An Arens and Royden Laugh In
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Dedicated to Goldie Hawn, on her birthday

Abstract *We attempt to deconstruct the Arens-Royden Theorem.*

Suppose A is a Banach algebra (by default complex, with identity 1): we shall write

0.1 $$A^{-1} = \{a \in A : 1 \in Aa \cap aA\}$$

for the open subgroup of *invertible* elements, and A_0^{-1} for the connected component of the identity in A^{-1}: it turns out ([8], [11] Theorem 7.11.4) that

0.2 $$A_0^{-1} = \mathrm{Exp}(A) = \{e^{c_1} e^{c_2} \ldots e^{c_k} : k \in \mathbf{N}, c \in A^k\}$$

coincides with the *generalized exponentials*, the subgroup generated by the exponentials. $\mathrm{Exp}(A)$ is open, relatively closed in A^{-1}, connected and a normal subgroup: thus we can form the quotient group,

0.3 $$\kappa(A) = A^{-1}/\mathrm{Exp}(A),$$

the *abstract index group* [4];[13] of A. Now we can state [2];[4];[6];[18];[20];[21];[22]

1. Theorem (Arens-Royden Mark I) *If A is commutative then*

1.1 $$\kappa(A) \cong H^1(\sigma(A), \mathbf{Z}),$$

the first Cech cohomology group of the "maximal ideal space" of A.

Specifically we shall interpret elements of the maximal ideal space $\sigma(A) \subseteq A^*$ as bounded multiplicative linear functionals on A; this includes sending $1 \in A$ to $1 \in \mathbf{C}$. We offer no formal definition of Cech cohomology: but if we believe the Arens-Royden theorem Mark I then it must apply to the algebra $C(\sigma(A))$ of continuous functions on $\sigma(A)$, which has of course the same maximal ideal space

1.2 $$\sigma\ C(\sigma(A)) \cong \sigma(A)\ ,$$

and whose abstract index group therefore offers an interpretation of the Cech cohomology. We arrive at

2. Theorem (Arens-Royden Mark II) *If A is commutative then*

2.1 $$\kappa(A) \cong \kappa\ C(\sigma(A)).$$

We can sharpen the statement a little more: the isomorphism is not any old isomorphism (remember the James space !), but a specific isomorphism derived from the Gelfand mapping. Stepping back a little, suppose $T : A \to B$ is a bounded multiplicative linear mapping of Banach algebras: in particular, for arbitrary $a, a' \in A$,

2.2 $$T(a'a) - T(a')T(a) = 0 = T(1) - 1.$$

For example if $B = \mathbf{C}$ then $T \in \sigma(A)$. It is clear - whether or not T is bounded - that

2.3 $$T(A^{-1}) \subseteq B^{-1};$$

if T is also bounded (or not [17]!) then $T(A_0^{-1}) \subseteq B_0^{-1}$ and also - of course the same thing -

2.4 $$T\ \mathrm{Exp}(A) \subseteq \mathrm{Exp}(B).$$

Thus $T : A \to B$ induces a mapping of abstract index groups,

2.5 $$\kappa(T) : \kappa(A) \to \kappa(B) :$$

κ god bless it is a functor. All this applies in particular to the *Gelfand mapping*: we define

2.6 $$\Gamma_A : A \to C(\sigma(A))$$

by setting - whether or not A is commutative -

2.7 $$\Gamma_A(a)(\varphi) = \varphi(a)\ (\varphi \in \sigma(A), a \in A).$$

It is this sort of thing that gives abstract linear analysis a bad name!

If $\sigma(A)$ is empty we will not trouble ourselves about the interpretation of (2.7) - take $C(\emptyset) = \mathbf{O}$?. Our sharpened version of the Arens-Royden theorem says that the isomorphism is induced by the Gelfand mapping:

3. Theorem (Arens-Royden Mark III) *If A is commutative then*

3.1 $$\kappa(\Gamma_A) : \kappa(A) \to \kappa\ C(\sigma(A)) \text{ is one-one onto.}$$

Stepping back again, suppose $T : A \to B$ is a homomorphism of Banach algebras. From (2.3) it follows that there is inclusion

3.2 $$A^{-1} \subseteq T^{-1}(B^{-1}) \subseteq A.$$

It is natural - think of the Calkin homomorphism and Atkinson's theorem - to describe ([9];[10];[11] Definition 7.6.1) $T^{-1}(B^{-1}) \subseteq A$ as the *T-Fredholm* elements of A. We are tempted to make a definition: we shall say that a homomorphism $T : A \to B$ *has the Gelfand property* ([11] (9.6.0.1)) iff

3.3 $$T^{-1}(B^{-1}) \subseteq A^{-1}.$$

Thus *Gelfand's theorem* can be succinctly stated:

4. Theorem (Gelfand) *If A is commutative then $\Gamma_A : A \to C(\sigma(A))$ has the Gelfand property.*

It is now tempting to try and deconstruct the Arens-Royden theorem, and - with their permission - to divide the statement into an "Arens theorem" and a "Royden theorem". Let us - tentatively - suggest that a homomorphism $T : A \to B$ have the *Arens property* if the index mapping $\kappa(T)$ is one-one, and the *Royden property* if $\kappa(T)$ is onto. Thus we say that $T : A \to B$ has the Arens property provided there is inclusion

4.1 $$A^{-1} \cap T^{-1}\mathrm{Exp}(B) \subseteq \mathrm{Exp}(A),$$

and that $T : A \to B$ has the Royden property provided

4.2 $$B^{-1} \subseteq T(A^{-1}) \cdot \mathrm{Exp}(B).$$

The Arens-Royden theorem therefore says that if A is commutative then the Gelfand mapping has both the Arens and the Royden properties.

5. Example $A = \mathrm{Holo}(\mathbf{S}) \subseteq C(\mathbf{S}) = B$ *the algebra of functions holomorphic in a neighbourhood of the circle* $\mathbf{S} = \partial\mathbf{D}$*, embedded $T : A \to B$ in the continuous functions.*

It is familiar ([15];[11] Theorem 7.10.7) that the abstract index group $\kappa(B) \cong \mathbf{Z}$ is essentially the integers. Now the "Arens condition" (4.1) says that if a function $b \in B$ invertible on $\mathbf{S}$ is holomorphic near $\mathbf{S}$ and has a continuous logarithm on $\mathbf{S}$ then that logarithm is holomorphic there.

In contrast the "Royden condition" (4.2) says that every continuous function $b \in B^{-1}$ invertible on the circle has holomorphic functions in its coset $b\mathrm{Exp}(B)$. Indeed if $b \in B^{-1}$ we can take $a = z^n$ with $n \in \mathbf{Z}$ given by the *topological degree* or "winding number" of $b/|b| : \mathbf{S} \to \mathbf{S}$.

6. Example *The Calkin homomorphism $T : A \to B$, where $A = B(X)$ is the bounded operators on a Banach space and $B = B(X)/K(X)$ is its quotient by the ideal of compact operators.*

Generally if $T : A \to B$ is onto there is ([8];[21] §4.8;[11] Theorem 7.11.5) equality

6.1 $$T\ \mathrm{Exp}(A) = \mathrm{Exp}(B);$$

for such T the "Arens condition" (4.1) takes the form

6.2 $$A^{-1} \cap (\mathrm{Exp}(A) + T^{-1}(0)) \subseteq \mathrm{Exp}(A),$$

while the "Royden condition" reduces to

6.3 $$B^{-1} \subseteq T(A^{-1}).$$

For example if $A = B(X)$ for a Hilbert space X then Kuiper's theorem ([5] Theorem I.6.1) says that the invertible group of $A = B(X)$ is connected: $A^{-1} = \mathrm{Exp}(A)$. This makes the "Arens property" (4.1) a triviality. The "Royden property" in this case reduces to the connectedness of B^{-1}, which never happens. If instead $B^{-1} = \mathrm{Exp}(B)$ is connected then the "Royden property" (4.2) becomes a triviality, and the "Arens property" only happens when A^{-1} is also connected.

The original Arens-Royden theorem has an extension to operator matrices [3],[21]: if A is commutative and $T = \Gamma_A$ is the Gelfand homomorphism then

6.4 $$\kappa(T^{n\times n}) : \kappa(A^{n\times n}) \to \kappa(C\sigma(A)^{n\times n})$$

is an isomorphism. Gonzalez and Aiena [1],[7] have used operator matrices to throw light on one way in which the invertible group of Banach space operators can fail to be connected:

7. Theorem *If* $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is a Banach algebra with blocks then*

7.1 $$1 - MN \subseteq A^{-1} \text{ and } 1 - NM \subseteq B^{-1}$$

if and only if there is equality

7.2 $$G^{-1} = \begin{pmatrix} A^{-1} & M \\ N & B^{-1} \end{pmatrix},$$

in which case

7.3 $$\mathrm{Exp}(G) \subseteq \begin{pmatrix} \mathrm{Exp}(A) & M \\ N & \mathrm{Exp}(B) \end{pmatrix}.$$

Proof. Recall [12] that for G to be a Banach algebra the diagonal blocks A and B must also be Banach algebras while the off diagonals M and N must be A B bimodules; products MN and NM lie in A and B respectively. Now if $1 - MN \subseteq A^{-1}$ and $1 - NM \subseteq B^{-1}$ then

$$\begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}\begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix} = \begin{pmatrix} 1-mn & 0 \\ 0 & 1-nm \end{pmatrix} = \begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}$$

and then

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & mb^{-1} \\ na^{-1} & 1 \end{pmatrix} ; \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}m \\ b^{-1}n & 1 \end{pmatrix};$$

also

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G^{-1} \Longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix} + \begin{pmatrix} 0 & -m \\ -n & 0 \end{pmatrix}$$

$$\in \begin{pmatrix} a & m \\ n & b \end{pmatrix}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} A & M \\ N & B \end{pmatrix}\begin{pmatrix} 0 & -m \\ -n & 0 \end{pmatrix}) = \begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} 1-Mn & -Am \\ -Bn & 1-Nm \end{pmatrix} \subseteq \begin{pmatrix} A & M \\ N & B \end{pmatrix}^{-1}.$$

This shows that (7.1) implies (7.2); conversely

$$\begin{pmatrix} 1-mn & 0 \\ 0 & 1-nm \end{pmatrix} = \begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}\begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix} \in \begin{pmatrix} A & M \\ N & B \end{pmatrix}^{-1} \Longrightarrow 1 - mn \in A^{-1} \ , \ 1 - nm \in B^{-1}.$$

Now if $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is in $\mathrm{Exp}(G)$ then there is $\begin{pmatrix} a_t & m_t \\ n_t & b_t \end{pmatrix}_{(0\le t\le 1)}$ connecting $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that (a_t) and (b_t) connect $a \in A^{-1}$ and $b \in B^{-1}$ to $1 \in A$ and $1 \in B$, giving (7.3) •

In fact each of the two conditions in (7.1) implies the other, and one of the inclusions in (7.2) implies the other. It is also clear from (7.1) that

7.4 $$1 - MN \subseteq \mathrm{Exp}(A) \text{ and } 1 - NM \subseteq \mathrm{Exp}(B);$$

thus also (cf [18]!) each of the two conditions in (7.4) implies the other.

These arguments can be used to show (cf [16]) that the invertible group on certain Banach spaces is not connected:

8. Example *If $X = Y \times Z$ with $Y = \ell_p$ and $Z = \ell_q$ with $q \neq p$ then*

8.1 $$T = \begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \in BL^{-1}(X, X) \setminus \mathrm{Exp}\ BL(X, X),$$

where u and v are the forward and backward shifts on Y and Z respectively and $w : Z \to Y$ is the rank one projection on the first co-ordinate.

Proof. If u' and v' are the forward and backward shifts on Z and Y respectively and $w' : Y \to Z$ the same projection then

8.2 $$v'u = 1 \neq uv' = 1 - ww' \text{ and } vu' = 1 \neq u'v = 1 - w'w,$$

so that T is invertible with

8.3 $$T^{-1} = \begin{pmatrix} v' & 0 \\ w' & u' \end{pmatrix}.$$

At the same time [1],[7] the whole of $BL(Y, Z)$ and of $BL(Z, Y)$ consist of inessential operators. By Theorem 7 therefore, for the Calkin quotient of T to be in the connected component of the identity it would be necessary for the Calkin quotients of u and v to be generalized exponentials, and hence in particular for

8.4 $$\mathrm{index}(u) = \mathrm{index}(v) = 0.$$

Since this is not the case T cannot be a generalized exponential •

Alternatively the Calkin mapping

$$\Phi : BL(X, X) = \begin{pmatrix} A' & M' \\ N' & B' \end{pmatrix} \to \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

has the property that for arbitrary $a' \in A' = BL(Y, Y)$, $b' \in B' = BL(Z, Z)$ there is ([10];[14];[11] Theorem 7.6.2) implication

8.5 $$\Phi(a') \in \mathrm{Exp}(A) \Longrightarrow a' \in a'(A')^{-1}a'\ ,\ \Phi(b') \in \mathrm{Exp}(B) \Longrightarrow b' \in b'(B')^{-1}b'\ ,$$

and now ([10];[14];[11] (9.3.4.3)) a left invertible element with an invertible generalized inverse must also be right invertible.

Going back to the inclusion (3.2), observe

8.6 $$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}(B^{-1});$$

we call ([9];[10];[11] Definition 7.6.1) $A^{-1} + T^{-1}(0)$ the T-*Weyl* elements of A: when T is a Calkin homomorphism these are the Fredholm operators of index zero. Thus we can enlarge the abstract index group and form [13] the quotient

8.7 $$\kappa_T(A) = (A^{-1} + T^{-1}(0))/\mathrm{Exp}(A) :$$

we can interpret this either as left cosets as right cosets, which may or may not be the same. Now there is inclusion $T(A^{-1}(0) + T^{-1}(0)) \subseteq B^{-1}$ and hence extension

8.8 $$\kappa(T) : \kappa_T(A) \to \kappa(B).$$

We can also consider separately left and right invertible elements:

8.9 $$A^{-1}_{left} = \{a \in A : 1 \in Aa\} \ ; \ A^{-1}_{right} = \{a \in A : 1 \in aA\}.$$

Evidently the invertibles are the intersection of the left and the right invertibles, which each form open sub semigroups; what is interesting ([13] Theorem 7) is that there is now a relationship between left and right cosets, while the generalized exponentials continue to be the connected component of the identity:

9. Theorem *If $a \in A^{-1}_{left}$ there is inclusion*

9.1 $$a\mathrm{Exp}(A) \subseteq \mathrm{Exp}(A)a.$$

The right cosets

9.2 $$\kappa_{left}(A) = A^{-1}_{left}/\mathrm{Exp}(A) = \{\mathrm{Exp}(A)a : a \in A^{-1}_{left}\}$$

form a multiplicative semigroup.

Proof. Suppose $a'a = 1 \in A$: then if $0 \neq \lambda \in \mathbf{C}$

9.3 $$aA^{-1}a' \subseteq A^{-1} + \lambda(1 - aa') \subseteq a'A^{-1}a$$

and

9.4 $$a\mathrm{Exp}(A)a' \subseteq \mathrm{Exp}(A) + \lambda(1 - aa') \subseteq a'\mathrm{Exp}(A)a.$$

For (9.3) observe that if $b \in A^{-1}$ then the inverse of $aba' - \lambda(1 - aa')$ is $ab^{-1}a' - \lambda^{-1}(1 - aa')$; to convert this to (9.1) note (cf [21] §4.2; [11] (9.11.3.4))

9.5 $$ae^{c}a' + 1 - aa' = e^{aca'}.$$

Now if $a'a = 1$ then (9.3) gives inclusion $aA^{-1} \subseteq A^{-1}a$, and (9.5) gives (9.4) and hence (9.1). From (9.3) we can unambiguously multiply right cosets

9.6 $$(A^{-1}a)(A^{-1}b) \subseteq A^{-1}(A^{-1}a)b,$$

and (9.1) enables us to do the same for right cosets modulo $\mathrm{Exp}(A)$ •

Finally if $a \in A^X$ is a system of Banach algebra elements, indexed by a set X, we can [13] extend the idea of left invertibles A^{-1}_{left} to systems

9.7 $$A^{-X}_{left} = \{a \in A^X : 1 \in \sum_{x \in X} Aa_x\},$$

and replace the "abstract left index semigroup" by the following object:

9.8 $$\kappa^X_{left}(A) = A^{-X}_{left}/\mathrm{Exp}(A) = \{\mathrm{Exp}(A)a : a \in A^{-X}_{left}\}.$$

It is clear that homomorphisms $T : A \to B$ induce mappings

9.9 $$\kappa^X_{left}(T) : \kappa^X_{left}(A) \to \kappa^X_{left}(B),$$

and we can now investigate separately "simultaneous" left and right Arens, Royden and indeed Gelfand properties.

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