{-1} + 1); \tag{3.23}$$

this estimate and condition (3.21) are unimprovable. Thus, the statement of Lemma 3.3 remains valid also for the $\rho_u^{2h}(\cdot)$ metric.

4. It seems attractive to find such meshes on which the function $u_0^h(x)$, $x \in \overline{D}_h$ converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.

Notice that in the class of meshes (3.3) satisfying only condition (3.18), there are no meshes on which the function $u_0^h(x)$, $x \in \overline{D}_h$ converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric. Nevertheless, it is interesting to clarify whether or not there exist meshes from class (3.3) (not satisfying condition (3.18)) on which the function $u_0^h(x)$, $x \in \overline{D}_h$ converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.

Consider the function $u_0^h(x)$, $x \in \overline{D}_h$ on the set

$$\overline{D}_h^\sigma = \overline{D}_h \cap [0, \sigma] \tag{3.24}$$

for $\sigma = \min\left[4^{-1}d, \psi(\varepsilon)\right]$, where $\overline{D}_h$ is a sufficiently arbitrary mesh from class (3.3), $\psi(\varepsilon) = \varepsilon \ln^\nu \varepsilon^{-1}$, ν is any number from the interval $(0, 1)$. The unimprovability of estimates (3.17a), (3.17b) implies that the condition

$$\sup_\varepsilon \max_i \left[\varepsilon^{-1} h^i\right] \to 0 \quad for \quad N \to \infty; \quad h^i = x^{i+1} - x^i, \quad x^i, x^{i+1} \in \overline{D}_h^\sigma$$

is necessary and sufficient for the ε-uniform convergence of the function $u_0^h(x)$, $x \in \overline{D}_h$ in the $\rho_u^{1h}(\cdot)$ metric. However, such a condition is impossible. We are thus led to the following lemma.

Lemma 3.4. *In the class of meshes* (3.3) *there do not exist meshes on which the function* $u_0^h(x)$, $x \in \overline{D}_h$ *converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.*

5. It is of interest to clarify whether or not there exist meshes on which the solution of finite difference scheme (3.4) converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric. We consider scheme (3.4), (3.3), assuming that the following condition satisfies on the mesh $\overline{D}_{h(3.24)}^\sigma$:

$$h^i \geq h^{i-1}, \tag{3.25}$$

that is, the mesh step does not decrease with moving away from the boundary Γ_1; $\sigma = \sigma_{(3.24)}$. On this mesh there is a node $x^{i*} \in \overline{D}_h$ such that the condition $\varepsilon^{-1} h^{i*} \gg 1$ holds for an appropriately chosen value of ε (sufficiently small) and for fixed N. Assuming that $\rho_u^{1h}(z)$ tends to zero ε-uniformly, we find that the derivative $\delta_{\overline{x}} z(x^{i*})$ satisfies the following relation

$$\left| \delta_{\overline{x}} z(x^{i*}) - \frac{d}{dx} V(x^{i*}) \right| = o(\varepsilon^{-1} \exp(-m\varepsilon^{-1} x^{i*})) \quad for \quad \varepsilon \to 0, \quad m = m_{1(3.17)}.$$

But then, by virtue of the difference equation from (3.4), we have the following formula for the derivative $\delta_x z(x^{i*})$:

$$\left| \left(\frac{d}{dx} V(x^{i*+1}) \right)^{-1} \left(\frac{d}{dx} V(x^{i*+1}) - \delta_x z(x^{i*}) \right) \right| \gg 1 \quad for \quad \varepsilon \to 0,$$

where $V(x)$, $x \in \overline{D}$ is the singular component of the solution to problem (3.2), (3.1). This formula contradicts to the assumption that the solution of scheme (3.4) on the mesh (3.3) satisfying (3.25) converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.

Lemma 3.5. *In the class of meshes* (3.3) *satisfying the condition* (3.25) *there do not exist meshes on which scheme* (3.4) *converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.*

Remark. The statement of Lemma 3.5 remains valid also in that case if on the interval $[0, \sigma]$, where $\sigma = \min[m, m_0 \varepsilon \ln \varepsilon^{-1}]$, $m_0 = m_{1(3.17)}$, there exists a subinterval of width $l(\varepsilon)$, where $l(\varepsilon) \to 0$, $\varepsilon^{-1} \ln(\varepsilon) \to \infty$ for $\varepsilon \to 0$, on which condition (3.25) holds.

4. Problem (2.2), (2.1). A classical finite difference scheme

1. In this section we give a classical difference scheme for problem (2.2), (2.1) and show some issues arising in the numerical solution. On the set $\overline{D}$ we introduce the rectangular mesh

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{4.1}$$

where $\overline{\omega}_1$ and ω_2 are meshes on the interval $[0, d]$ and on the axis x_2 respectively; ω_2 is a uniform mesh with the stepsize $h_2 = N_2^{-1}$; $\overline{\omega}_1$ is a mesh with an arbitrary distribution of nodes satisfying only the condition $h_1 \leq M N_1^{-1}$, $h_1 = \max_i h_1^i$, where $h_1^i = x_1^{i+1} - x_1^i$, $x_1^i, x_1^{i+1} \in \overline{\omega}_1$.

Here $N_1 + 1$ and $N_2 + 1$ are the number of nodes in the mesh $\overline{\omega}_1$ and on the interval of unit length in the mesh ω_2; assume $N = \min[N_1, N_2]$. To construct numerical methods, it seems interesting to use sufficiently simple meshes

$$\overline{D}_h = \overline{D}_{h(4.1)}, \quad where \ \overline{\omega}_1 \ is \ a \ piecewise \ uniform \ mesh. \tag{4.2}$$

To solve the problem, we use the monotone scheme with upwind difference derivatives [14]

$$\Lambda z(x) \equiv \left\{ \varepsilon \sum_{s=1,2} a_s(x)\delta_{\overline{xs}\widehat{xs}} + \sum_{s=1,2} \left[b_s^+(x)\delta_{xs} + b_s^-(x)\delta_{\overline{xs}} \right] - c(x) \right\} z(x) =$$

$$= f(x,t), \quad x \in D_h, \tag{4.3}$$

$$z(x) = \varphi(x), \quad x \in \Gamma_h.$$

where $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\delta_{\overline{xs}\widehat{xs}}z(x)$ and $\delta_{xs}z(x)$, $\delta_{\overline{xs}}z(x)$ are the second and first (forward and back) difference derivatives, for example, $\delta_{\overline{x1}\widehat{x1}}z(x) = 2(h_1^i + h_1^{i-1})^{-1}[\delta_{x1}z(x) - \delta_{\overline{x1}}z(x)]\,z(x)$, $x = (x_1^i, x_2)$; $v^+(x) = 2^{-1}(v(x) + |v(x)|)$, $v^-(x) = 2^{-1}(v(x) - |v(x)|)$.

For the solution of difference scheme (4.3), (4.1) we have the error estimate

$$| u(x) - z(x) | \leq M \left[(\varepsilon^2 + N_1^{-1})^{-1}N_1^{-1} + N_2^{-1} \right], \quad x \in \overline{D}_h. \tag{4.4}$$

Let

$$\overline{D}_h \quad be \ an \ uniform \ mesh \tag{4.5}$$

with the stepsize in x_1 equal to $h_1 = d\,N_1^{-1}$. For the solutions of finite difference scheme (4.3) on meshes (4.2) or (4.5) we have the error bounds

$$| u(x) - z(x) | \leq M \left[(\varepsilon + N_1^{-1})^{-1}N_1^{-1} + N_2^{-1} \right], \quad x \in \overline{D}_h. \tag{4.6}$$

For the derivatives in the case of scheme (4.3), (4.5) we have

$$\left| \frac{\partial}{\partial x_1}u(x) - \delta_{x1}\,z(x) \right| \leq \tag{4.7a}$$

$$\leq M \left[\varepsilon^{-1}\,(\varepsilon + N_1^{-1})^{-1}N_1^{-1} + N_2^{-1} \right], \quad x \in \overline{D}_h^-;$$

$$\left| \frac{\partial^2}{\partial x_1^2}u(x) - \delta_{\overline{x1}\widehat{x1}}\,z(x) \right| \leq \tag{4.7b}$$

$$\leq M \left[\varepsilon^{-2}\,(\varepsilon + N_1^{-1})^{-1}N_1^{-1} + N_2^{-1} \right], \quad x \in \overline{D}_h;$$

$$\left| \frac{\partial}{\partial x_2}u(x) - \delta_{x2}\,z(x) \right|, \quad \left| \frac{\partial^2}{\partial x_2^2}u(x) - \delta_{\overline{x2}\widehat{x2}}\,z(x) \right| \leq \tag{4.8}$$

$$\leq M \left[(\varepsilon + N_1^{-1})^{-1}N_1^{-1} + N_2^{-1} \right], \quad x \in \overline{D}_h.$$

Estimates (4.7) are unimprovable w.r.t. the values of N_s, ε, while estimates (4.6), (4.8) are unimprovable under the (unimprovable) condition

$$N_1^{-1} = \mathcal{O}\left(\varepsilon\right). \tag{4.9}$$

When deriving these estimates we used the explicit form of the main terms in the asymptotic (in ε) for the regular and singular parts of the solutions to problems (2.2), (2.1) and (4.3), (4.5).

For the function $\widetilde{z}_{(2.6)}(x)$, $x \in \overline{D}$ in the case of scheme (4.3), (4.5) we have the estimates

$$\left\|\frac{\partial^{k_1}}{\partial x_1^{k_1}}(u - \widetilde{z})\right\|, \quad \|u - \widetilde{z}\|_{C_*^k} \leq \tag{4.10}$$

$$\leq M \left[\varepsilon^{-k_1}\left(\varepsilon + N_1^{-1}\right)^{-1} N_1^{-1} + N_2^{-1}\right], \quad k_1 = k = 1, 2;$$

$$\left\|\frac{\partial^{k_2}}{\partial x_2^{k_2}}(u - \widetilde{z})\right\|, \quad \|u - \widetilde{z}\| \leq$$

$$\leq M \left[\left(\varepsilon + N_1^{-1}\right)^{-1} N_1^{-1} + N_2^{-1}\right], \quad k_2 = 0, 1, 2;$$

these estimates are unimprovable w.r.t. the values of N, ε.

Thus, the function $\widetilde{z}(x)$ converges in the norms $\|\cdot\|$ and $\|\cdot\|_{C_*^k}$ if the following unimprovable condition holds:

$$N_1^{-1} = o(\varepsilon^{1+k_0}) \tag{4.11}$$

where $k_0 = 0$ and $k_0 = k$ in the case of the norms $\|\cdot\|$ and $\|\cdot\|_{C_*^k}$ respectively; $k = 1, 2$.

Theorem 4.1. *Let estimates (9.2), (9.3) be satisfied for the solutions of boundary value problem (2.2), (2.1) and their components from representation (9.1). Then, in the case of finite difference scheme (4.3), (4.5), the condition (4.11) is necessary and sufficient for the convergence of the function $\widetilde{z}(x)$, $x \in \overline{D}$ in the norms $\|\cdot\|$ and $\|\cdot\|_{C_*^k}$. Estimates (4.4), (4.6), (4.7), (4.8), (4.10) are valid for the mesh solutions and derivatives; estimates (4.7), (4.10), and also estimates (4.6), (4.8), if condition (4.9) holds, are unimprovable with respect to the values of N_s, ε.*

2. We now consider the convergence of scheme (4.3), (4.5) in the $\rho_u^{kh}(\cdot)$ metric.

The solutions of this scheme are not bounded (N_s, ε)-uniformly in the $\rho_u^{kh}(\cdot)$ metric. The condition

$$N_1^{-1} = \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)\right) \tag{4.12}$$

is necessary and sufficient for the numerical solutions to be bounded in the $\rho_u^{kh}(\cdot)$ metric. Under this condition, we have the following estimate

$$\rho_u^{kh}(z) \leq M \left[N_1^{-1}\varepsilon^{-1}\ln(\varepsilon^{-1} + 1) + N_2^{-1}\right], \quad k = 1, 2, \tag{4.13}$$

which is unimprovable w.r.t. the values of N, ε.

The condition

$$N_1^{-1} = o(\varepsilon \ \ln^{-1}(\varepsilon^{-1} + 1)) \tag{4.14}$$

is necessary and sufficient for the convergence of scheme (4.3), (4.5) in the $\rho_u^{kh}(\cdot)$ metric. The convergence defect of this scheme in the $\rho_u^{kh}(\cdot)$ metric is $\mathcal{O}\left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1)\right)$.

Theorem 4.2. *Let the hypothesis of Theorem 4.1 be fulfilled. Then condition (4.12) (condition (4.14)) is necessary and sufficient for the solutions of finite difference scheme (4.3), (4.5) to be (N_s, ε)-uniformly bounded (to be convergent) in the $\rho_u^{kh}(\cdot)$ metric; the convergence defect of this scheme is $\mathcal{O}\left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1)\right)$. Under condition (4.12), the estimate (4.13) unimprovable with respect to the values of N, ε is satisfied for the solutions of the difference scheme.*

3. As above in the case of the model boundary value problem (3.2), (3.1) and finite difference scheme (3.4), (3.1), we establish the following non-existence result.

Theorem 4.3. *In the class of meshes (4.1) satisfying condition (3.25) on the interval $[0, d]$, there do not exist meshes on which scheme (4.3) converges ε-uniformly in the $\rho_u^{1h}(\cdot)$ metric.*

5. Piecewise uniform mesh. The model problem

For the boundary value problem (3.2), (3.1) we give a scheme on piecewise uniform meshes which converges ε-uniformly in the norm $\| \cdot \|$ and discuss the approximation of derivatives.

On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h^* = \overline{D}_h^* (\sigma(l)) = \overline{D}_h^*(l) = \overline{\omega}_0^*. \tag{5.1a}$$

Here $\overline{\omega}_0^*$ is a mesh with a piecewise constant stepsize. To construct the mesh $\overline{\omega}_0^*$, we divide the interval $[0, d]$ into two parts $[0, \sigma]$ and $[\sigma, d]$, in each of them the mesh stepsize is constant and equal to $h^{(1)} = 2\sigma N^{-1}$ and $h^{(2)} = 2(d - \sigma)N^{-1}$, respectively. Assume

$$\sigma = \sigma(\varepsilon, N, d; l, m) = \min \left[2^{-1}d, \ lm^{-1}\varepsilon \ln N \right], \tag{5.1b}$$

where $m = a^{-1}b$, $l > 0$ is a mesh parameter. The mesh $\overline{D}_h^*$ is thus constructed. Let us introduce auxilary parameters γ_k used in the sequel for constructions

$$\gamma_k = \gamma_k(\varepsilon; m) = k \, m^{-1} \varepsilon \ \ln(\varepsilon^{-1} + 1), \quad m = m_{(5.1)}, \quad k = 1, 2. \tag{5.2}$$

Note that the meshes

$$\overline{D}_h^* = \overline{D}_{h(5.1)}^*(l) \tag{5.3}$$

have been introduced in [4] (see also [5, 9, 6]), where l is an arbitrary number satisfying the condition $l > 1$.

For the solutions of difference scheme (3.4), (5.1) we have the estimates

$$\|u - \tilde{z}\| \le M \left\{ N^{-1} \min \left[\ln N, \varepsilon^{-1} \right] + N^{-l-1} \left(\varepsilon + N^{-1} \right)^{-1} \right\} \equiv \qquad (5.4a)$$

$$\equiv M \, \mu_0(N, \varepsilon; l);$$

$$\|u - \tilde{z}\| \le M \left(N^{-1} \ln N + N^{-l} \right), \quad M = M(l). \qquad (5.4b)$$

Estimates (5.4a) and (5.4b) are unimprovable w.r.t. the values of N, ε and N respectively. We obtain the best ε-uniform order of convergence

$$\|u - \tilde{z}\| \le M \, N^{-1} \ln N \qquad (5.5)$$

if the following condition holds

$$l \ge 1. \qquad (5.6)$$

For the derivatives in the case of scheme (3.4), (5.1), we have

$$\left| \frac{d}{dx} u(x) - \delta_x z(x) \right| \le M \, \varepsilon^{-1} \, \mu_0 \left(N, \, \varepsilon; l \right), \quad x \in \overline{D}_h^-;$$

$$\left| \frac{d^2}{dx^2} u(x) - \delta_{\widehat{xx}} z(x) \right| \le M \, \varepsilon^{-2} \, \mu_0 \left(N, \, \varepsilon; l \right), \quad x \in D_h,$$

where $\mu_0 \left(N, \, \varepsilon; l \right) = \mu_{0(5.4)} \left(N, \, \varepsilon; l \right);$ these estimates are unimprovable. For the function $\tilde{z}(x)$, $x \in \overline{D}$, we have the unimprovable estimate

$$\left\| \frac{d^k}{dx^k} (u - \tilde{z}) \right\|, \; \|u - \tilde{z}\|_{C_*^k} \le M \, \varepsilon^{-k} \, \mu_0 \left(N, \, \varepsilon; l \right), \quad k = 1, 2. \qquad (5.7)$$

The error $\left\| \dfrac{d^k}{dx^k} (u - \tilde{z}) \right\|, \quad k = 1, 2$ is bounded under the condition

$(N^{-1} \varepsilon^{-k} \min \left[\ln N, \varepsilon^{-1} \right] + N^{-l-1} \varepsilon^{-k} (\varepsilon + N^{-1})^{-1} \le M)$:

$$\begin{aligned}
N^{-1} &= \mathcal{O}\left(\varepsilon^{2/(l+1)} \right) && \text{for} \quad l < 1, \\
N^{-1} &= \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1) \right) && \text{for} \quad l \ge 1, \quad k = 1; \\
N^{-1} &= \mathcal{O}\left(\varepsilon^{3/(l+1)} \right) && \text{for} \quad l < 2^{-1}, \\
N^{-1} &= \mathcal{O}\left(\varepsilon^2 \ln^{-1}(\varepsilon^{-1} + 1) \right) && \text{for} \quad l \ge 2^{-1}, k = 2;
\end{aligned} \qquad (5.8)$$

this condition is unimprovable. The condition

$$\begin{aligned}
N^{-1} &= o(\varepsilon^{(k+1)/(l+1)}) && \text{for} \quad l < k^{-1}; \\
N^{-1} &= o(\varepsilon^k \ln^{-1}(\varepsilon^{-1} + 1)) && \text{for} \quad l \ge k^{-1}, \quad k = 1, 2
\end{aligned} \qquad (5.9)$$

is necessary and sufficient for the convergence of the derivatives $(d^k/dx^k)\tilde{z}, \; x \in \overline{D}$.

The convergence defect of the scheme in the norm $\| \cdot \|_{C_*^k}$ is

$$\mathcal{O}\left(\varepsilon^{-(k+1)/(l+1)}\right) \quad \text{for} \quad l < k^{-1} \quad \text{and} \quad \mathcal{O}\left(\varepsilon^{-k}\ln(\varepsilon^{-1}+1)\right) \quad \text{for} \quad l \geq k^{-1}.$$

Lemma 5.1. *The solution of finite difference scheme* (3.4), (5.1) *converges ε-uniformly in the norm $\| \cdot \|$. Condition* (5.8) *(condition* (5.9)*) is necessary and sufficient for the boundedness of $\|u - \tilde{z}\|_{C_*^k}$, $k = 1, 2$ (for the convergence in the norm $\| \cdot \|_{C_*^k}$). Estimates* (5.4), (5.7) *and also estimate* (5.5), *if condition* (5.6) *holds, are satisfied for the mesh solutions. Estimates* (5.4a), (5.7) *and estimate* (5.4b) *are unimprovable with respect to the values of N, ε and N respectively; the convergence defect for this scheme in the norm $\| \cdot \|_{C_*^k}$ is not lower than $\mathcal{O}\left(\varepsilon^{-k}\ln(\varepsilon^{-1}+1)\right)$, $k = 1, 2$.*

2. Let us consider the convergence of scheme (3.4), (5.1) in the $\rho_u^{kh}(\cdot)$ metric.

Taking into account estimate (3.17), we verify that the solution of grid problem (3.4), (5.1) is not bounded (N, ε)-uniformly in the $\rho_u^{1h}(\cdot)$ metric. With regard of estimates (3.16), (3.17) and the explicit form of the singular components of the solutions to the differential and grid problems, we establish that the condition (either $\varepsilon^{-2}h^{(1)}\gamma_1 \leq M$ for $\varepsilon^{-1}(\gamma_1 - \sigma) < M_0$, or $N^{-1}\varepsilon^{-1}(\varepsilon + N^{-1})^{-1}(\gamma_1 - \sigma) \leq M$ for $\gamma_1 - \sigma - M_0\varepsilon \geq 0$) :

$$N^{-1} = \mathcal{O}\left(\varepsilon^2(\gamma_1 - \sigma)^{-1}\right) \quad \text{for} \quad \varepsilon^{-1}(\gamma_1 - \sigma) \geq M_0, \tag{5.10a}$$

$$\sigma N^{-1} = \mathcal{O}\left(\varepsilon^2\gamma_1^{-1}\right) \qquad \text{for} \quad \varepsilon^{-1}(\gamma_1 - \sigma) < M_0, \tag{5.10b}$$

is necessary and sufficient for the mesh solutions to be bounded in the $\rho_u^{1h}(\cdot)$ metric, where $\sigma = \sigma_{(5.1)}$, $\gamma_1 = \gamma_{1(5.2)}$, M_0 is any constant. In condition (5.10a) we have $M_0 \leq \varepsilon^{-1}(\gamma_1 - \sigma) \leq M \ln(\varepsilon^{-1} + 1)$.

In the case of condition (5.10a), we obtain the estimate

$$\rho_u^{1h}(z) \leq M\left\{N^{-1}\left(\min\left[\ln N, \varepsilon^{-1}\right]\right)^2 + N^{-1}\left(\varepsilon + N^{-1}\right)^{-1}\varepsilon^{-1}(\gamma_1 - \sigma)\right\}; \tag{5.11}$$

while under condition (5.10b) we have

$$\rho_u^{1h}(z) \leq M\left\{N^{-1}\min\left[\ln N, \varepsilon^{-1}\right]\varepsilon^{-1}\gamma_1 + \right. \tag{5.12}$$

$$\left. + N^{-1}\left(\varepsilon + N^{-1}\right)^{-1}\exp(-\varepsilon^{-1}(\sigma - \gamma_1))\right\}.$$

Estimates (5.11), (5.12) are unimprovable.

Note that under the condition $\varepsilon^{-1}(\gamma_1 - \sigma) \geq M_0$ (see condition (5.10a) and estimate (5.11)) we have $\varepsilon^{-1}(\gamma_1 - \sigma) \leq M \ln \varepsilon^{-1} + 1$.

In the case of the $\rho_u^{2h}(\cdot)$ metric, the condition

$$N^{-1} = \mathcal{O}\left(\varepsilon^2(\gamma_2 - \sigma)^{-1}\right) \quad \text{for} \quad \varepsilon^{-1}(\gamma_2 - \sigma) \geq M_0, \tag{5.13a}$$

$$\sigma N^{-1} = \mathcal{O}\left(\varepsilon^2\gamma_2^{-1}\right) \qquad \text{for} \quad \varepsilon^{-1}(\gamma_2 - \sigma) < M_0, \tag{5.13b}$$

is necessary and sufficient for the mesh solutions to be bounded. Under condition (5.13a) we have

$$\rho_u^{2h}(z) \leq M \left\{ N^{-1} \left(\min \left[\ln N, \varepsilon^{-1} \right] \right)^2 + N^{-1} \left(\varepsilon + N^{-1} \right)^{-1} \varepsilon^{-1} (\gamma_2 - \sigma) \right\}. \tag{5.14}$$

Under condition (5.13b) and the additional condition

$$\left\| \frac{d^2}{dx^2} U(0) \right\| \leq M \left(\varepsilon + \delta \right)^\lambda, \quad when \quad k = 2, \tag{5.15}$$

where λ is any constant from the interval $[0, 1]$, the value $\delta = \delta(N)$ tends to zero as $N \to \infty$, the following estimate holds

$$\rho_u^{2h}(z) \leq M \left\{ N^{-1} \min \left[\ln N, \varepsilon^{-1} \right] \varepsilon^{-1} \gamma_2 + N^{-1} \left(\varepsilon + N^{-1} \right)^{-1} (\varepsilon + \delta)^\lambda + \right.$$

$$\left. + N^{-1} \left(\varepsilon + N^{-1} \right)^{-1} \exp \left(-\varepsilon^{-1} (\sigma - \gamma_2) \right) \right\}. \tag{5.16}$$

Estimates (5.14), (5.16) are unimprovable.

It follows from estimates (5.11), (5.12), (5.14) and (5.16) that the condition of convergence in the $\rho_u^{kh}(\cdot)$ metric for scheme (3.4), (5.1) depends on the value l. The scheme converges if the following unimprovable condition holds:

$$\begin{aligned} N^{-1} &= o(\varepsilon^2 (\gamma_k - \sigma)^{-1}), & l < k; \\ N^{-1} &= o(\varepsilon^{k/l}), & l \geq k; \quad k = 1, 2 \quad and \quad \lambda > 0 \quad for \quad k = 2, \end{aligned} \tag{5.17a}$$

$$\begin{aligned} N^{-1} &= o(\varepsilon^2 (\gamma_k - \sigma)^{-1}), & l < k; \\ N^{-1} &= o(\varepsilon), & l \geq k; \quad k = 2 \quad and \quad \lambda = 0. \end{aligned} \tag{5.17b}$$

If $l < k$ in condition (5.17), then we have

$$m \, \varepsilon \, \ln^{-1}(\varepsilon^{-1} + 1) \leq \varepsilon^2 \left(\gamma_k - \sigma \right)^{-1} \leq M_0^{-1} \, \varepsilon.$$

Thus, the convergence defect for scheme (3.4), (5.1) in the $\rho_u^{kh}(\cdot)$ metric is not higher than $\mathcal{O} \left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1) \right)$ for $l < k$; for $l \geq k$ the defect is $\mathcal{O} \left(\varepsilon^{-k/l} \right)$ if $k = 1$ and also if $k = 2$ and $\lambda > 0$; the defect is $\mathcal{O} \left(\varepsilon^{-1} \right)$ if $\lambda = 0$ for $k = 2$. The defect $\mathcal{O} \left(\varepsilon^{-1} \ln(\varepsilon^{-1} + 1) \right)$ is achieved, for example, if the condition $\gamma_k \geq (1 + m)\sigma$ holds.

If the following condition holds

$$l \geq k \nu^{-1} \quad and \quad \{\lambda > 0, \ if \ k = 2\}, \quad k = 1, 2, \tag{5.18}$$

where $\nu = \nu_{(2.10)}$, $\nu \leq 1$, then the convergence defect of scheme (3.4), (5.1) in the $\rho_u^{kh}(\cdot)$ metric is not higher than $\mathcal{O} \left(\varepsilon^{-\nu} \right)$, i.e. the scheme converges almost ε-uniformly in the $\rho_u^{kh}(\cdot)$ metric; the convergence defect is unimprovable with respect to ν. Under the condition

$$l \geq k \quad and \quad N^{-1} = \mathcal{O} \left(\varepsilon^{k/l} \right), \quad \{\lambda > 0 \ if \ k = 2\} \tag{5.19}$$

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the following unimprovable estimate is valid:

$$\rho_u^{kh}(z) \leq \begin{cases} M\left\{N^{-1}\min\left[\ln N, \varepsilon^{-1}\right]\ln(\varepsilon^{-1}+1) + N^{-l}\varepsilon^{-1}\right\}, & k = 1, \\ M\left\{N^{-1}\min\left[\ln N, \varepsilon^{-1}\right]\ln(\varepsilon^{-1}+1) + N^{-l}\varepsilon^{-2}+ \right. \\ \left. \quad +N^{-1}(\varepsilon + N^{-1})^{-1}(\varepsilon + \delta)^{\lambda}\right\}, & k = 2. \end{cases} \tag{5.20}$$

Under the condition

$$l \equiv k = 2, \quad \lambda = 0 \quad and \quad N^{-1} = \mathcal{O}\left(\varepsilon\right) \tag{5.21}$$

the following unimprovable estimate holds

$$\rho_u^{2h}(z) \leq M\,N^{-1}(\varepsilon + N^{-1})^{-1}. \tag{5.22}$$

Lemma 5.2. *In the case of finite difference scheme (3.4), (5.1) the condition (5.10), (5.13) (condition (5.17)) is necessary and sufficient for the mesh solutions to be (N, ε)-uniformly bounded (for the convergence of the scheme) in the $\rho_u^{kh}(\cdot)$ metric. Under condition (5.18) the scheme converges almost ε-uniformly in $\rho_u^{kh}(\cdot)$; the convergence defect is $\mathcal{O}\left(\varepsilon^{-k/l}\right)$ if $l \geq k$. Under condition (5.21) the scheme converges in $\rho_u^{2h}(\cdot)$ with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$. The mesh solutions satisfy estimates (5.11), (5.12), (5.14), (5.16), (5.20) and (5.22) if conditions (5.10a), (5.10b), (5.13a), {(5.13a), (5.15)}, (5.19) and (5.21) hold, respectively. These estimates are unimprovable with respect to the values of N, ε.*

6. Finite difference scheme with improved ρ_u^{kh} convergence. The model problem

The magnitude σ from mesh (5.1), which locates the transition point $x = \sigma$ of the mesh, depends on the values of N, ε, l. On such meshes the scheme converges in the $\rho_u^{kh}(\cdot)$ metric under condition (5.17), in particular, under the following condition (unimprovable with respect to the value of ε):

$$N^{-1} = o(\varepsilon^{k/l}) \quad for \quad l \geq k, \quad k = 1, 2. \tag{6.1}$$

It is of our interest to construct piecewise-uniform meshes on which scheme (3.4) converges in the $\rho_u^{kh}(\cdot)$ metric under a weaker condition than condition (6.1).

1. We consider schemes on piecewise-uniform meshes in the case when the transition point $x = \sigma$ satisfies the following condition

$$\varepsilon^{-1}\sigma = \psi(\varepsilon), \quad where \quad \psi(\varepsilon) \to \infty, \quad \varepsilon\,\psi(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0.$$

On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h = \overline{D}_{h(5.1a)}^{\,*}(\sigma(\eta)), \tag{6.2a}$$

where

$$\sigma = \sigma(\varepsilon; \eta, m) = \min\left[2^{-1}d, \ \eta\, m^{-1}\varepsilon \ln(\varepsilon^{-1} + M)\right], \quad m = m_{(5.1)}, \tag{6.2b}$$

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$\eta > 0$ is a mesh parameter; the constant M is chosen to satisfy the condition $\ln(1 + M) \geq d$, $M > e$. For such values of σ we have

$$|V(x)| \leq M \, \varepsilon^\eta, \quad x \geq \sigma, \quad x \in \overline{D}.$$

By using the explicit form of the components (the singular part and the main term of the regular part) of the solutions to problems (3.2), (3.1) and (3.4), (6.2), we establish the estimate

$$\|u - \tilde{z}\| \leq M \left\{ \min \left[N^{-1} \ln(\varepsilon^{-1} + 1), \, 1 \right] + N^{-1} \varepsilon^\eta \left(\varepsilon + N^{-1} \right)^{-1} \right\} \equiv \tag{6.3}$$

$$\equiv M \, \mu_1(N, \varepsilon; \eta);$$

this estimate is unimprovable w.r.t. the values of N, ε. The error $\|u - \tilde{z}\|$ is ε-uniformly bounded under the condition

$$\forall N, \quad \forall \varepsilon. \tag{6.4}$$

The scheme converges in the norm $\| \cdot \|$ if the following unimprovable condition holds:

$$N^{-1} = o(\ln^{-1}(\varepsilon^{-1} + 1)); \tag{6.5}$$

thus, the convergence defect of the scheme is $\mathcal{O}\left(\ln(\varepsilon^{-1} + 1)\right)$.

In the case when the solution of the difference scheme converges under the condition $N^{-1} \ln^r(\varepsilon^{-1} + 1) \to 0$, where $r > 0$ is some constant, we say that the scheme converges ε-uniformly up to a logarithmic factor (namely, up to the factor $\ln^r(\varepsilon^{-1} + 1)$).

Thus, scheme (3.4), (6.2) converges in the norm $\| \cdot \|$ ε-uniformly up to the logarithmic factor $\ln^r(\varepsilon^{-1} + 1)$.

For the derivatives we get the unimprovable estimates

$$\left| \frac{d}{dx} u(x) - \delta_x z(x) \right| \leq M \varepsilon^{-1} \mu_1(N, \varepsilon; \eta), \quad x \in \overline{D}_h^-;$$

$$\left| \frac{d^2}{dx^2} u(x) - \delta_{\widehat{x}\widehat{x}} z(x) \right| \leq M \varepsilon^{-2} \mu_1(N, \varepsilon; \eta), \quad x \in D_h;$$

where $\mu_1(N, \varepsilon; \eta) = \mu_{1(6.3)}(N, \varepsilon; \eta)$. For the function $\tilde{z}(x)$, $x \in \overline{D}$ we have the unimprovable estimate

$$\left\| \frac{d^k}{dx^k}(u - \tilde{z}) \right\| \leq M \varepsilon^{-k} \mu_1(N, \varepsilon; \eta), \quad k = 1, 2. \tag{6.6}$$

The condition $\varepsilon^{-k} \mu_1(N, \varepsilon; \eta) \leq M$, i.e.

$$N^{-1} = \mathcal{O}\left(\varepsilon^k \ln^{-1}(\varepsilon^{-1} + 1) \right) \quad \text{for} \quad \eta \geq 1;$$

$$N^{-1} = \mathcal{O}\left(\varepsilon^{1+k-\eta} \right) \qquad \qquad \text{for} \quad \eta < 1; \quad k = 1, 2 \tag{6.7}$$

is necessary and sufficient for the error $\left\| \dfrac{d^k}{dx^k}(u - \tilde{z}) \right\|$ to be bounded. The derivatives $(d^k/dx^k)\tilde{z}(x),\ x \in \overline{D}$ converge if the following unimprovable condition holds:

$$
\begin{aligned}
N^{-1} &= o(\varepsilon^k \ln^{-1}(\varepsilon^{-1} + 1)) && \text{for}\quad \eta \geq k; \\
N^{-1} &= o(\varepsilon^{1+k-\eta}) && \text{for}\quad \eta < k; \quad k = 1, 2.
\end{aligned}
\tag{6.8}
$$

The convergence defect of the scheme in the $\| \cdot \|_{C_*^k}$ norm, $k = 1, 2$ is $\mathcal{O}\left(\varepsilon^{-1-k+\eta}\right)$ for $\eta < 1$ and $\mathcal{O}\left(\varepsilon^{-k} \ln(\varepsilon^{-1} + 1)\right)$ for $\eta \geq 1$.

Lemma 6.1. *Conditions* (6.4) *and* (6.7) *(conditions* (6.5) *and* (6.8)*) are necessary and sufficient for the errors in the solutions of finite difference scheme* (3.4), (6.2) $\|u - \tilde{z}\|$ *and* $\|u - \tilde{z}\|_{C_*^k}$ *to be bounded (for the convergence of the scheme in the norms* $\| \cdot \|$ *and* $\| \cdot \|_{C_*^k}$*) respectively. Estimates* (6.3), (6.6) *unimprovable with respect to the values of* N, ε *are satisfied for the mesh solutions. The convergence defect of the scheme is* $\mathcal{O}\left(\ln(\varepsilon^{-1} + 1)\right)$ *in the norm* $\| \cdot \|$ *and not lower than* $\mathcal{O}\left(\varepsilon^{-k} \ln(\varepsilon^{-1} + 1)\right)$ *in the norm* $\| \cdot \|_{C_*^k}$.

2. We now consider the appoximation of the solutions to the boundary value problem by the discrete solutions in the $\rho_u^{kh}(\cdot)$ metric.

The solution of difference scheme (3.4), (6.2) is not bounded (N, ε)-uniformly in the $\rho_u^{1h}(\cdot)$ metric. The condition (either $h \leq M\,\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)$ for $\eta < 1$, or $h \leq M \ln^{-2}(\varepsilon^{-1} + 1)$ for $\eta \geq 1$) :

$$
\begin{aligned}
N^{-1} &= \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)\right) && \text{for}\quad \eta < 1; \\
N^{-1} &= \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1} + 1)\right) && \text{for}\quad \eta \geq 1
\end{aligned}
\tag{6.9}
$$

is necessary and sufficient for $\rho_u^{1h}(z)$ to be bounded.

In the case of condition (6.9) we have the unimprovable estimate

$$
\rho_u^{1h}(z) \leq
\begin{cases}
M\left[N^{-1}\left(\varepsilon + N^{-1}\right)^{-1} + N^{-1}\varepsilon^{-1}(1 - \eta)\ln(\varepsilon^{-1} + 1)\right] & \text{for}\quad \eta \leq 1, \\
M\left[N^{-1}\ln^2(\varepsilon^{-1} + 1) + N^{-1}\varepsilon^{\eta-1}(\varepsilon + N^{-1})^{-1}\right] & \text{for}\quad \eta > 1.
\end{cases}
\tag{6.10}
$$

It follows from this estimate that the scheme converges in the $\rho_u^{1h}(\cdot)$ metric under the following unimprovable condition:

$$
\begin{aligned}
N^{-1} &= o(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)) && \text{for}\quad \eta < 1, \\
N^{-1} &= o(\varepsilon) && \text{for}\quad \eta = 1, \\
N^{-1} &= o(\ln^{-2}(\varepsilon^{-1} + 1)) && \text{for}\quad \eta > 1.
\end{aligned}
\tag{6.11}
$$

In the case of the $\rho_u^{2h}(\cdot)$ metric, the condition

$$
\begin{aligned}
N^{-1} &= \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)\right) && \text{for}\quad \eta < 2; \\
N^{-1} &= \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1} + 1)\right) && \text{for}\quad \eta \geq 2
\end{aligned}
\tag{6.12}
$$

is necessary and sufficient for the mesh solutions to be bounded. If this condition and also the following additional condition hold

$$\left| \frac{d^2}{dx^2} U(0) \right| \leq M \left(\varepsilon \ln(\varepsilon^{-1} + 1) + \delta \right)^\lambda, \tag{6.13}$$

where λ is a number from the interval $[0, 1]$, the value $\delta = \delta(N)$ tends to zero as $N \to \infty$, then we have the unimprovable estimate

$$\rho_u^{2h}(z) \leq \tag{6.14}$$

$$\leq \begin{cases} M \left[N^{-1} \left(\varepsilon + N^{-1} \right)^{-1} + N^{-1} \varepsilon^{-1} (2 - \eta) \ln(\varepsilon^{-1} + 1) \right] & for \quad \eta \leq 2, \\ M \left\{ \min \left[N^{-1} \ln^2(\varepsilon^{-1} + 1), 1 \right] + N^{-1} \left(\varepsilon + N^{-1} \right)^{-1} (\varepsilon \ln(\varepsilon^{-1} + 1) + N^{-1})^\lambda + \right. \\ \left. + N^{-1} \varepsilon^{\eta-2} \left(\varepsilon + N^{-1} \right)^{-1} \right\} & for \quad \eta > 2. \end{cases}$$

The scheme converges in the $\rho_u^{2h}(\cdot)$ metric under the unimprovable condition

$$\begin{cases} N^{-1} = o(\varepsilon \ln^{-1}(\varepsilon^{-1} + 1)) & for \quad \eta < 2; \\ N^{-1} = o(\varepsilon) & for \quad \eta = 2; \\ N^{-1} = o(\ln^{-2}(\varepsilon^{-1} + 1)) & for \quad \eta > 2, \quad if \quad \lambda \neq 0; \\ N^{-1} = o(\varepsilon) & for \quad \eta > 2, \quad if \quad \lambda = 0. \end{cases} \tag{6.15}$$

Thus, scheme (3.4), (6.2) under the condition

$$\eta > k, \quad \{\lambda > 0, \quad when \quad k = 2\} \quad and \quad N^{-1} = o(\ln^{-2}(\varepsilon^{-1} + 1)), \quad k = 1, 2 \tag{6.16}$$

converges in the $\rho_u^{kh}(\cdot)$ metric, that is, the scheme converges ε-uniformly up to a logarithmic factor (the convergence defect is $\mathcal{O}\left(\ln^2(\varepsilon^{-1} + 1)\right)$). Under the condition

$$\eta \geq k = 2, \quad \lambda = 0 \quad and \quad N^{-1} = o(\varepsilon), \tag{6.17}$$

the scheme converges in the $\rho_u^{2h}(\cdot)$ metric; the convergence defect is $\mathcal{O}\left(\varepsilon^{-1}\right)$.

Under the condition

$$\eta > k, \quad \{\lambda > 0, \quad when \quad k = 2\} \quad and \quad N^{-1} = \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1} + 1)\right), \quad k = 1, 2, \tag{6.18}$$

we have the estimate

$$\rho_u^{kh}(z) \leq \begin{cases} M \left[N^{-1} \ln^2(\varepsilon^{-1} + 1) + N^{-\eta+1} \right], & k = 1, \\ M \left[N^{-1} \ln^2(\varepsilon^{-1} + 1) + N^{-\lambda} \ln N + N^{-\eta+2} \right], & k = 2, \end{cases} \tag{6.19}$$

and, under the additional condition

$$\eta \geq k + 1, \quad k = 1, 2, \tag{6.20}$$

we have the estimate

$$\rho_u^{kh}(z) \leq \begin{cases} M \, N^{-1} \ln^2(\varepsilon^{-1} + 1), & k = 1, \\ M \, N^{-1} \ln^2(\varepsilon^{-1} + 1) + N^{-\lambda} \ln N, & k = 2. \end{cases} \qquad (6.21)$$

The scheme converges in the $\rho_u^{kh}(\cdot)$ with the first-order accuracy for $k = 1$ and with the order λ (up to the factor $\ln N$) for $k = 2$, moreover, the convergence for $k = 1$ and $k = 2$ is ε-uniform up to the factor $\ln^2(\varepsilon^{-1} + 1)$.

In the case of the condition

$$\eta \geq k = 2, \quad \lambda = 0 \quad and \quad N^{-1} = \mathcal{O}\left(\varepsilon\right) \qquad (6.22)$$

we have the estimate

$$\rho_u^{2h}(z) \leq M \left[N^{-1}(\varepsilon + N^{-1})^{-1} + N^{-\eta+2} \right], \qquad (6.23)$$

and, under the additional condition (6.20), we have

$$\rho_u^{2h}(z) \leq M \, N^{-1}(\varepsilon + N^{-1})^{-1}, \qquad (6.24)$$

i.e. the scheme converges in the $\rho_u^{2h}(\cdot)$ metric with the first order of accuracy and with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$.

Lemma 6.2. *In the case of finite difference scheme (3.4), (6.2) conditions (6.9), (6.12) (conditions (6.11), (6.15)) are necessary and sufficient for the mesh solutions to be (N, ε)-uniformly bounded (for the convergence of the scheme) in the $\rho_u^{kh}(\cdot)$ metric. Under condition (6.16) the scheme converges in the $\rho_u^{kh}(\cdot)$ metric ε-uniformly up to the factor $\ln^2(\varepsilon^{-1}+1)$. Under condition (6.17) the scheme converges in the $\rho_u^{2h}(\cdot)$ metric with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$. The mesh solutions satisfy estimates (6.10), (6.14), (6.19), (6.21), (6.23), (6.24) if conditions (6.9), $\{(6.12), (6.13)\}$, (6.18), $\{(6.18), (6.20)\}$, (6.22), $\{(6.22), (6.20)\}$ hold respectively. These estimates are unimprovable w.r.t. the values of N, ε.*

7. Special scheme on piecewise-uniform meshes. Problem (2.2), (2.1)

1. For boundary value problem (2.2), (2.1) we consider the approximation of the solutions and derivatives in the case of finite difference scheme (4.3) on piecewise-uniform meshes. On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h^* = \overline{D}_h^*(\sigma(l)) = \overline{D}_h^*(l) = \overline{\omega}_1^* \times \omega_2. \qquad (7.1a)$$

Here ω_2 is a uniform mesh and $\overline{\omega}_1^*$ is a piecewise-uniform mesh. To construct the mesh $\overline{\omega}_1^*$, we divide the interval $[0, d]$ in two parts $[0, \sigma]$ and $[\sigma, d]$; in each part the mesh stepsize is constant and equal to $h^{(1)} = 2\sigma N_1^{-1}$ and $h^{(2)} = 2(d - \sigma)N_1^{-1}$ respectively. Assume

$$\sigma = \sigma(\varepsilon, N_1, d; l, m) = \min \left[2^{-1}d, \, l \, m^{-1} \varepsilon \ln N_1 \right], \qquad (7.1b)$$

where $m = a^{-1}b$, $l > 0$ is a parameter of the mesh. The auxilary parameter γ_k is defined by

$$\gamma_k = \gamma_k(\varepsilon; m) = k \, m^{-1} \varepsilon \ln(\varepsilon^{-1} + 1), \quad m = m_{(7.1)}, \quad k = 1, 2. \qquad (7.1c)$$

For the solutions of difference scheme (4.3), (7.1) (taking into account the explicit form of the singular components from representation (9.3)), we find the following estimates

$$\|u - \tilde{z}\| \leq M \left[N_1^{-1} \min[\ln N_1, \varepsilon^{-1}] + N_1^{-l-1}(\varepsilon + N_1^{-1})^{-1} + N_2^{-1} \right] \equiv \tag{7.2a}$$

$$\equiv M \left[\mu_0(N_1, \varepsilon; l) + N_2^{-1} \right] \equiv M \mu_1(N, \varepsilon; l);$$

$$\|u - \tilde{z}\| \leq M \left[N_1^{-1} \ln N_1 + N_1^{-l} + N_2^{-1} \right] \equiv M \mu_2(N; l). \tag{7.2b}$$

Estimates (7.2a) and (7.2b) are unimprovable w.r.t. the values of N_s, ε and N_s respectively. Under the condition

$$l \geq 1 \tag{7.3a}$$

we achieve the best ε-uniform order of convergence

$$\|u - \tilde{z}\| \leq M \left[N_1^{-1} \ln N_1 + N_2^{-1} \right]. \tag{7.3b}$$

For the derivatives we have the estimates

$$\left\| \frac{\partial^{k_1}}{\partial x_1^{k_1}} (u - \tilde{z}) \right\|, \quad \|u - \tilde{z}\|_{C_*^k} \leq M \left[\varepsilon^{-k_1} \mu_0 (N_1, \varepsilon; l) + N_2^{-1} \right],$$

$$k_1 = k = 1, 2; \tag{7.4}$$

$$\left\| \frac{\partial^{k_2}}{\partial x_2^{k_2}} (u - \tilde{z}) \right\| \leq M \mu_1 (N, \varepsilon; l), \quad k_2 = 1, 2; \tag{7.5}$$

these estimates are unimprovable w.r.t. the values of N_s, ε. The errors in the solutions of finite difference scheme (4.3), (7.1) are bounded in the norm $\| \cdot \|_{C_*^k}$ under the condition

$$N_1^{-1} = \mathcal{O}\left(\varepsilon^{(k+1)/(l+1)} \right) \qquad for \quad l < k^{-1};$$
$$N_1^{-1} = \mathcal{O}\left(\varepsilon^k \ln^{-1}(\varepsilon^{-1} + 1) \right) \quad for \quad l \geq k^{-1}; \quad k = 1, 2. \tag{7.6}$$

The scheme converges in the norm $\| \cdot \|_{C_*^k}$ under the condition

$$N_1^{-1} = o\left(\varepsilon^{(k+1)/(l+1)} \right) \qquad for \quad l < k^{-1};$$
$$N_1^{-1} = o\left(\varepsilon^k \ln^{-1}(\varepsilon^{-1} + 1) \right) \quad for \quad l \geq k^{-1}; \quad k = 1, 2. \tag{7.7}$$

Conditions (7.6), (7.7) are unimprovable.

The convergence defect of the scheme in the norm $\| \cdot \|_{C_*^k}$ is

$$\mathcal{O}\left(\varepsilon^{-(k+1)/(l+1)} \right) \quad for \quad l < k^{-1} \quad and \quad \mathcal{O}\left(\varepsilon^{-k} \ln(\varepsilon^{-1} + 1) \right) \quad for \quad l \geq k^{-1}.$$

Under the condition

$$l \geq k^{-1} \tag{7.8a}$$

we have the estimate

$$\|u - \tilde{z}\|_{C_*^k} \leq M \left[N_1^{-1} \varepsilon^{-k} \min[\ln N_1, \varepsilon^{-1}] + N_2^{-1} \right], \quad k = 1, 2; \tag{7.8b}$$

that is, the scheme converges for fixed values of the parameter ε with the first order up to a logarithmic factor.

Theorem 7.1. *Let the hypothesis of Theorem 4.1 be fulfilled. Then the finite difference scheme (4.3), (7.1) converges ε-uniformly in the norm $\|\cdot\|$. Condition (7.6) (condition (7.7)) is necessary and sufficient for the boundedness of the error (for the convergence of the scheme) in the $\|\cdot\|_{C_*^k}$ norm. The mesh solutions satisfy estimates (7.2), (7.3), (7.4), (7.5), (7.8); estimates (7.2a), (7.4), (7.5), (7.8) and estimates (7.2b), (7.3) are unimprovable with respect to the values of N_s, ε and N_s respectively.*

2. We now give the estimates of convergence for scheme (4.3), (7.1) in the case of the $\rho_u^{kh}(\cdot)$ metric. The solution of problem (4.3), (7.1) is not bounded (N, ε)-uniformly in this metric. Taking into account the explicit form of the singular components of the solutions for the differential and discrete problems, we establish the condition

$$N_1^{-1} = \mathcal{O}\left(\varepsilon^2 (\gamma_k - \sigma)^{-1}\right) \quad for \;\; \varepsilon^{-1}(\gamma_k - \sigma) \geq M_0, \tag{7.9a}$$

$$\sigma \, N_1^{-1} = \mathcal{O}\left(\varepsilon^2 \, \gamma_k^{-1}\right) \qquad for \;\; \varepsilon^{-1}(\gamma_k - \sigma) < M_0; \;\; k = 1, 2, \tag{7.9b}$$

where $\sigma = \sigma_{(7.1)}$, $\gamma_k = \gamma_{k(7.1)}$, M_0 is any constant. This condition is necessary and sufficient for the boundness of $\rho_u^{kh}(z)$.

Under condition (7.9a) we obtain

$$\rho_u^{kh}(z) \leq M \left\{ N_1^{-1} \left(\min \left[\ln N_1, \varepsilon^{-1} \right] \right)^2 + \right. \tag{7.10}$$

$$\left. + N_1^{-1} \left(\varepsilon + N_1^{-1} \right)^{-1} \varepsilon^{-1}(\gamma_k - \sigma) + N_2^{-1} \right\}, \quad k = 1, 2.$$

Under condition (7.9b) we have the estimate

$$\rho_u^{1h}(z) \leq M \left\{ \varepsilon^{-1} \gamma_1 \, N_1^{-1} \min \left[\ln N_1, \varepsilon^{-1} \right] + \right. \tag{7.11}$$

$$\left. + N_1^{-1} \left(\varepsilon + N_1^{-1} \right)^{-1} \exp \left(-\varepsilon^{-1}(\sigma - \gamma_1) \right) + N_2^{-1} \right\},$$

and, under the additional condition

$$\left| \frac{\partial^2}{\partial x_1^2} U(x) \right| \leq M \left(\varepsilon + \delta \right)^\lambda, \quad x \in \Gamma_1, \tag{7.12}$$

where λ is a number from the interval $[0, 1]$, $\delta = \delta(N)$ tends to zero as $N \to \infty$, we have

$$\rho_u^{2h}(z) \leq M \left\{ \varepsilon^{-1} \gamma_2 \, N_1^{-1} \min \left[\ln N_1, \varepsilon^{-1} \right] + \right. \tag{7.13}$$

$$+ N_1^{-1} \left(\varepsilon + N_1^{-1} \right)^{-1} (\varepsilon + \delta)^\lambda + + N_1^{-1} \left(\varepsilon + N_1^{-1} \right)^{-1} \exp \left(-\varepsilon^{-1}(\sigma - \gamma_2) \right) + N_2^{-1} \right\}.$$

Estimates (7.10), (7.11), (7.13) are unimprovable. Note that in the case of the condition $\varepsilon^{-1}(\gamma_k - \sigma) \geq M_0$ (see condition (7.9a) and estimate (7.10)) the inequality $\varepsilon^{-1}(\gamma_k - \sigma) \leq M \ln(\varepsilon^{-1} + 1)$ is valid.

These estimates imply that the condition

$$N_1^{-1} = o(\varepsilon^2(\gamma_k - \sigma)^{-1}), \quad l < k;$$
$$N_1^{-1} = o(\varepsilon^{k/l}), \qquad\qquad l \geq k; \quad k = 1, 2 \quad and \quad \lambda > 0 \quad for \quad k = 2; \tag{7.14a}$$

$$N_1^{-1} = o(\varepsilon^2(\gamma_k - \sigma)^{-1}), \quad l < k;$$
$$N_1^{-1} = o(\varepsilon), \qquad\qquad l \geq k; \quad \lambda = 0 \quad and \quad k = 2 \tag{7.14b}$$

is necessary and sufficient for the convergence of the scheme in the $\rho_u^{kh}(\cdot)$ metric. Note that in the above condition (7.14) for $l < k$, the following estimate occurs

$$m\,\varepsilon\,\ln^{-1}(\varepsilon^{-1} + 1) \;\leq\; \varepsilon^2\,(\gamma_k - \sigma)^{-1} \;\leq\; M_0^{-1}\,\varepsilon.$$

Thus, the convergence defect of the scheme in the $\rho_u^{kh}(\cdot)$ metric is not higher than $\mathcal{O}\left(\varepsilon^{-1}\ln(\varepsilon^{-1} + 1)\right)$ for $l < k$; the defect is $\mathcal{O}\left(\varepsilon^{-k/l}\right)$ for $l \geq k$ for $k = 1$ and also for $k = 2$ when $\lambda > 0$; the defect is $\mathcal{O}\left(\varepsilon^{-1}\right)$ for $k = 2$ and $\lambda = 0$. The defect $\mathcal{O}\left(\varepsilon^{-1}\ln(\varepsilon^{-1} + 1)\right)$ is achieved, for example, under the condition $\gamma_k \geq (1 + m)\sigma$.

If the following condition holds

$$l \geq k\,\nu^{-1} \quad and \quad \{\lambda > 0, \quad when \quad k = 2\}, \quad k = 1, 2, \tag{7.15}$$

where $\nu = \nu_{(2.10)}$, $\nu \leq 1$, the scheme converges almost ε-uniformly in the $\rho_u^{kh}(\cdot)$ metric, moreover, the convergence defect is not higher than $\mathcal{O}\left(\varepsilon^{-\nu}\right)$ and unimprovable with respect to ν. Under the condition

$$l \geq k \quad and \quad N_1^{-1} = \mathcal{O}\left(\varepsilon^{k/l}\right), \quad \{\lambda > 0, \quad when \quad k = 2\}, \quad k = 1, 2 \tag{7.16}$$

we have the unimprovable estimate

$$\rho_u^{kh}(z) \leq \begin{cases} M\left\{N_1^{-1}\min\left[\ln N_1, \varepsilon^{-1}\right]\ln(\varepsilon^{-1} + 1) + N_1^{-l}\varepsilon^{-1} + N_2^{-1}\right\}, & k = 1, \\[2mm] M\left\{N_1^{-1}\min\left[\ln N_1, \varepsilon^{-1}\right]\ln(\varepsilon^{-1} + 1) + N_1^{-l}\varepsilon^{-2} + \right. \\[2mm] \left. + N^{-1}(\varepsilon + N_1^{-1})^{-1}(\varepsilon + \delta)^{\lambda} + N_2^{-1}\right\}, & k = 2. \end{cases} \tag{7.17}$$

Under the condition

$$l \geq k = 2, \quad \lambda = 0 \quad and \quad N_1^{-1} = \mathcal{O}\left(\varepsilon\right) \tag{7.18}$$

the following unimprovable estimate holds:

$$\rho_u^{2h}(z) \;\leq\; M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_2^{-1}\right]. \tag{7.19}$$

Theorem 7.2. *Let the hypothesis of Theorem 4.1 be fulfilled. Then condition (7.9) (condition (7.14)) is necessary and sufficient in order that the mesh solutions of finite difference scheme (4.3), (7.1) are (N, ε)-uniformly bounded (the scheme is convergent) in the $\rho_u^{kh}(\cdot)$ metric. Under condition (7.15) the scheme converges almost ε-uniformly in the $\rho_u^{kh}(\cdot)$ metric; the*

convergence defect is $\mathcal{O}\left(\varepsilon^{-k/l}\right)$ for $l \geq k$. Under condition (7.18) the scheme converges in the $\rho_u^{2h}(\cdot)$ metric with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$. The mesh solutions satisfy estimates (7.10), (7.11), (7.13), (7.17) and (7.19) if conditions (7.9a), (7.9b), {(7.9b), (7.12)}, (7.16) and (7.18) hold respectively; these estimates are unimprovable with respect to the values of N_s, ε.

Remark. In the case of the meshes from [4, 5, 9] the value l is choosen to satisfy only the condition $l > 1$. Thus, scheme (4.3) on the meshes from [4, 5, 9] converges in the $\rho_u^{1h}(\cdot)$ metric if the condition $N_1^{-1} = o(\varepsilon^\nu)$ holds, where $\nu = l^{-1} < 1$, moreover, ν may take arbitrary values as much as desired close to 1. On the same meshes, under $\lambda > 0$ the scheme (4.3) converges in the $\rho_u^{2h}(\cdot)$ metric if the condition $N_1^{-1} = o(\varepsilon^{2/l})$ holds when the value l insignificantly exceeds 1.

8. Finite difference schemes with improved $\rho_u^{kh}(\cdot)$ convergence. Problem (2.2), (2.1)

1. In this section we construct a scheme which converges in the $\rho_u^{kh}(\cdot)$ metric ε-uniformly up to a logarithmic factor.

On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h = \overline{D}^*_{h(7.1a)}(\sigma(\eta)), \tag{8.1a}$$

where
$$\sigma = \sigma(\varepsilon; \eta, m) = \min[2^{-1}d, \eta m^{-1}\varepsilon \ln(\varepsilon^{-1} + M)], \tag{8.1b}$$

$m = m_{(9.6)}$, $\eta > 0$ is a parameter of the mesh, M is a constant satisfying the condition $\ln(1 + M) \geq d$, $M > e$.

Using the a-priori estimates of the solutions to the boundary value problem, for difference scheme (4.3), (8.1) we establish the estimate

$$\|u - \tilde{z}\| \leq M \left\{ \min\left[N_1^{-1}\ln(\varepsilon^{-1} + 1), 1\right] + N_1^{-1}\varepsilon^\eta(\varepsilon + N_1^{-1})^{-1} + N_2^{-1} \right\} \equiv \tag{8.2}$$

$$\equiv M\left[\mu_0(N_1, \varepsilon; \eta) + N_2^{-1}\right] \equiv M\,\mu_1(N, \varepsilon; \eta),$$

this estimate is unimprovable w.r.t. the values of N_s, ε. The error $\|u - \tilde{z}\|$ is (N, ε)-uniformly bounded under the condition

$$\forall N_s, \quad \forall \varepsilon, \quad s = 1, 2; \tag{8.3}$$

the scheme converges under the unimprovable condition

$$N_1^{-1} = o(\ln^{-1}(\varepsilon^{-1} + 1)). \tag{8.4}$$

The convergence defect of the scheme is $\mathcal{O}\left(\ln(\varepsilon^{-1} + 1)\right)$.

For the derivatives we have the unimprovable estimates

$$\left\|\frac{\partial^{k_2}}{\partial x_2^{k_2}}(u - \tilde{z})\right\| \leq M\,\mu_1\left(N, \varepsilon; \eta\right), \quad k_2 = 1, 2; \tag{8.5}$$

$$\left\|\frac{\partial^{k_1}}{\partial x_1^{k_1}}(u - \tilde{z})\right\|, \ \|u - \tilde{z}\|_{C_*^k} \leq M\left[\varepsilon^{-k_1}\mu_0\left(N_1, \varepsilon; \eta\right) + N_2^{-1}\right], \quad k = k_1 = 1, 2.$$

Errors of the mesh solutions are bounded in the norm $\| \cdot \|_{C_*^k}$ under the condition

$$\begin{aligned}
N_1^{-1} &= \mathcal{O}\left(\varepsilon^k \ln^{-1}(\varepsilon^{-1}+1)\right) && for \quad \eta \geq 1; \\
N_1^{-1} &= \mathcal{O}\left(\varepsilon^{1+k-\eta}\right) && for \quad \eta < 1; \quad k = 1, 2;
\end{aligned} \tag{8.6}$$

the scheme converges under the condition

$$\begin{aligned}
N_1^{-1} &= o\left(\varepsilon^k \ln^{-1}(\varepsilon^{-1}+1))\right) && for \quad \eta \geq 1; \\
N_1^{-1} &= o\left(\varepsilon^{1+k-\eta}\right) && for \quad \eta < 1; \quad k = 1, 2;
\end{aligned} \tag{8.7}$$

conditions (8.6), (8.7) are unimprovable. The convergence defect of the scheme in the norm $\| \cdot \|_{C_*^k}$ is $\mathcal{O}\left(\varepsilon^{-1-k+\eta}\right)$ for $\eta < 1$ and $\mathcal{O}\left(\varepsilon^{-k} \ln(\varepsilon^{-1}+1)\right)$ for $\eta \geq 1$.

Theorem 8.1. *Let the hypothesis of Theorem* 4.1 *be fulfilled. Then conditions* (8.3) *and* (8.6) *(conditions* (8.4) *and* (8.7)*) are necessary and sufficient for the boundedness of the errors* $\|u - \tilde{z}\|$ *and* $\|u - \tilde{z}\|_{C_*^k}$ *in the solutions of difference scheme* (4.3), (8.1) *(for the convergence of the scheme in the norms* $\| \cdot \|$ *and* $\| \cdot \|_{C_*^k}$ *) respectively. The mesh solutions satisfy estimates* (8.2), (8.5) *which are unimprovable with respect to the values of* N_s, ε; *the convergence defect of the scheme is* $\mathcal{O}\left(\ln(\varepsilon^{-1}+1)\right)$ *in the norm* $\| \cdot \|$ *and not lower than* $\mathcal{O}\left(\varepsilon^{-k} \ln(\varepsilon^{-1}+1)\right)$ *in the norm* $\| \cdot \|_{C_*^k}$.

2. In the case of the $\rho_u^{kh}(\cdot)$ metric, the solutions of finite difference scheme (4.3), (8.1) are bounded under the (unimprovable) condition

$$\begin{aligned}
N_1^{-1} &= \mathcal{O}\left(\varepsilon \ln^{-1}(\varepsilon^{-1}+1)\right) && for \quad \eta < k; \\
N_1^{-1} &= \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1}+1)\right) && for \quad \eta \geq k; \quad k = 1, 2.
\end{aligned} \tag{8.8}$$

Under this condition we have the unimprovable estimate

$$\rho_u^{1h}(z) \leq M\,N_2^{-1} + \tag{8.9a}$$

$$+ \begin{cases}
M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_1^{-1}\varepsilon^{-1}(1-\eta)\ln(\varepsilon^{-1}+1)\right] & for \quad \eta \leq 1, \\
M\left\{N_1^{-1}\ln^2(\varepsilon^{-1}+1) + N_1^{-1}\varepsilon^{\eta-1}(\varepsilon + N_1^{-1})^{-1}\right\} & for \quad \eta > 1,
\end{cases}$$

and, under the additional condition

$$\left\|\frac{\partial^2}{\partial x_1^2}U(x)\right\| \leq M\left(\varepsilon \ln(\varepsilon^{-1}+1) + \delta\right)^\lambda, \quad x \in \Gamma_1 \ for \ k = 2, \tag{8.10}$$

where λ is a number from $[0, 1]$, $\delta = \delta(N)$ tends to zero ε-uniformly as $N \to \infty$, we have the following unimprovable estimate

$$\rho_u^{2h}(z) \leq M\,N_2^{-1} + \tag{8.9b}$$

$$+ \begin{cases}
M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_1^{-1}\varepsilon^{-1}(2-\eta)\ln(\varepsilon^{-1}+1)\right] & for \quad \eta \leq 2, \\
M\left\{N_1^{-1}\ln^2(\varepsilon^{-1}+1) + N_1^{-1}(\varepsilon + N_1^{-1})^{-1}(\varepsilon\ln(\varepsilon^{-1}+1) + \delta)^\lambda + \right. \\
\left. \qquad\qquad + N_1^{-1}\varepsilon^{\eta-2}(\varepsilon + N_1^{-1})^{-1}\right\} & for \quad \eta > 2.
\end{cases}$$

The scheme converges in the $\rho_u^{1h}(\cdot)$ metric under the unimprovable condition

$$
\begin{aligned}
N_1^{-1} &= o(\varepsilon \ \ln^{-1}(\varepsilon^{-1}+1)) \quad for \ \ \eta < 1; \\
N_1^{-1} &= o(\varepsilon) \qquad\qquad\qquad for \ \ \eta = 1; \\
N_1^{-1} &= o(\ln^{-2}(\varepsilon^{-1}+1)) \quad for \ \ \eta > 1;
\end{aligned}
\tag{8.11a}
$$

while in the $\rho_u^{2h}(\cdot)$ metric the scheme in question converges under the unimprovable condition

$$
\begin{aligned}
N_1^{-1} &= o(\varepsilon \ \ln^{-1}(\varepsilon^{-1}+1)) \quad for \ \ \eta < 2; \\
N_1^{-1} &= o(\varepsilon) \qquad\qquad\qquad for \ \ \eta = 2; \\
N_1^{-1} &= o(\ln^{-2}(\varepsilon^{-1}+1)), \quad if \ \ \lambda \neq 0 \\
N_1^{-1} &= o(\varepsilon), \qquad\qquad\qquad if \ \ \lambda = 0 \ \ \eta > 2.
\end{aligned}
\tag{8.11b}
$$

Thus, scheme (4.3), (8.1) under the condition

$$
\eta > k, \ \ \{\lambda > 0, \ \ when \ \ k = 2\} \ \ and \ \ N_1^{-1} = o(\ln^{-2}(\varepsilon^{-1}+1)), \ \ k = 1, 2
\tag{8.12a}
$$

converges in the $\rho_u^{kh}(\cdot)$ metric, and the convergence is ε-uniform up to the logarithmic factor $\ln^2(\varepsilon^{-1}+1)$.

Under the condition

$$
\eta \geq k = 2, \ \ \lambda = 0 \ \ and \ \ N_1^{-1} = o(\varepsilon),
\tag{8.12b}
$$

the scheme converges in the $\rho_u^{2h}(\cdot)$ metric; the convergence defect is $\mathcal{O}\left(\varepsilon^{-1}\right)$.

In the case of the condition

$$
\eta > k, \ \ \{\lambda > 0, \ \ when \ \ k = 2\} \ \ and \ \ N_1^{-1} = \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1}+1)\right), \ \ k = 1, 2,
\tag{8.13}
$$

we have the estimate

$$
\rho_u^{kh}(z) \leq \ M \ N_2^{-1} +
\tag{8.14}
$$

$$
+ \begin{cases}
M\left[N_1^{-1}\ln^2(\varepsilon^{-1}+1) + N_1^{-\eta+1}\right], & k = 1, \\
M\left[N_1^{-1}\ln^2(\varepsilon^{-1}+1) + N_1^{-\eta+2} + N_1^{-\lambda}\ln N_1\right], & k = 2,
\end{cases}
$$

and, under the additional condition

$$
\eta \geq k + 1, \ \ k = 1, 2,
\tag{8.15}
$$

the estimate

$$
\rho_u^{kh}(z) \leq \ M \ N_2^{-1} + \begin{cases}
M \ N_1^{-1}\ln^2(\varepsilon^{-1}+1), & k = 1, \\
M\left\{N_1^{-1}\ln^2(\varepsilon^{-1}+1) + N_1^{-\lambda}\ln N_1\right\}, & k = 2.
\end{cases}
\tag{8.16}
$$

The scheme converges in the $\rho_u^{kh}(\cdot)$ metric with the first-order accuracy with respect to N_1 for $k = 1$ and with the order of accuracy λ (up to the factor $\ln N_1$) for $k = 2$ ε-uniformly up to the factor $\ln^2(\varepsilon^{-1}+1)$.

Under the condition

$$\eta \geq k = 2, \quad \lambda = 0 \quad and \quad N_1^{-1} = \mathcal{O}\left(\varepsilon\right), \tag{8.17}$$

we obtain the estimate

$$\rho_u^{2h}(z) \leq M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_1^{-\eta+2} + N_2^{-1}\right], \tag{8.18a}$$

and, under the additional condition (8.15), the estimate

$$\rho_u^{2h}(z) \leq M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_2^{-1}\right], \tag{8.18b}$$

that is, the scheme converges with the first order of accuracy and with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$.

Theorem 8.2. *Let the hypothesis of Theorem 4.1 be fulfilled. Then condition (8.8) (condition (8.11)) is necessary and sufficient in order that the solutions $z(x)$, $x \in \overline{D}_h$ of finite difference scheme (4.3), (8.1) are (N, ε)-uniformly bounded (the solutions are convergent) in the $\rho_u^{kh}(\cdot)$ metric. Under condition (8.12a) the scheme converges in the $\rho_u^{kh}(\cdot)$ metric ε-uniformly up to the factor $\ln^2(\varepsilon^{-1}+1)$. Under condition (8.12b) the scheme converges in the $\rho_u^{2h}(\cdot)$ metric with the convergence defect $\mathcal{O}\left(\varepsilon^{-1}\right)$. The mesh solutions satisfy estimates (8.9a), (8.9b), (8.14), (8.16), (8.18a), (8.18b) if conditions (8.8), {(8.8), (8.10)}, (8.12a), {(8.12a), (8.15)}, (8.12b), {(8.12b), (8.15)} hold respectively; estimates (8.9) are unimprovable with respect to the values of N_s, ε.*

9. Appendix

1. In this section we give *a-priori* estimates for the solutions of boundary value problem (2.2), (2.1) used in the constructions; the technique of deriving the estimates is similar to that from [4].

The solution of the problem can be decomposed into the sum of two functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{9.1}$$

where $U(x)$ and $V(x)$ are the regular and singular parts of the solution. The function $U(x)$, $x \in \overline{D}$ is a restriction onto $\overline{D}$ of the function $U^0(x)$, which is the solution of the following problem on the half-plane $\overline{D}^0$:

$$L^0 \, U^0(x) = f^0(x), \quad x \in D^0, \quad U^0(x) = \varphi(x), \quad x \in \Gamma^0.$$

Here

$$D^0 = \{x : \quad x_1 \in (-\infty, d), \quad x_2 \in R\},$$

the operator L^0 and the function $f^0(x)$ are smooth continuations of the operator $L_{(2.2)}$ and the function $f(x)$, which preserve the properties of the data of problem (2.2), (2.1). The function $V(x)$ is the solution of the problem

$$L \, V(x) = 0, \quad x \in D, \quad V(x) = \varphi(x) - U(x), \quad x \in \Gamma.$$

It is convenient to represent the function $U^0(x)$, $x \in \overline{D}^0$ as the sum of functions which is an expansion with respect to the parameter ε:

$$U^0(x) = \sum_{n=0}^{3} \varepsilon^n U_n^0(x) + v_{U^0}(x), \quad x \in \overline{D}^0,$$

where $v_{U^0}(x)$ is the remainder term; the functions $U_n^0(x)$ are the solutions of the (initial) problems for the hyperbolic equations

$$L_1^0 U_0^0(x) \equiv \left\{ \sum_{s=1,2} b_s^0(x) \frac{\partial}{\partial x_s} - c^0(x) \right\} U_0^0(x) = f^0(x), \quad x \in D^0,$$

$$U_0^0(x) = \varphi(x), \quad x \in \Gamma^0;$$

$$L_1^0 U_n^0(x) = - \sum_{s=1,2} a_s^0(x) \frac{\partial^2}{\partial x_s^2} U_{n-1}^0(x), \quad x \in D^0,$$

$$U_0^n(x) = 0, \quad x \in \Gamma^0, \quad n = 1, 2, 3.$$

Here the functions $a_s^0(x)$, $b_s^0(x)$, $c^0(x)$ are continuations of the functions $a_s(x)$, $b_s(x)$, $c(x)$.

If the data of boundary value problem (2.2), (2.1) are sufficiently smooth, we have the following estimate for the function $U(x)$, $x \in \overline{D}$:

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} U(x) \right| \leq M \left[1 + \varepsilon^{3-k} \right], \quad x \in \overline{D}, \quad k \leq 5. \tag{9.2}$$

The function $V(x)$ can be written as the sum of functions

$$V(x) = \sum_{n=0,1,2} \varepsilon^n V_n(x) + v_V(x), \quad x \in \overline{D}, \tag{9.3}$$

where $v_V(x)$ is the remainder term. The functions $V_n(x)$, $x \in \overline{D}$ are restrictions onto $\overline{D}$ of the functions $V_n^1(x)$, $x \in \overline{D}^1$, which are the (bounded) solutions of the problems

$$L_2^1 V_0^1(x) \equiv \left\{ \varepsilon a_1(x^*) \frac{\partial^2}{\partial x_1^2} + b_1(x^*) \frac{\partial}{\partial x_1} \right\} V_0^1(x) = 0, \quad x \in D^1,$$

$$V_0^1(x) = \varphi(x) - U_0(x), \quad x \in \Gamma^1;$$

$$L_2^1 V_1^1(x) = \left\{ -\varepsilon \frac{\partial}{\partial x_1} a_1(x^*) x_1 \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_1} b_1(x^*) x_1 \frac{\partial}{\partial x_1} - \right.$$

$$\left. -\varepsilon a_2(x^*) \frac{\partial^2}{\partial x_1^2} - b_2(x^*) \frac{\partial}{\partial x_2} + c(x^*) \right\} V_0^1(x),$$

$$L_2^1 V_2^1(x) = \left\{ -\varepsilon 2^{-1} \frac{\partial^2}{\partial x_1^2} a_1(x^*) x_1^2 \frac{\partial^2}{\partial x_1^2} - \ldots + \frac{\partial}{\partial x_1} c(x^*) x_1 \right\} V_0^1(x) +$$

$$+ \left\{ -\varepsilon \frac{\partial}{\partial x_1} a_1(x^*) x_1 \frac{\partial^2}{\partial x_1^2} + \ldots + c(x^*) \right\} V_1^1(x), \quad x \in D^1,$$

$$V_n^1(x) = -U_n(x), \quad x \in \Gamma^1, \quad n = 1, 2;$$

the functions $V_n^1(x)$ exponentially decrease for $\varepsilon^{-1}x_1 \to \infty$. Here $D^1 = (0, \infty) \times R$, $\Gamma^1 = \overline{D}^1 \backslash D^1$, $x^* = (0, x_2)$. The functions $V_n(x)$ can be written explicitly in terms of the functions $\varphi(x^*)$, $U_0(x^*)$ and $U_1(x^*)$. Note that $\varphi(x) - U_0(x) = 0$, $U_n(x) = 0$, $n \geq 1$ for $x \in \Gamma_2$.

When estimating the components in representation (9.3), we find

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V(x) \right| \leq M \varepsilon^{-k_1} \left[1 + \varepsilon^{3-k_2} \right] \exp(-m\varepsilon^{-1}x_1), \quad x \in \overline{D}, \quad k \leq 5, \tag{9.4}$$

where m is any number from the interval $(0, m)$, $m = \min_{\overline{D}}[a_1^{-1}(x)b_1(x)]$.

The following estimates are also valid:

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V_n(x) \right| \leq M \varepsilon^{-k_1} \exp(-m_0 \varepsilon^{-1}x_1), \quad x \in \overline{D}; \tag{9.5}$$

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} v_V(x) \right| \leq M \varepsilon^{3-k_1} \left[1 + \varepsilon^{3-k_2} \right] \exp(-m\varepsilon^{-1}x_1), \quad x \in \overline{D}; \tag{9.6}$$

$$k \leq 5, \quad n = 0, 1, 2, \quad m_0 = \min_{\Gamma_1}[a_1^{-1}(x)b_1(x)], \quad m = m_{(9.4)}.$$

Theorem 9.1. *Let* a_s, b_s, $c, f \in C^{9+\alpha}(\overline{D})$, $\varphi \in C^{9+\alpha}(\Gamma)$, $\alpha > 0$. *Then the components from representations* (9.1), (9.3) *satisfy estimates* (9.2), (9.4) – (9.6).

2.   Generally speaking, $\lambda = 0$ in estimates (7.12) and (8.10). In the ρ_u^{2h} metric this fact results in the convergence defect $\mathcal{O}(\varepsilon^{-1})$ for $l \geq 2$ in the case of scheme (4.3), (7.1) (estimate (7.19)) and for $\eta \geq 2$ in the case of scheme (4.3), (8.1) (estimate (8.18)). We give the modification of a numerical method which allows us to weaken the convergence defect.

The solution of problem (2.2), (2.1) can be written as the sum of functions

$$u(x) = u^{(1)}(x) + u^{(2)}(x), \quad x \in \overline{D}, \tag{9.7}$$

where $u^{(i)}(x)$, $x \in \overline{D}$ are the solutions of the problems

$$L^{(1)} u^{(1)}(x) \equiv \left\{ \sum_{s=1,2} b_s(x) \frac{\partial}{\partial x_s} - c(x) \right\} u^{(1)}(x) = f(x), \quad x \in \overline{D} \backslash \Gamma_2, \tag{9.8a}$$

$$u^{(1)}(x) = \varphi(x), \quad x \in \Gamma^2;$$

$$L_{(2.2)} u^{(2)}(x) = f^{(2)}(x) \equiv -\varepsilon \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} u^{(1)}(x), \quad x \in D, \tag{9.8b}$$

$$u^{(2)}(x) = \varphi(x) - u^{(1)}(x), \quad x \in \Gamma.$$

The data of problem (9.8a) as well as the solution $u^{(1)}(x)$ itself are assumed to be smoothly extended beyond the boundary Γ_1 onto the left m^1-neighbourhood of the set $\overline{D}$. The function $u^{(1)}(x)$, $x \in \overline{D}$ is regular. The function $U^{(2)}(x)$, $x \in \overline{D}$, i.e. the regular part of the solution to problem (9.8b), satisfies the following condition

$$\left| \frac{\partial^2}{\partial x_1^2} U^{(2)}(x) \right| \leq M \varepsilon, \quad x \in \Gamma_1.$$

To solve problem (9.8), we use the difference scheme

$$\Lambda^{(1)} z^{(1)}(x) \equiv \left\{ \sum_{s=1,2} [b_s^+(x)\delta_{xs} + b_s^-(x)\delta_{\overline{x}s}] - c(x) \right\} z^{(1)}(x) = f(x), \tag{9.9a}$$

$$x \in \overline{D}_h^{(1)} \setminus \Gamma_2,$$

$$z^{(1)}(x) = \varphi(x), \quad x \in \Gamma_{2h};$$

$$\Lambda_{(4.3)} z^{(2)}(x) = \widetilde{f}^{(2)h}(x) \equiv -\varepsilon \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} \widetilde{z}^{(1)}(x), \quad x \in D_h^{(2)}, \tag{9.9b}$$

$$z^{(2)}(x) = \varphi(x) - \widetilde{z}^{(1)}(x), \quad x \in \Gamma_h.$$

Here

$$\overline{D}_h^{(2)} = \overline{\omega}_1^{(2)} \times \omega_2^{(2)} \quad is \ either \ \overline{D}_{h(7.1)} \ or \ \overline{D}_{h(8.1)}; \tag{9.10a}$$

$$\overline{D}_h^{(1)} = \overline{\omega}_1^{(1)} \times \omega_2^{(1)} \quad is \ a \ uniform \ mesh, \tag{9.10b}$$

$\omega_2^{(1)} = \omega_2^{(2)}$, the stepsize of the mesh $\overline{\omega}_1^{(1)}$ is equal to the stepsize $h_1^{(2)}$ of the mesh $\overline{\omega}_1^{(2)}$; the mesh $\overline{D}_h^{(1)}$ is introduced on the set $\overline{D}$ with its left m^1-neighbourhood. Note that the meshes $\overline{D}_h^{(1)}$ and $\overline{D}_h^{(2)}$ coincide outside the σ-neighbourhood of the set Γ_1. The solution of problem (9.9), (9.10) is defined by

$$\widetilde{z}^h(x) \equiv \widetilde{z}^{(1)}(x) + \widetilde{z}^{(2)}(x), \quad x \in \overline{D}. \tag{9.9c}$$

For the function $\widetilde{z}^h(x)$, $x \in \overline{D}$, in the case of the mesh $\overline{D}_h^{(2)} = \overline{D}_{h(7.1)}$ provided that

$$l \geq k \quad and \quad N_1^{-1} = \mathcal{O}\left(\varepsilon^{k/l}\right), \quad k = 1, 2. \tag{9.11}$$

the following estimate holds

$$\rho_u^{kh}(\widetilde{z}^h) \leq M \left\{ N_1^{-1} \min[\ln N_1, \varepsilon^{-1}] \ln(\varepsilon^{-1} + 1) + N_1^{-l}\varepsilon^{-k} + N_2^{-l} \right\}, \quad k = 1, 2. \tag{9.12}$$

For the case of the mesh $D_h^{(2)} = \overline{D}_{h(8.1)}$ provided that

$$\eta > k \quad and \quad N_1 = \mathcal{O}\left(\ln^2(\varepsilon^{-1} + 1)\right), \tag{9.13}$$

we have the estimate

$$\rho_u^{kh}(\widetilde{z}^h) \leq M \left\{ N_1^{-1} \ln^2(\varepsilon^{-1} + 1) + N_1^{-\eta+k} + N_2^{-1} \right\}, \quad k = 1, 2. \tag{9.14}$$

Thus, the scheme (9.9), (9.10) converges in the $\rho_u^{kh}(\cdot)$ metric almost ε-uniformly with the convergence defect $\mathcal{O}\left(\varepsilon^{-\nu}\right)$ in the case of mesh (7.1) under the condition $l = k\nu^{-1}$ and ε-uniformly up to the factor $\ln^2\left(\varepsilon^{-1} + 1\right)$ in the case of mesh (8.1) under condition (9.13).

3. In the $\rho_u^{1h}(\cdot)$ metric the difference scheme (4.3), (8.1) has the convergence defect $\mathcal{O}\left(\ln(\varepsilon^{-1} + 1)\right)$ (for large values of η), while scheme (4.3), (7.1) has the convergence defect

$\mathcal{O}\left(\varepsilon^{-1/l}\right)$ (for large values of l). However, scheme (4.3), (7.1), as opposed to scheme (4.3), (8.1), converges ε-uniformly in the norm $\|\cdot\|$. It is possible to show that, in the class of piecewise uniform meshes having one transition point of the mesh $\overline{\omega}_1$, there do not exist meshes on which scheme (4.3) converges ε-uniformly in the norm $\|\cdot\|$ and ε-uniformly up to a logarithmic factor in the $\rho_u^{1h}(\cdot)$ metric.

4. We give a scheme which converges ε-uniformly in the norm $\|\cdot\|$ and ε-uniformly (up to a logarithmic factor) in the $\rho_u^{1h}(\cdot)$ metric.

On the set $\overline{D}_{(2.1)}$ we construct the mesh

$$\overline{D}_h^* \tag{9.15a}$$

in the following way. Under the condition

$$N_1^{-l} \leq \varepsilon^{1+\lambda}, \tag{9.16a}$$

we set

$$\overline{D}_{h(9.15a)}^* = \overline{D}_{h(7.1)}^*, \quad l = l_{(7.1)},$$

$\lambda > 0$ is an arbitrary number. Under the condition

$$N_1^{-l} > \varepsilon^{1+\lambda}, \tag{9.16b}$$

we construct the mesh $\overline{D}_{h(9.15)}^*$ in such a way. The interval $[0, d]$ is divided onto the parts $[0, \sigma_N]$, $[\sigma_N, \sigma_\varepsilon]$ and $[\sigma_\varepsilon, d]$. On each of these intervals the stepsize of the mesh $\overline{\omega}_1$ is constant and equal to $h_1^{(1)}$, $h_1^{(2)}$, $h_1^{(3)}$; $h_1^{(1)} = 3\sigma_N N_1^{-1}$, $h_1^{(2)} = 3(\sigma_\varepsilon - \sigma_N)N_1^{-1}$, $h_1^{(3)} = 3(d - \sigma_\varepsilon)N_1^{-1}$. We assume

$$\begin{aligned}
\sigma_N &= \sigma_{N(9.15)}(\varepsilon, N_1, d; l, m) = \min\left[3^{-1}d, \ lm^{-1}\varepsilon \ln N\right], \\
\sigma_\varepsilon &= \sigma_{\varepsilon(9.15)}(\varepsilon, N_1, d; l, m) = \sigma_N + \min\left[3^{-1}d, \ \eta m^{-1}\varepsilon \ln(\varepsilon^{-1} + M)\right],
\end{aligned} \tag{9.15b}$$

where $m = m_{(7.1)}$, $M = M_{(8.1)}$, $l = l_{(7.1)}$, $\eta = \eta_{(8.1)}$, $\eta > 1$.

The mesh $\overline{D}_{h(9.15)}^*$ has been constructed.

The solution of difference scheme (4.3), (9.15) is (N, ε)-uniformly bounded in the norm $\|\cdot\|$, while in the $\rho_u^{1h}(\cdot)$ metric it is bounded under the unimprovable condition

$$N_1^{-1} = \mathcal{O}\left(\ln^{-2}(\varepsilon^{-1} + 1)\right). \tag{9.17}$$

In the norm $\|\cdot\|$ under (9.17) we have the error bound

$$\|u - \tilde{z}\| \leq M\left[N_1^{-l} + N_1^{-1} \ln N_1 + N_2^{-1}\right]. \tag{9.18}$$

Let condition (9.17) be satisfied. Then, in the case of condition (9.16a), we obtain the estimate

$$\rho_u^{1h}(z) \leq M\left[N_1^{-1} \ln N_1 + N_1^{-l} + N_1^{-\lambda} + N_2^{-1}\right]; \tag{9.19a}$$

under condition (9.16b) we have the estimate

$$\rho_u^{1h}(z) \leq M \left[N_1^{-1} \ln^2(\varepsilon^{-1}+1) + N_1^{-\nu} + N_2^{-1} \right], \tag{9.19b}$$

where $\nu = \min[l, \eta - 1]$.

Thus, scheme (4.3), (9.15) converges in the $\rho_u^{1h}(\cdot)$ metric under the (unimprovable) condition

$$N_1^{-1} = o(\ln^{-2}(\varepsilon^{-1}+1)), \tag{9.20}$$

i.e. the scheme converges ε-uniformly up to the factor $\ln^2(\varepsilon^{-1}+1)$. Under the condition

$$l = \lambda = 1, \quad \eta = 2 \tag{9.21}$$

this gives the estimate

$$\|u - \tilde{z}\| \leq M \left[N_1^{-1} \ln N_1 + N_2^{-1} \right], \tag{9.22}$$

and under the additional condition (9.17) we have

$$\rho_u^{1h}(z) \leq M \left\{ N_1^{-1} \left[\ln N_1 + \ln^2(\varepsilon^{-1}+1) \right] + N_2^{-1} \right\}. \tag{9.23}$$

Theorem 9.2. *Let the hypothesis of Theorem 4.1 be fulfilled. Then the solution of finite difference scheme (4.3), (9.15) converges ε-uniformly in the norm $\|\cdot\|$. Condition (9.17) (condition (9.20)) is necessary and sufficient for the (N, ε)-uniform boundedness (for the convergence) of the mesh solutions in the $\rho_u^{1h}(\cdot)$ metric; the convergence defect of the scheme is $\mathcal{O}\left(\ln^2(\varepsilon^{-1}+1)\right)$. The mesh solutions satisfy estimate (9.19) and, in the case of conditions (9.17), (9.21) and $\{(9.17), (9.21)\}$, also estimates (9.19), (9.22) and (9.23) respectively.*

Acknowledgements

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