Skew exactness perturbation

Robin Harte and David Larson

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Abstract *We offer a perturbation theory for finite ascent and descent properties of bounded operators.*

There are various degrees of "skew exactness" ([10];[7] (10.9.0.1), (10.9.0.2)) between compatible pairs of operators, bounded and linear between normed spaces:

1. Definition *Suppose* $T : X \to Y$ *and* $S : Y \to Z$ *are bounded and linear between normed spaces; then we may classify the pair* (S,T) *as left skew exact if there is inclusion*

1.1 $$S^{-1}(0) \cap T(X) = \{0\} \ ,$$

strongly left skew exact if there is $k > 0$ *for which*

1.2 $$\|T(\cdot)\| \le k\|ST(\cdot)\| \ ,$$

and splitting left skew exact if there is $R \in BL(Z,Y)$ *for which*

1.3 $$T = RST \ .$$

Also we may classify the pair (S,T) *as right skew exact if there is inclusion*

1.4 $$S^{-1}(0) + T(X) = Y \ ,$$

strongly right skew exact if there is $k > 0$ *for which: for every* $y \in Y$ *there is* $x \in X$ *for which*

1.5 $$Sy = STx \text{ with } \|x\| \le k\|y\| \ ,$$

and splitting right skew exact if there is $R \in BL(Y,X)$ *for which*

1.6 $$S = STR \ .$$

It is easy to see that

2. Theorem *In the notation of Definition 1, there is implication*

2.1 $$(1.3) \Longrightarrow (1.2) \Longrightarrow (1.1)$$

and

2.2 $$(1.6) \Longrightarrow (1.5) \Longrightarrow (1.4) \ .$$

Proof. Most of this holds slightly more generally ([7] Theorems 10.1.2, 10.1.4), with a general operator $R' : X \to Z$ in place of the product ST . Note that (cf [3] (6.1)) (1.1) holds iff

2.3 $$(ST)^{-1}(0) \subseteq T^{-1}(0) \ ,$$

and that (1.4) holds iff

2.4 $$S(Y) \subseteq ST(X) \ \bullet$$

For Hilbert spaces X, Y, Z there is ([7] Theorem 10.8.1) implication (1.2)$\Longrightarrow$(1.3) and (1.5)$\Longrightarrow$(1.6).

A slightly stronger version of the condition (1.1) asks that

2.5 $$S^{-1}(0) \cap \text{cl } T(X) = \{0\} \ ,$$

which says that the operator $K_M J_N$ is one one, where (*cf* Yang [11];[5]) $K_M : Y \to Y/M$ and $J_N : N \to Y$ are the natural quotient and injection induced by the subspaces $M = \text{cl } TX$ and $N = S^{-1}(0)$. Stronger again is the condition that there be $k > 0$ for which there is implication

2.6 $$y \in S^{-1}(0) \Longrightarrow \|y\| \leq k \text{ dist}(y, T(X)) \ ,$$

which says that the same operator $K_M J_N$ is bounded below. Evidently

2.7 $$(1.2) \Longrightarrow (2.6) \Longrightarrow (2.5) \Longrightarrow (1.1) \ :$$

if $k > 0$ satisfies (1.2) and if $Sy = 0$ then

$$\|y\| \leq \|y - Tx\| + \|Tx\| \leq \|y - Tx\| + k\|S(Tx - y)\| \leq (1 + k\|S\|)\|y - Tx\| \bullet$$

Condition (2.6), with $k = 1$, has been noticed by Anderson [1], who describes it by calling $T(X)$ *orthogonal to* $S^{-1}(0)$. Turnsek [13] has observed that it holds for certain operators on Banach algebras:

3. Theorem *If $S \in BL(Y, Y)$ then (2.6) holds with $k = 1$ for (S, S) provided*

3.1 $$\|I - S\| \leq 1 \ .$$

Proof. Following the argument of Turnsek ([13] Theorem 1.1) write

$$S = I - U \text{ and } V_n = I + U + \ldots + U^n \ ,$$

so that

3.2 $$SV_n = I - U^{n+1} = V_n S$$

and we have

$$Sy = 0 \Longrightarrow (n + 1)y = V_n y = (I - U^{n+1})x + V_n(y - Sx)$$

and hence

$$\|y\| \leq \frac{2}{n + 1}\|x\| + \|y - Sx\| \ ;$$

now let $n \to \infty$ $\bullet$

The argument of Theorem 3 suggests - wrongly - that we are using a weakened version of the condition (1.3): we call the pair (S, T) *almost left skew exact* if there are (R_n) in $BL(Z, Y)$ with

3.3 $$\|T - R_n ST\| \to 0 \text{ and } \sup_n \|R_n\| < \infty \ ,$$

and *almost right skew exact* if instead (R_n) in $BL(Y, X)$ with

3.4 $$\|S - STR_n\| \to 0 \text{ and } \sup_n \|R_n\| < \infty \ .$$

Also call (S, T) *almost strongly right skew exact* if there is $k > 0$ for which: for every $y \in Y$ there is (x_n) in X for which

3.5 $$\|Sy - STx_n\| \to 0 \text{ with } \sup_n \|x_n\| \leq k\|y\| \ .$$

Evidently (cf [10] Theorem 10.1.2)

3.6 $$(1.3) \Longrightarrow (3.3) \Longrightarrow (1.2)$$

and

3.7 $$(1.6) \Longrightarrow (3.4) \Longrightarrow (3.5) \ ;$$

thus (3.3) implies (2.6). We do not however derive (3.3) for (S, S) from the condition (3.1). We also remark that, whenever the space Z is complete, there is implication

3.8 $$(1.4) \Longrightarrow (3.5) \ :$$

this ([2];[4] Theorem 1.1; [7] Theorem 10.5.5) uses Baire's theorem.

Under certain circumstances the "left" and "right" skew exactnesses are equivalent; we begin (cf [3] Lemma 6.2) by extending the finite ascent/descent characterizations:

4. Theorem *Suppose, under the conditions of Definition 1, that $W \subseteq X$ with $T(W) \subseteq S^{-1}(0)$, and that $V \subseteq Y$ with $T(X) \subseteq S^{-1}(V)$. Then each of the following conditions is equivalent to (1.1):*

4.1 $$T^\vee : X/T^{-1}(0) \to Y/S^{-1}(0) \text{ one one ;}$$

4.2 $$S^\wedge : T(X) \to V \text{ is one one .}$$

Also each of the following conditions is equivalent to the condition (1.4):

4.3 $$S^\wedge : T(X) \to S(Y) \text{ onto ;}$$

4.4 $$T^\vee : X/W \to Y/S^{-1}(0) \text{ is onto .}$$

Proof. The equivalences (1.1)$\Longleftrightarrow$(4.1) and (1.4)$\Longleftrightarrow$(4.3) are clear. We claim that (1.1) is equivalent to (4.2) with $V = Z$, and that this in turn is equivalent to (4.2) for arbitrary V for which $T(X) \subseteq S^{-1}V$. The second equivalence is clear; for the first note that for arbitrary $x \in X$ there is implication

$$S(Tx) \in S^{-1}(0) \Longleftrightarrow STx = 0 .$$

We also claim that (1.4) is equivalent to (4.4) with $W = \{0\}$, and that this in turn is equivalent to (4.4) for arbitrary W for which $T(W) \subseteq S^{-1}(0)$. The second equivalence is clear; for the first note that for arbitrary $y \in Y$ there is implication

$$y \in S^{-1}(0) + T(X) \Longleftrightarrow Sy \in S(TX) \bullet$$

If in particular $X = Y = Z$ and $ST = TS$ then (4.2) applies with $V = T(X)$, and (4.4) applies with $W = S^{-1}(0)$. We apply this in particular with $S = T^k$ for some $k \in \mathbf{N}$:

5. Theorem *If $X = Y = Z$ and $S = T^k : Y \to Y$, with T in the "commutative closure" of the invertibles, in the sense that there are (R_n) in $BL(X, X)$ with*

5.1 $$R_n \in BL^{-1}(X, X) \ ; \ R_nT = TR_n \ ; \ \|R_n - T\| \to 0 ,$$

then the following are equivalent:

5.2 $$(ST)^{-1}(0) \subseteq T^{-1}(0) \text{ and } T(X) = \text{cl } T(X) \ ;$$

5.3 $$S(Y) \subseteq ST(X) \text{ and } T(X) = \text{cl } T(X) \ .$$

Proof. We recall ([5];[7] Theorem 3.5.1) that for bounded linear operators $T : X \to Y$ between (possibly incomplete) normed spaces

5.4 $$T \text{ bounded below and a limit of dense} \implies T \text{ almost open ,}$$

and hence ([5];[7] Theorem 5.5.6) by duality

5.5 $$T \text{ almost open and a limit of bounded below} \implies T \text{ bounded below .}$$

Now if R_n commutes with T then it leaves both $T(X)$ and $S^{-1}(0)$ invariant, and if R_n is invertible then (cf [7] Theorem 3.11.1) its restriction $R_n^\wedge$ to $T(X)$ will be bounded below and its quotient on $Y/S^{-1}(0)$ will be onto. Thus if we assume (5.2) then by (4.1) and closed range $T^\vee$ will be bounded below and the limit of onto $R_n^\vee$, therefore onto, giving (5.3). If instead we assume (5.3) then by (4.3) $S^\wedge$ will be onto and by closed range almost open, and the limit of bounded below $(R_n^k)^\wedge$, therefore bounded below, giving (5.2) $\bullet$

(5.2) and (5.3) are together equivalent to the condition that $T \in BL(X,X)$ is *polar* ([7] Definition 7.5.2), in the sense that $0 \in \mathbf{C}$ is at worst a pole of the resolvent function $(zI-T)^{-1}$. If we relax the closed range condition we can still [12] get one of the implications, provided we further tighten the approximation by commuting invertible operators:

6. Theorem *Suppose that $S = T^k$ and that $0 \notin \text{int}\,\sigma(T)$. If the finite descent condition (1.4) holds then so also does the finite ascent condition (5.2), including closed range.*

Proof. This is shown on Hilbert space ([12] Lemma 2.5) by Herrero, Larson and Wogen. Alternatively, since we are assuming that 0 is at worst on the boundary of the spectrum then we can take the approximating invertible operators $R_n = T - \lambda_n I$ to be scalar perturbations of the operator T. Now if (1.4) holds, then the quotient operator $T^\vee$ on $X/S^{-1}(0)$ is (4.5) onto, and the limit of operators $(T-\lambda_n I)^\vee$, which we claim are invertible. As in Theorem 5 it is clear that the quotient $(T-\lambda_n I)^\vee$ is onto: we claim it is also one one. To see this recall that the operator $T-\lambda_n I$ is one-one and the restriction $(T-\lambda_n I)^\wedge = (-\lambda_n I)^\wedge$ to the subspace $T^{-1}(0)$ is onto, so that ([4] Theorem 3.11.2) the induced quotient is also one one. For the closed range note that $T(X)$ now has a closed complement, and appeal to the "Lemma of Neuberger" ([7] Theorem 4.8.2) •

Theorem 6 does not reverse:

7. Example *If*

7.1 $$S = I - \lambda U \text{ or } S = I - \lambda V \text{ or } S = \lambda W \ ,$$

where $|\lambda| = 1$, U and V are the forward and backward shifts on ℓ_2, and W the standard weight,

7.2 $$(Ux)_1 = 0 \ , \ (Ux)_{n+1} = x_n \ ; \ (Vx)_n = x_{n+1} \ ; \ (Wx)_n = (1/n)x_n \ ,$$

then S is one one and not onto, therefore of finite descent and not of finite ascent, while

7.3 $$\|I - S\| = 1 \text{ so that } 0 \notin \text{int}\,\sigma(S) \ .$$

Proof. This is easily checked: note that, extended to all sequences, there is equivalence, for arbitrary $x \in X^{\mathbf{N}}$,

7.4 $$x \in (I-\lambda U)^{-1} \iff x \in (I - \overline{\lambda} V)^{-1} \iff x = x_1(1, \lambda, \lambda^2, \ldots) \ \bullet$$

We need some auxiliary subspaces:

8. Definition *If $T \in BL(X,X)$ write*

8.1 $$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0) \text{ and } T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X)$$

for the hyperkernel and the hyperrange of T, and

8.2 $$E_X(T) = \sum_{\lambda \in \mathbf{C}} (T-\lambda I)^{-\infty}(0) \text{ and } F_X(T) = \bigcap_{\lambda \in \mathbf{C}} (T-\lambda I)^{\infty}(X) \ .$$

Each of the subspaces in Definition 8 is linear, not necessarily closed, and *hyperinvariant* under T. We recall that $T \in BL(X,X)$ is called *algebraic* if there is a nontrivial *polynomial* $0 \neq p \in \text{Poly}$ for which

8.3 $$p(T) = 0 \ ;$$

more generally T is said to be *locally algebraic* if

8.4 $$X = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} \ .$$

For the record

9. Theorem *If $T \in BL(X,X)$ for a Banach space X then*

9.1 $$T \text{ locally algebraic } \Longrightarrow T \text{ algebraic} \ .$$

Necessary and sufficient for T to have finite descent is that

9.2 $$E_X(T) + T(X) = X \ .$$

Proof. The first part of this is known as *Kaplansky's Lemma*; the proof [9] is a combination of Baire's theorem and the Euclidean algorithm for polynomials. The Euclidean algorithm also gives equality

9.3 $$E_X(T) = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} = \{x \in X : \dim \text{Poly}(T)x < \infty\} \ ,$$

and dually

9.4 $$F_X(T) = \bigcap \{p(T)(X) : 0 \neq p \in \text{Poly}\} \ .$$

Then again with a combination of Baire's theorem and the Euclidean algorithm, if $T \in BL(X,X)$ there is ([12] Lemma 2.4) $k \in \mathbf{N}$ for which

9.5 $$E_X(T) + T(X) = T^{-\infty}(0) + T(X) = T^{-k}(0) + T(X) \bullet$$

Dually, using the Euclidean algorithm, we get half way:

9.6 $$F_X(T) \cap T^{-1}(0) = T^{\infty}(X) \cap T^{-1}(0) \ .$$

For the essence of a possible spectral mapping theorem (cf [10]), we have

10. Theorem *If $S, T \in BL(X,X)$ satisfy $ST = TS$ and either*

10.1 $$S \in \{T^k : k \in \mathbf{N}\}$$

or

10.2 $$VS - TU = I \text{ with } \{U, V\} \subseteq \text{comm}(S,T) \ ,$$

then there is equivalence

10.3 $$ST \text{ of finite ascent } \Longleftrightarrow S \ , \ T \text{ of finite ascent} \ ,$$

and equivalence

10.4 $$ST \text{ of finite descent } \Longleftrightarrow S \ , \ T \text{ of finite descent} \ .$$

Proof. The backward implications are easy ([7] Theorem 7.9.2): if S and T commute and satisfy $S^{-k}(0) = S^{-k-1}(0)$ and $T^{-k}(0) = T^{-k-1}(0)$ then

$$(ST)^{-k}(0) = S^{-k}T^{-k}(0) = S^{-k}T^{-k-1}(0) = T^{-k-1}S^{-k}(0) = T^{-k-1}S^{-k-1}(0) = (ST)^{-k-1}(0) \ ,$$

if instead $ST = TS$ with $S^kX = S^{k+1}X$ and $T^kX = T^{k+1}X$ then

$$(ST)^kX = S^kT^k(X) = S^kT^{k+1}X = T^{k+1}S^kX = T^{k+1}S^{k+1}X = (ST)^{k+1}X \ .$$

Also the forward implications are clear when (10.1) $S = T^k$ is a power of T; if instead we assume (10.2) then we argue

10.5 $$(ST)^{-1}(0) \subseteq T^{-1}(0) + T(X) \text{ and } (ST)X \supseteq T^{-1}(0) \cap T(X) \ ,$$

while if (U,V) satisfies (10.2) then for arbitrary $k \in \mathbf{N}$

10.6 $$V_kS^k - T^kU_k = I \text{ with } \{U_k, V_k\} \subseteq \text{comm}(S^k, T^k) \ .$$

To verify (10.5) argue

$$STx = 0 \Longrightarrow x + TUx = VSx \in T^{-1}(0) \ ; T(Tx) = 0 \Longrightarrow Tx = TVSx - TUTx = (ST)(Vx) \ .$$

For (10.6) note that for arbitrary $k \in \mathbf{N}$

$$VS - TU = I \Longrightarrow V^{k+1}S^{k+1} - TU(I + VS + \ldots + V^kS^k) = I \bullet$$

For an induced "spectrum" to be a closed set we have

11. Theorem *$T \in BL(X,X)$ is of finite descent then so is $T - \lambda I$ for sufficiently small $\lambda \in \mathbf{C}$.*

Proof. This has been shown on Hilbert space by Han/Larson/Pan ([11] Lemma 2.2, Theorem 2.4). It is clear from the open mapping theorem (applied to the condition (4.4) with $W = \{0\}$) that if the condition (1.4) holds then also

$$S^{-1}(0) + (T-U)(X) = Y$$

whenever $T - U \in BL(X,Y)$ is sufficiently close to $T \in BL(X,Y)$: the problem is that we must also perturb S. However if $S = T^k$ and $U = \lambda I$, so that $E_X(T-U) = E_X(T)$, then we can argue

$$E_X(T-U) + (T-U)(X) = E_X(T) + (T-U)(X) \supseteq S^{-1}(0) + (T-U)(X) = X \bullet$$

The subspaces of Definition 8 lead to certain special kinds of operator:

12. Definition *We shall call $T \in BL(X,X)$ triangular if the subspace $E_X(T)$ is dense:*

12.1 $$\mathrm{cl}\ E_X(T) = X\ .$$

Dually $T \in BL(X,X)$ is co-triangular if the subspace $F_X(T)$ is trivial:

12.2 $$F_X(T) = \{0\}\ .$$

The shifts of Example 7 are either triangular or co-triangular:

13. Example *On each of the spaces c_0 and ℓ_p $(1 \le p < \infty)$, the forward shift U is triangular, the backward shift V is co-triangular and the standard weight W is both triangular and co triangular.*

Proof. The hyperkernel of the backward shift is dense, since it includes all the "terminating" sequences:

13.1 $$V^{-\infty}(0) \supseteq c_{00}\ .$$

Thus

13.2 $$E(V) \supseteq V^{-\infty}(0) \textit{ is dense}$$

and also

13.3 $$F(V) = \bigcap_{|\lambda|=1} (V - \lambda I)^{\infty}(X) \supseteq \sum_{|\lambda|<1} (V-\lambda I)^{-\infty}(0) \supseteq V^{-\infty}(0) \textit{ is dense}\ .$$

Since $U - \lambda I$ is one one for every $\lambda \in \mathbf{C}$ we have

13.4 $$E(U) = \{0\} \textit{ is trivial}$$

and also

13.5 $$F(U) \subseteq U^{\infty}(X) = \{0\} \textit{ is trivial}\ .$$

Finally we notice that the weight W commutes with the projection UV, and more generally

13.6 $$WU^nV^n = U^nV^nW\ (n \in \mathbf{N})\ ;$$

also for each $n \in \mathbf{N}$

13.7 $$(\frac{1}{n}I - W)^{-1}(0) = U^{n-1}(I-UV)V^{n-1}(X) \textit{ and } (\frac{1}{n}I - W)(X) = (U^{n-1}(I-UV)V^{n-1})^{-1}(0)\ ,$$

so that $E(W)$ is dense and $F(W)$ is trivial $\bullet$

Triangularity and Fredholmness co-operate to generate finite ascent or descent:

14. Theorem *If $T \in BL(X,X)$ then*

14.1 $$T \text{ upper semi-Fredholm and co-triangular} \Longrightarrow T \text{ of finite ascent}$$

and

14.2 $$T \text{ lower semi-Fredholm and triangular} \Longrightarrow T \text{ of finite descent} .$$

Proof. If $T \in BL(X,X)$ is upper semi-Fredholm then the finite ascent condition can be written in the form

14.3 $$F_X(T) \cap T^{-1}(0) = \{0\} .$$

Indeed since ([7] Theorem 6.12.2) each power T^m is also upper semi-Fredholm then $T^{-m}(0)$ is finite dimensional for each $m \in \mathbf{N}$ and $T^m(X)$ is closed; thus if for each $m \in \mathbf{N}$ we have $T^m(X) \cap T^{-1}(0) \neq \{0\}$ then there is (x_m) in X for which

$$\|T^m(x_m)\| = 1 \text{ and } T^{m+1}x_m = 0 .$$

By local compactness there is a subsequence

$$(y_m) = T^{\phi(m)}(x_{\phi(m)}) \to y_\infty \in T^{\phi(m)}(X) ,$$

using the closedness of all the ranges, so that $\|y_\infty\| = 1$ and $y_\infty \in F_X(T) \cap T^{-1}(0)$. This proves (14.1); towards (14.2) we claim that for subspaces $Y, Z \subseteq X$

14.4 $$Y \text{ closed of finite codimension and } Z \text{ dense} \Longrightarrow Y + Z = X :$$

because if $\dim(X/Y) = n$ find successively $e_1, e_2, \ldots, e_n$ with $e_{j+1} \in Z \setminus (Y + \mathbf{C}e_1 + \mathbf{C}e_2 + \ldots + \mathbf{C}e_j)$. Applying this with $Y = T(X)$ and $Z = E_X(T)$ gives (14.2) •

It is clear that in (14.1) we can replace the "co-triangular" condition (12.2) by the weaker condition (14.3); dually in (14.2) we can replace the triangular condition (12.1) by the weaker condition

14.5 $$\text{cl } E_X(T) + T(X) = X .$$

For operators which are both upper semi Fredholm and of finite ascent, or lower semi Fredholm of finite descent ("semi Browder" in the sense of [7] Definition 7.9.1) the conditions of Theorem 11 can be replaced by simple commutivity ([7] Theorem 7.9.2) .

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