Solutions to Dilation Equations*

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Declaration

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Or, somewhat less anapæsticly[†]:

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David Malone (September 7, 2001).

[†] See `http://www.sfu.ca/~finley/discussion.html` for more details.

Summary

This thesis aims to explore part of the wonderful world of dilation equations. Dilation equations have a convoluted history, having reared their heads in various mathematical fields. One of the early appearances was in the construction of continuous but nowhere differentiable functions. More recently dilation equations have played a significant role in the study of subdivision schemes and in the construction of wavelets. The intention here is to study dilation equations as entities of interest in their own right, just as the similar subjects of differential and difference equations are often studied.

It will often be $L^p(\mathbb{R})$ properties we are interested in and we will often use Fourier Analysis as a tool. This is probably due to the author's original introduction to dilation equations through wavelets.

A short introduction to the subject of dilation equations is given in Chapter 1. The introduction is fleeting, but references to further material are given in the conclusion.

Chapter 2 considers the problem of finding all solutions of the equation which arises when the Fourier transform is applied to a dilation equation. Applying this result to the Haar dilation equation allows us first to catalogue the $L^2(\mathbb{R})$ solutions of this equation and then to produce some nice operator results regarding shift and dilation operators. We then consider the same problem in $\mathbb{R}^n$ where, unfortunately, techniques using dilation equations are not as easy to apply. However, the operator results are retrieved using traditional multiplier techniques.

In Chapter 3 we attempt to do some hands-on calculations using the results of Chapter 2. We discover a simple 'factorisation' of the solutions of the Haar dilation equation. Using this factorisation we produce many solutions of the Haar dilation equation. We then examine how all these results might be applied to the solutions of other dilation equations.

A technique which I have not seen exploited elsewhere is developed in Chapter 4. This technique examines a left-hand or right-hand 'end' of a dilation equation. It is initially developed to search for refinable characteristic functions and leads to a characterisation of refinable functions which are constant on intervals of the form $[n, n+1)$. This left-hand end method is then applied successfully to the problem of 2- and 3-refinable functions and used to obtain bounds on smoothness and boundedness.

Chapter 5 is a collection of smaller results regarding dilation equations. The relatively simple problem of polynomial solutions of dilation equations is covered, as are some methods for producing new solutions and equations from known solutions and equations. Results regarding when self-similar tiles can be of a simple form are also presented.

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Chapter 1

What are these things called Dilation Equations?

1.1 Introduction

In this chapter we will try to get a basic feel for dilation equations. We will see how they arise in the construction of wavelets, investigate some examples and briefly outline some of the techniques used to analyse them.

1.2 Where do they come from?

A wavelet basis for $L^2(\mathbb{R})$ is an orthonormal basis of the form:

$$\left\{2^{-\frac{n}{2}} w\left(2^n x - k\right) : k, n \in \mathbb{Z}\right\}.$$

The function w is usually referred to as the *mother wavelet*. In an effort to produce a theory which facilitated the construction and analyses of these bases, the notion of a *Multiresolution Analysis* (MRA) was conceived.

Definition 1.1. A multiresolution analysis of $L^2(\mathbb{R})$ is a collection of subsets $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that:

1. $\exists\, g \in L^2(\mathbb{R})$ so that V_0 consists of all (finite) linear combinations of $\{g(\cdot - k) : k \in \mathbb{Z}\}$,

2. the $g(\cdot - k)$ are an orthonormal series in V_0,

3. for any V_j we have $f(\cdot) \in V_j \iff f(2\cdot) \in V_{j+1}$,

4. $\bigcup_{j=-\infty}^{+\infty} V_j$ is dense in $L^2(\mathbb{R})$,

5. $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$,

6. $V_j \subset V_{j+1}$.

This structure can be viewed in an intuitive way. Consider trying to approximate some function f by choosing a function in V_0. This amounts to choosing coefficients so that:

$$f(x) \approx \sum_k a_k g(x-k),$$

which is a common mathematical problem.

Now consider what happens when we move from V_0 to V_1. This corresponds to allowing the choice of twice as many functions which are half as wide as before. This should result in a better approximation, and part 6 ensures that our choice of function in V_1 can be at least as good as the choice in V_0.

As we move along the chain V_n, we expect improving approximations of f, corresponding to improving resolution. Parts 4 and 5 ensure that these improving approximations converge in $L^2(\mathbb{R})$ and are not in some sense degenerate.

Once you have one of these MRA structures, there exist* recipes for constructing wavelets (eg. [33]). It is reasonably clear that the construction of the MRA rests heavily on locating a suitable g.

What can we say about g using Definition 1.1? Well, first $V_0 = \text{span}\{g(\cdot - k) : k \in \mathbb{Z}\}$ so, using part 3 of the definition we know that $V_1 = \text{span}\{g(2\cdot - k) : k \in \mathbb{Z}\}$. Noting that $g \in V_0 \subset V_1$ we conclude that:

$$g(x) = \sum_k c_k g(2x-k).$$

This equation, where $g(x)$ is expressed in terms of translates of $g(2x)$, is a *dilation equation* or *two scale difference equation*. A function satisfying such an equation is said to be *refinable*, or to emphasise the scale: *2-refinable*.

1.3 The Haar dilation equation

The Haar dilation equation is the most simple example which illuminates the structure of what is going on here. Consider $\chi_{[0,1)}$, the characteristic function of the interval $[0,1)$.

*Not all wavelets arise from MRAs, see Chapter 4 of [5] for some more exotic wavelets.

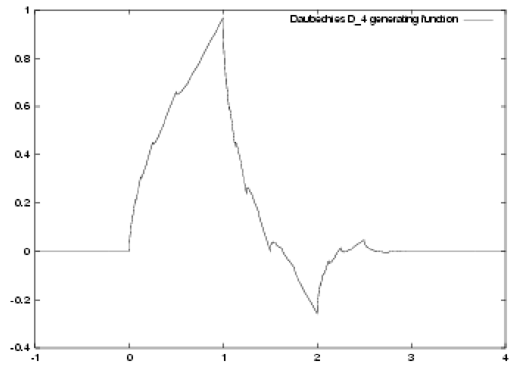


Figure 1.1: Daubechies's D_4 generating function.

Clearly $\chi_{[0,1)} = \chi_{[0,\frac{1}{2})} + \chi_{[\frac{1}{2},1)}$, however as $\chi_{[0,\frac{1}{2})}(x) = \chi_{[0,1)}(2x)$ and $\chi_{[\frac{1}{2},1)}(x) = \chi_{[0,1)}(2x-1)$ we see $\chi_{[0,1)}$ is a solution of:

$$g(x) = g(2x) + g(2x-1).$$

This choice of g actually leads to a well-behaved MRA and in turn to the Haar wavelet basis of $L^2(\mathbb{R})$ given by the mother wavelet:

$$w(x) = \begin{cases} +1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases}.$$

This basis has been known since at least 1910. More recently, people have begun to produce wavelet bases by solving carefully-chosen dilation equations with the aim of producing wavelets with particular properties. In particular in [9], Daubechies produces a whole family of orthonormal compactly-supported wavelets using this method. This family, usually labeled D_{2N}, uses $2N$ non-zero coefficients in the dilation equation to achieve smoothness of roughly $C^{\frac{N}{5}}$. For small N, the functions are actually significantly smoother; Figure 1.1 shows D_4 which is roughly 0.55 times differentiable.

1.4 Relating properties and coefficients

Some properties of g place simple conditions on the coefficients of the dilation equation. For example, if g is in $L^1(\mathbb{R})$ and has non-zero mean, then integrating both sides of the dilation equation gives:

$$2 = \sum_k c_k.$$

Orthonormality of $g(\cdot - k)$ in $L^2(\mathbb{R})$ can also be applied to give:

$$2\delta_{0m} = \sum_k c_k \overline{c_{k-2m}},$$

for any $m \in \mathbb{Z}$.

The Strang-Fix condition, which tests for the ability to approximate x^m, can also be used to give the condition:

$$0 = \sum_k c_k (-1)^k k^m.$$

These are the most common conditions imposed on coefficients in order to produce wavelets. So, given that we have chosen some set of coefficients, how do we go about finding a solution to the dilation equation with these coefficients? If it exists, will it be unique? Will it have the properties which we wanted?

1.5 Fourier techniques

Many of those working on wavelets had a signal processing background and for them the application of the Fourier transform to dilation equations seems to have been a natural step. The Fourier transform takes a function and provides frequency information. On $L^1(\mathbb{R})$ the Fourier transform can† be defined by:

$$\begin{aligned} \mathcal{F} : L^1(\mathbb{R}) &\to L^\infty(\mathbb{R}) \\ f(x) &\mapsto \hat{f}(\omega) = (\mathcal{F} f)(\omega) := \int f(x) e^{-i\omega x}\, dx. \end{aligned}$$

The Fourier transform has many nice properties: it is bijective on $L^2(\mathbb{R})$; it scales the usual inner product $(f, g) = 2\pi(\hat{f}, \hat{g})$; and it turns convolution‡ into pointwise multiplication. Most interesting, for the study of dilation equations, is how it interacts with translation and dilation:

$$\begin{aligned} f(x) &\mapsto \hat{f}(\omega), \\ f(\lambda x) &\mapsto |\lambda|^{-1} \hat{f}(\lambda^{-1}\omega), \\ f(x-k) &\mapsto e^{-i\omega k} \hat{f}(\omega). \end{aligned}$$

†The normalisation of the Fourier transform is irksomely nonstandard. For example [40] define it with an extra factor of 2π inside the exponential and [11] doesn't bother with the minus sign.

‡The convolution of two functions f and g is given by $(f * g)(x) = \int f(t) g(x-t)\, dt$.

Applying this to:

$$g(x) = \sum_k c_k g(2x - k),$$

we get:

$$\hat{g}(\omega) = \hat{g}\left(\frac{\omega}{2}\right)\left(\frac{1}{2}\sum_k c_k e^{-i\frac{\omega}{2}k}\right).$$

Letting $p(\omega) = \frac{1}{2}\sum c_k e^{-i\omega k}$ we can rewrite this as:

$$\hat{g}(\omega) = \hat{g}\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{2}\right).$$

The trigonometric polynomial p is referred to as the *symbol* of the equation.

The transformed equation has been used by many authors and will be used frequently in Chapters 2 and 3. Most authors are concerned with the case where g is integrable, which ensures the continuity of $\hat{g}$, allowing the iteration of the transformed equation until it becomes an infinite product. By estimating the decay of this product, [11] shows that if the function is compactly-supported and the equation has N coefficients, then the support of the function will be of length $N - 1$.

1.6 Matrix methods

Another technique commonly applied to dilation equations involves rewriting the dilation equation in matrix form. The most obvious way of introducing linear operators into the picture is to define an operator $\mathcal{V}$ by:

$$(\mathcal{V}f)(x) = \sum_k c_k f(2x - k).$$

Then a solution to the dilation equation corresponds to a fixed point of this operator. Solutions to dilation equations can be produced by choosing some initial function f_0 and examining the sequence $\mathcal{V}^n f_0$. This process need not converge, but for suitably chosen f_0 can converge quite rapidly. The iteration of this operator is sometimes referred to as the *cascade algorithm*.

If we are searching for g, a compactly-supported solution (say on $[0, N]$), then we may write out the dilation equation for $x = 0, 1, \ldots, N$. We get:

$$\begin{aligned}
g(0) &= c_0 g(0), \\
g(1) &= c_2 g(0) + c_1 g(1) + c_0 g(2), \\
g(2) &= c_4 g(0) + c_3 g(1) + c_2 g(2) + c_1 g(3) + c_0 g(4), \\
g(3) &= c_6 g(0) + c_5 g(1) + c_4 g(2) + c_3 g(3) + c_2 g(1) + \ldots, \\
&\vdots \\
g(N-1) &= c_N g(N-2) + c_{N-1} g(N-1) + c_{N-2} g(N), \\
g(N) &= c_N g(N).
\end{aligned}$$

This is an eigenvalue problem of the form:

$$\vec{g} = M\vec{g}.$$

Solving this problem tells us the values of g at the integers and can be used to produce good guesses for f_0. If we further assumed that g was C^m, then by differentiating both sides of the dilation equation, we can show that $1, \frac{1}{2}, \frac{1}{4}, \ldots, 2^{-m}$ must be eigenvalues of M.

This idea, of splitting a solution into a vector, can be taken further and has proved to be a powerful tool. Consider $\Phi g : [0, 1] \to \mathbb{C}^N$ given by:

$$\Phi g(x) = \begin{pmatrix} g(x) \\ g(x+1) \\ \vdots \\ g(x+N-1) \end{pmatrix}.$$

We can then rewrite the dilation equation as:

$$\Phi g(x) = \begin{cases} T_0 \Phi g(2x) & x \in [0, \frac{1}{2}) \\ T_1 \Phi g(2x-1) & x \in [\frac{1}{2}, 1) \end{cases}$$

where T_0 and T_1 are matrices given by:

$$T_0 = \left(c_{2j-k}\right)_{j,k} \text{ and } T_1 = \left(c_{2j-k+1}\right)_{j,k}.$$

We can neaten the form of this equation by considering the binary expansion of $x \in [0, 1]$:

$$x = 0.\epsilon_1 \epsilon_2 \epsilon_3 \ldots$$

and using the map $\tau : x \mapsto 2x \mod 1$. We can now represent the dilation equation as:

$$\Phi g(x) = T_{\epsilon_1} \Phi g(\tau x).$$

By iterating this relation we get:

$$\Phi g(x) = T_{\epsilon_1} T_{\epsilon_2} \dots T_{\epsilon_n} \Phi g(\tau^n x).$$

Suppose g is smooth, then by varying x in digits past ϵ_n we can make a small change in $\Phi g(x)$. However, this can correspond to a large change in $\Phi g(\tau^n x)$. This means that the product of matrices must have a dampening effect on this change. To get a hold on this idea people have defined quantities such as the *Joint Spectral Radius* of a collection of matrices:

$$\rho\left(M_0, M_1, \dots, M_{q-1}\right) = \lim_{n\to\infty} \sup_{(\epsilon_1,\dots,\epsilon_n)\in\{0,\dots,q-1\}^n} \left\|M_{\epsilon_1} \dots M_{\epsilon_n}\right\|^{\frac{1}{n}}.$$

For example, for a continuous solution we expect that $\rho(T_0, T_1) < 1$ when T_0, T_1 are considered as operators on some appropriate space.

These matrix techniques are not used that frequently later in this work, but the results of Chapter 4 could be viewed as a variation on the idea of producing the vector Φg from g.

1.7 Conclusion

We have just completed a whirlwind introduction to dilation equations. We have seen how they arise naturally in the study of multiresolution analyses and got a flavour of the most basic techniques used in their study. There are many explorations of these and similar ideas — see [4, 11, 12, 20, 47, 31] for a taster. Generalisations of dilation equations exist where the function is vector valued, the coefficients are matrices and dilation becomes a matrix [39, 51]. Chapters 1 and 2 of [3] also provide an introduction to these ideas and the later chapters go on to generalise this work to higher dimensional situations.

More about Multiresolution Analysis and Wavelets can be found in any one of the multitude of books about Wavelets; [10] is considered one of the classic works and appendices 1 and 2 of [41] are reprints of papers which provide 'popular' introductions to the subject area.

The Fourier transform is a fundamental piece of mathematics with many practical and elegant applications. Practical details can be found in most engineering mathematics texts, for example see [27]. More theoretical details can be found in such books as [40]. Despite being a practical tool and a nice piece of theory the Fourier transform naturally shows up in the physics of waves; it is actually possible to build an optical system which effects the Fourier transform (see Chapter 11 of [19]).

Chapter 2

Maximal solutions to transformed dilation equations

2.1 Introduction

We saw in Chapter 1 that if g satisfies the dilation equation:

$$g(x) = \sum_k c_k g(2x - k),$$

then the Fourier transform of g satisfies:

$$\hat{g}(\omega) = p\left(\frac{\omega}{2}\right) \hat{g}\left(\frac{\omega}{2}\right)$$

for almost all $\omega \in \mathbb{R}$ (providing it has a Fourier transform), where $p(\omega) = \frac{1}{2}\sum c_k e^{-ik\omega}$. Note that $p(\omega)$ depends only on the dilation equation. It is easy to show that we can redefine $\hat{g}$ on a set of measure zero so that it satisfies this equation everywhere (see [32] Lemma 4.6).

We also note that if π is some function satisfying $\pi(\omega) = \pi(2\omega)$, then $\pi\hat{g}$ also satisfies the above equation, and if there were some g_1 such that $\mathcal{F}(g_1) = \pi\hat{g}$, then g_1 would also be a solution of the dilation equation.

Let us try to formulate a converse of this result. Imagine we can find a function m so that if g is any solution of the dilation equation (with a Fourier transform), then we can find a function π so that $\pi m = \mathcal{F}(g)$ and $\pi(\omega) = \pi(2\omega)$. This would give us some sort of characterisation of all solutions with Fourier transforms. We can hope that m would be the Fourier transform of some function, and so would be a universal solution in some sense.

2.2 What does maximal look like?

Definition 2.1. Given $p : \mathbb{R} \to \mathbb{F}$ with $\mathbb{F}$ a field, define $\Phi_2(p)$ be the set of all functions which satisfy:

$$\Phi_2(p) = \left\{ \phi : \mathbb{R} \to \mathbb{F} : \phi(\omega) = p\left(\frac{\omega}{2}\right)\phi\left(\frac{\omega}{2}\right), \forall \omega \in \mathbb{R} \right\}.$$

For suitable choices of p this will be a transformed dilation equation, but for the moment we place no restrictions on p. Note that $\Phi_2(p)$ is never empty as it always contains $\phi = 0$.

Definition 2.2. For $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{F}$ we write $\phi_1 \preccurlyeq \phi_2$ if we can find ρ so that $\phi_1(\omega) = \rho(\omega)\phi_2(\omega)$, for all ω.

Again, for the moment, we will not place any restrictions on ρ. In the long run, we will not be looking for ρ but for a π such that $\pi(\omega) = \pi(2\omega)$.

Lemma 2.1. *As defined above $\preccurlyeq$ is a pre-order.*

Proof. We need to show $\phi \preccurlyeq \phi$ for any ϕ and $\phi_1 \preccurlyeq \phi_2, \phi_2 \preccurlyeq \phi_3$ implies $\phi_1 \preccurlyeq \phi_3$. The former follows by taking $\rho(\omega) = 1$, the latter by using the product of the two ρ functions. ∎

The following lemma gives us some sort of idea about what the relation $\preccurlyeq$ means.

Lemma 2.2. *$\phi_1 \preccurlyeq \phi_2$ is equivalent to:*

$$\{\omega : \phi_1(\omega) \neq 0\} \subset \{\omega : \phi_2(\omega) \neq 0\}.$$

Proof. First we show $\phi_1 \preccurlyeq \phi_2 \Rightarrow \{\omega : \phi_1(\omega) \neq 0\} \subset \{\omega : \phi_2(\omega) \neq 0\}$. As $\phi_1 \preccurlyeq \phi_2$ we can find ρ so that $\phi_1(\omega) = \rho(\omega)\phi_2(\omega)$. So,

$$\begin{aligned} \phi_2(\omega) = 0 \quad &\Rightarrow \quad \phi_1(\omega) = 0, \\ \{\omega : \phi_2(\omega) = 0\} \quad &\subset \quad \{\omega : \phi_1(\omega) = 0\}, \\ \{\omega : \phi_1(\omega) \neq 0\} \quad &\subset \quad \{\omega : \phi_2(\omega) \neq 0\}, \end{aligned}$$

as required.

Now we show $\{\omega : \phi_1(\omega) \neq 0\} \subset \{\omega : \phi_2(\omega) \neq 0\} \Rightarrow \phi_1 \preccurlyeq \phi_2$. We begin by setting:

$$\rho(\omega) = \begin{cases} \frac{\phi_1(\omega)}{\phi_2(\omega)} & \text{if } \phi_2(\omega) \neq 0 \\ 0 & \text{if } \phi_2(\omega) = 0 \end{cases}.$$

If we take ω so that $\phi_2(\omega) \neq 0$, then clearly $\phi_1(\omega) = \rho(\omega)\phi_2(\omega)$. If we take ω so that $\phi_2(\omega) = 0$, then $\phi_1(\omega)$ must be zero, because the contrapositive of our hypothesis is $\{\omega : \phi_2(\omega) = 0\} \subset \{\omega : \phi_1(\omega) = 0\}$. So in this case $\phi_1(\omega) = \rho(\omega)\phi_2(\omega)$. ■

We want to use this relation to partially order a set of functions. Unfortunately there are functions for which $\phi_1 \preccurlyeq \phi_2$ and $\phi_2 \preccurlyeq \phi_1$ but $\phi_1 \neq \phi_2$. For instance take $\phi_1(\omega) = \omega$ and $\phi_2(\omega) = \omega^2$.

We get around this in the usual way: by taking equivalence classes. We say $\phi_1 \sim \phi_2$ iff $\phi_1 \preccurlyeq \phi_2$ and $\phi_2 \preccurlyeq \phi_1$. It is straightforward to show that this is an equivalence relation and, if we take equivalence classes, that the inherited relation $\preccurlyeq$ is a partial order. We note that Lemma 2.2 shows that two functions are equivalent iff they are zero on the same set. We will use $[\phi]$ to denote the equivalence class containing ϕ.

We will take equivalence classes of functions in $\Phi_2(p)$, but up to this stage could have used any collection of functions taking values in some arbitrary field.

Now that we have a partially-ordered set, an obvious thing to do is to use Zorn's Lemma to show that it has a maximal element. We could use Theorem 2.3 and Corollary 2.4, which follow.

Theorem 2.3. *Let E be a chain of equivalence classes of $\Phi_2(p)$ with the equivalence relation and order described above. Then there exists a function $m \in \Phi_2(p)$ whose equivalence class is an upper bound for E.*

Corollary 2.4. *$\Phi_2(p)$ has a maximal element with respect to the pre-order on it, and in fact $[\Phi_2(p)]$ is a complete lattice.*

However, we can actually construct a maximal element directly, without using the axiom of choice (Lemma 2.5).

Lemma 2.5. *We can construct a maximal element in $\Phi_2(p)$ with respect to the pre-order defined above.*

Proof. Our plan is as follows: for all $y \in \pm[1, 2)$ we define m at some $2^l y$ (with $l \in \mathbb{Z}$), and then use the two relations:

$$\begin{aligned} m(\omega) &= \frac{m(2\omega)}{p(\omega)} \qquad \text{and} \\ m(\omega) &= m\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{2}\right) \end{aligned}$$

to extend m to $\mathbb{R} \setminus \{0\}$. Finally we give m a value at zero and check that it is maximal using Lemma 2.2.

The only problem that could arise in this scheme is that $p(\omega)$ might be zero when we want to divide by it. To avoid this we carefully choose l as follows. For our $y \in \pm[1,2)$ we examine the set:

$$\{n \in \mathbb{Z} : p(2^n y) = 0\}.$$

If this set has no lower bound, we set $m(2^n y) = 0$ for all $n \in \mathbb{Z}$. If it has a lower bound, then we take l to be its least element, set $m(2^l y) = 1$, and use our relations to find $m(2^n y)$. If the set is empty, we set $m(y) = 1$.

Now we do not have problems dividing by zero, since if we are using the rule:

$$m(2^n y) = \frac{m(2^{n+1}y)}{p(2^n y)},$$

then $|2^{n+1}y| \leq |2^l y|$ (since this relation chains towards the origin). Dividing by 2 we get $|2^n y| \leq |2^{l-1}y|$, and so by the definition of l, $p(2^n y) \neq 0$ when $n < l$.

It only remains to define m at 0, where we want $m(0) = p(0)m(0)$, so we set $m(0) = 1$ if $p(0) = 1$, and $m(0) = 0$ otherwise.

By its construction, m satisfies:

$$m(\omega) = m\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{2}\right),$$

and hence $m \in \Phi_2(p)$. It remains to be shown that m is maximal, which by Lemma 2.2 is equivalent to showing that $m(\omega) = 0 \Rightarrow \phi(\omega) = 0$ for all $\phi \in \Phi_2(p)$.

Suppose $m(\omega) = 0$.

First we dispose of the case $\omega = 0$. If $m(\omega) = 0$, we know $p(0) \neq 1$ which means that $\phi(0) = 0$ because of the constraint $\phi(0) = p(0)\phi(0)$. If $\omega \neq 0$ we may write $\omega = 2^n y$ with $y \in \pm[1,2)$. We re-examine the set:

$$\left\{k \in \mathbb{Z} : p(2^k y) = 0\right\}.$$

and consider three cases:

- This set has no lower bound. In this case we can choose a $k < n$ such that $p(2^k y) = 0$, and using the fact that $\phi \in \Phi_2(p)$:

$$\phi(\omega) = \phi(2^n y) = p(2^{n-1}y)p(2^{n-2}y)\ldots p(2^k y)\phi(2^k y) = 0.$$

- This set has a lower bound. Let l be its least element; we know that $m(2^l y) = 1$ and $p(2^l y) = 0$. We also know for $k < l$:

$$m(2^k y) = \frac{m(2^l y)}{p(2^k y)p(2^{k-1}y)\dots p(2^{l-1}y)} = \frac{1}{p(2^k y)p(2^{k+1}y)\dots p(2^{l-1}y)} \neq 0.$$

But, since $m(\omega) = 0$ and $\omega = 2^n y$, we conclude that $n > l$. This means:

$$\phi(\omega) = \phi(2^n y) = p(2^{n-1}y)p(2^{n-2}y)\dots p(2^l y)\phi(2^l y) = 0$$

as $p(2^l y) = 0$.

- This set is empty. Now $m(\omega)$ cannot be zero, as its value will be the product or quotient of non-zero values of $p(\omega)$.

So we have constructed a maximal $m \in \Phi_2(p)$. ∎

Lemma 2.6. *Given $\phi, m \in \Phi_2(p)$ with $\phi \preccurlyeq m$ we may find π so that $\phi = \pi m$ and $\pi(\omega) = \pi(2\omega)$.*

Proof. By the definition of $\phi \preccurlyeq m$, we can find ρ so that:

$$\phi(\omega) = \rho(\omega)m(\omega).$$

However this ρ does not have to fulfil $\rho(\omega) = \rho(2\omega)$. We define π by:

$$\pi(\omega) = \frac{\phi(\omega)}{m(\omega)} \text{ or } \frac{\phi(\omega/2)}{m(\omega/2)} \text{ or } \frac{\phi(\omega/4)}{m(\omega/4)} \text{ or } \frac{\phi(\omega/8)}{m(\omega/8)} \text{ or } \dots \text{ or } 0$$

depending on which one is the first to have $m(\omega/2^n) \neq 0$. If $m(\omega/2^n) = 0$ for all $n = 0, 1, 2, 3, \dots$, then we set $\pi(\omega) = 0$.

First we check if π is a valid substitute for ρ. If $m(\omega) \neq 0$, then $\phi(\omega) = \pi(\omega)m(\omega)$ by π's definition, and if $m(\omega) = 0$, then $\phi(\omega) = \rho(\omega)m(\omega) = 0$, so the value of $\pi(\omega)$ doesn't matter.

Now we have to check if $\pi(\omega) = \pi(2\omega)$.

First consider the case $m(2\omega) \neq 0$ then, as $m(2\omega) = p(\omega)m(\omega)$, neither $p(\omega)$ or $m(\omega)$ can be zero so:

$$\pi(2\omega) = \frac{\phi(2\omega)}{m(2\omega)} = \frac{\phi(\omega)p(\omega)}{m(\omega)p(\omega)} = \frac{\phi(\omega)}{m(\omega)} = \pi(\omega),$$

as required.

On the other hand, if $m(2\omega) = 0$, then:

- either $m(2\omega/2^n) = 0$ for all $n = 1, 2, 3, \ldots$, which means $\pi(2\omega) = 0$ and $\pi(\omega) = 0$, as required,
- or for some $n > 0$ we can write $\pi(2\omega) = \frac{\phi(\omega/2^n)}{m(\omega/2^n)}$, and $\pi(\omega)$ will be the same, also as required.

■

Theorem 2.7. *Given a dilation equation:*

$$f(x) = \sum c_k f(2x - k),$$

we may find a function $m(\omega)$ such that for any solution $g(x)$ of the dilation equation whose Fourier transform converges almost everywhere, we can write:

$$\hat{g}(\omega) = \pi(\omega) m(\omega) \quad a.e. \quad \omega \in \mathbb{R},$$

for some function π with the property $\pi(\omega) = \pi(2\omega)$.

Proof. Under the Fourier transform the dilation equation becomes:

$$\hat{f}(\omega) = p\left(\frac{\omega}{2}\right) \hat{f}\left(\frac{\omega}{2}\right),$$

where $p(\omega) = \frac{1}{2} \sum c_k e^{-ik\omega}$ depends only on the dilation equation. By Lemma 2.5 we can find $m \in \Phi_2(p)$ so that $\phi \preccurlyeq m$ for all $\phi \in \Phi_2(p)$.

However $\hat{g}$ may not be in $\Phi_2(p)$ as it may diverge at some points and fail to satisfy the transformed dilation equation on a set of measure zero. We work around this by changing $\hat{g}$ on a set of measure zero.

First we alter $\hat{g}$ so that it is zero where it diverges. We may need to further redefine $\hat{g}$ on a second set of measure zero so that it satisfies:

$$\hat{g}(\omega) = p\left(\frac{\omega}{2}\right) \hat{g}\left(\frac{\omega}{2}\right)$$

everywhere. This can be achieved by setting $\hat{g}(2^n \omega) = 0, \forall n \in \mathbb{Z}$, for any ω which fails to satisfy the transformed dilation equation. This procedure changes $\hat{g}$ on at most a countable union of sets of measure zero, and so $\hat{g}$ is essentially unchanged.

Now $\hat{g}$ is a member of $\Phi_2(p)$. Thus $\hat{g} \preccurlyeq m$, so by Lemma 2.6 we can find π so that:

$$\hat{g} = m\pi$$

and $\pi(\omega) = \pi(2\omega)$. ■

It would be nice to have a way to check if a given function in $\Phi_2(p)$ is maximal. The following result provides a simple sufficient condition, which should be applicable if p is not too complicated.

Theorem 2.8. *Suppose we have $\phi \in \Phi_2(p)$ with the following properties:*

- *if $p(0) = 1$, then $\phi(0)$ is non-zero,*
- *there exists some $\epsilon > 0$ so that ϕ is non-zero on $\pm(0, \epsilon)$.*

Then ϕ is maximal in $\Phi_2(p)$.

Proof. First note that the second condition imposes a condition on p, so it is not always possible to find a ϕ with these properties*.

Suppose $\psi \in \Phi_2(p)$. The first condition tells us exactly that $\phi(0) = 0 \Rightarrow \psi(0) = 0$. If $\phi(\omega) = 0$ for some $\omega \neq 0$, then we can choose l so that $\phi(\omega)$ is of the form:

$$\phi(\omega) = p\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{4}\right) p\left(\frac{\omega}{8}\right) \dots p\left(\frac{\omega}{2^l}\right) \phi\left(\frac{\omega}{2^l}\right)$$

and $\frac{\omega}{2^l} \in \pm(0, \epsilon)$. As, $\phi\left(\frac{\omega}{2^l}\right) \neq 0$ we know $p\left(\frac{\omega}{2^n}\right) = 0$ for some n between 1 and l. But any $\psi(\omega)$ is also of the form:

$$\psi(\omega) = p\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{4}\right) p\left(\frac{\omega}{8}\right) \dots p\left(\frac{\omega}{2^l}\right) \psi\left(\frac{\omega}{2^l}\right),$$

and so will have the same 0 factor in it, and accordingly will be zero. ■

This result and variations of it are easily applied to any analytic ϕ we find, as we know a lot about the zeros of ϕ. For instance, if p is continuous and ϕ is analytic, then either ϕ is maximal or identically zero. Daubechies and Lagarias show in [11] that if p arises from the dilation equation:

$$f(\omega) = \sum_{k=0}^{N} c_k f(2\omega - \beta_k)$$

*For example, consider $p(\omega) = \sin \log \omega$, which takes the value 0 at infinitely many points in $(-\epsilon, \epsilon)$.

and if $p(0) = 1$ we can define an analytic function $\hat{f}_0$ by:

$$\hat{f}_0(\omega) = \prod_{j=1}^{\infty} p(2^{-j}\omega).$$

This function will then have $\hat{f}(0) = 1$, will be non-zero around 0 as it is continuous, will be in $\Phi_2(p)$, and so by the previous theorem will be maximal. If $|p(0)| > 1$, then the authors use a maximal solution of the form:

$$\hat{f}_0(\omega) = |\omega|^{\log_2 p(0)} \prod_{j=1}^{\infty} \frac{p(2^{-j}\omega)}{p(0)}.$$

Both factors in this expression must be non-zero around the origin, so it is also maximal.

2.3 Solutions to $f(x) = f(2x) + f(2x-1)$ and Fourier-like transforms

We can apply the results of the previous section in a quite straightforward manner to classify all $L^2(\mathbb{R})$ solutions of the dilation equation:

$$f(x) = f(2x) + f(2x-1).$$

In this case it is well known that $\chi_{[0,1)}$ is the only $L^1(\mathbb{R})$ solution (up to scale). Its Fourier transform:

$$\hat{\chi}_{[0,1)} = \frac{1 - e^{-i\omega}}{i\omega},$$

satisfies the conditions of Theorem 2.8 and so this solution is maximal, in the sense of Theorem 2.7.

Theorem 2.9. *The $L^2(\mathbb{R})$ solutions of:*

$$f(x) = f(2x) + f(2x-1)$$

are in a natural one-to-one correspondence with the functions in $L^2(\pm[1,2))$.

Proof. We simply classify the solutions of the Fourier transform of the dilation equation, and use the fact that the Fourier transform is bijective. As $\hat{\chi}_{[0,1)}$ is maximal we know that any solution of the transformed equation is of the form:

$$\hat{g} = \pi\hat{\chi}_{[0,1)}$$

with $\pi(\omega) = \pi(2\omega)$. We show that $\pi \in L^2(\pm[1,2))$ iff $\hat{g}$ is in $L^2(\mathbb{R})$.

If $\hat{g} \in L^2(\mathbb{R})$, noting that $|\hat{\chi}_{[0,1)}| > 0.1$ on $[1,2]$ allows us to write:

$$\pi(\omega) = \frac{\hat{g}(\omega)}{\hat{\chi}_{[0,1)}(\omega)} \qquad \omega \in [1,2].$$

This means that $\pi|_{[1,2)}$ is measurable as the ratio of well-behaved functions. So:

$$\infty > \int_1^2 |\hat{g}(\omega)|^2 \, d\omega = \int_1^2 |\pi(\omega)\hat{\chi}_{[0,1)}(\omega)|^2 \, d\omega > (0.1)^2 \int_1^2 |\pi(\omega)|^2 \, d\omega,$$

So $\pi \in L^2([1,2))$. The same argument works to show $\pi \in L^2(-[1,2))$.

Conversely, if π is in $L^2(\pm[1,2))$, then we may use the fact that $\hat{\chi}_{[0,1)}$ is bounded near zero and $\hat{\chi}_{[0,1)}$ decays like $2/\omega$ away from zero. Again we do $\mathbb{R}^+$ first.

$$\begin{aligned}
\int_0^\infty |\hat{g}(\omega)|^2 \, d\omega &= \int_0^\infty |\pi(\omega)\hat{\chi}_{[0,1)}(\omega)|^2 \, d\omega \\
&= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} |\pi(\omega)\hat{\chi}_{[0,1)}(\omega)|^2 \, d\omega \\
&\leq \sum_{n\in\mathbb{Z}} \sup_{[2^n,2^{n+1}]} |\hat{\chi}_{[0,1)}(\omega)|^2 \int_{2^n}^{2^{n+1}} |\pi(\omega)|^2 \, d\omega \\
&\leq \sum_{n\leq 0} \int_{2^n}^{2^{n+1}} |\pi(\omega)|^2 \, d\omega + \sum_{n>0} \frac{4}{2^{2n}} \int_{2^n}^{2^{n+1}} |\pi(\omega)|^2 \, d\omega \\
&= \sum_{n\leq 0} 2^n \int_1^2 |\pi(\omega)|^2 \, d\omega + \sum_{n>0} \frac{4}{2^{2n}} 2^n \int_1^2 |\pi(\omega)|^2 \, d\omega \\
&= 6 \left\| \pi|_{[1,2)} \right\|_2^2 .
\end{aligned}$$

Applying the same argument to $\mathbb{R}^-$ we see that $\|\hat{g}\|_2 \leq \sqrt{12} \left\| \pi|_{\pm[1,2)} \right\|_2$. ■

We can now examine what happens if our function π is in $L^2(\mathbb{R})$ but not in $L^\infty(\mathbb{R})$. In fact this gives us some information about what happens if either the solution of the dilation equation $\hat{g}$, or our multiplier π, does not belong to $L^\infty(\mathbb{R})$.

Lemma 2.10. *If $f \in L^2(\mathbb{R})$ and $f \notin L^\infty(\mathbb{R})$, then we can find $h \in L^2(\mathbb{R})$ so that $fh \notin L^2(\mathbb{R})$.*

Proof. We may define B_n for $n \in \mathbb{N}$ by:

$$B_n = \{x \in \mathbb{R} : n \leq |f(x)| < n+1\}.$$

As $f \notin L^\infty(\mathbb{R})$ an infinite number of these sets have non-zero measure. As $f \in L^2(\mathbb{R})$ none have infinite measure. Let[†] $\epsilon_n = |B_n|$. Now define h by:

$$h(x) = \begin{cases} \frac{1}{n\sqrt{\epsilon_n}} & x \in B_n \text{ and } \epsilon_n > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\|h\|_2^2 = \sum_{n,\epsilon_n>0} \left(\frac{1}{n\sqrt{\epsilon_n}}\right)^2 \epsilon_n < \sum_n \frac{1}{n^2} < \infty;$$

however,

$$\|fh\|_2^2 \geq \sum_{n,\epsilon_n>0} \left(\frac{n}{n\sqrt{\epsilon_n}}\right)^2 \epsilon_n = \sum_{n,\epsilon_n>0} 1 = \infty.$$

■

Consider next what happens if we have a solution of a dilation equation which is not essentially bounded.

Lemma 2.11. *Suppose $g \in L^2(\mathbb{R})$ is the solution of a dilation equation where $\hat{g}$ is essentially unbounded. We may find $\pi \in L^2(\pm[1,2))$ which when extended by $\pi(\omega) = \pi(2\omega)$ leads to a function $\pi\hat{g} \notin L^2(\mathbb{R})$.*

Proof. Taking $f = \hat{g}$ we define the B_n as in Lemma 2.10. Passing to a subsequence if necessary, we can assume that $|B_n| > 0$ for all n. Examine $B'_n = [B_n] \subset \pm[1,2)$, the set of representatives of points in B_n. Next form the sequence a_n using the rules:

$$a_1 = |B'_n|, \qquad a_{n+1} = \min\left(a_n/3, B'_{n+1}\right),$$

[†]We use $|X|$ for the measure of the set X.

and then choose $E'_n \subset B'_n$ so that $|E'_n| = a_n$. Now:

$$\left| \bigcup_{r=n+1}^{\infty} E'_r \right| \leq \sum_{r=n+1}^{\infty} a_r \leq \sum_{r=1}^{\infty} \frac{a_n}{3^r} < a_n,$$

so $D'_n = E'_n \setminus \bigcup_{r=1}^{\infty} E'_{n+1}$ has non-zero measure for all n. Note that the D'_n are all disjoint.

If we take D_n to be the set of points in B_n whose representatives are in D'_n, then the D_n have non-zero measure and are a disjoint collection of sets with disjoint representatives. We use the construction of h on $\hat{g}\chi_{\bigcup D_n}$ in Lemma 2.10 to give the values of π on D_n. Extending π in the usual way we see:

$$\|\pi\hat{g}\|_2 \geq \left\|\pi\hat{g}\chi_{\bigcup D_n}\right\|_2 = \infty.$$

■

By considering unbounded multipliers we can produce a slightly different result.

Theorem 2.12. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which commutes with integer translations and dilations by 2^n. Then $\mathcal{A}$ is of the form:*

$$\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$$

where $\pi(\omega) = \pi(2\omega)$ and $\pi \in L^\infty(\mathbb{R})$. Conversely any such π gives rise to a bounded linear $\mathcal{A}$ which commutes with all translations and dilations by 2^n.

Proof. Consider the image of $g = \chi_{[0,1)}$. We know that g is a solution of:

$$f(x) = f(2x) + f(2x - 1),$$

and as $\mathcal{A}$ commutes with dilation by 2 and translation by 1 we know $\mathcal{A}g$ must also be an $L^2(\mathbb{R})$ solution of this dilation equation. Theorem 2.9 tells us that $\mathcal{F}\mathcal{A}g = \pi\mathcal{F}g$, where $\pi(\omega) = \pi(2\omega)$ and $\pi \in L^2(\pm[1, 2))$.

We note that $\chi_{[\frac{n}{2^m}, \frac{n+1}{2^m})}$ can be obtained via integer translations and dilations of scale

2^n applied to g. This allows us to make the following calculation:

$$\begin{aligned}\mathcal{A}\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})} &= \mathcal{A}\mathcal{D}_{2^m}\mathcal{T}_{-n}g \\ &= \mathcal{D}_{2^m}\mathcal{T}_{-n}\mathcal{A}g \\ &= \mathcal{D}_{2^m}\mathcal{T}_{-n}\mathcal{F}^{-1}\pi\mathcal{F}g \\ &= \mathcal{F}^{-1}2^{-m}\mathcal{D}_{2^{-m}}e^{in\cdot}\pi\mathcal{F}g,\end{aligned}$$

and remembering that $\pi(\omega) = \pi(2\omega)$ and so will commute with dilation by a power of 2:

$$\begin{aligned}\mathcal{A}\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})} &= \mathcal{F}^{-1}\pi 2^{-m}\mathcal{D}_{2^{-m}}e^{in\cdot}\mathcal{F}g \\ &= \mathcal{F}^{-1}\pi\mathcal{F}\mathcal{D}_{2^m}\mathcal{T}_{-n}g \\ &= \mathcal{F}^{-1}\pi\mathcal{F}\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})}.\end{aligned}$$

Thus, as both $\mathcal{A}$ and $\mathcal{F}^{-1}\pi\mathcal{F}$ are linear, we can see that $\mathcal{A}f = \mathcal{F}^{-1}\pi\mathcal{F}f$ for any f in the Haar Multiresolution Analysis on $L^2(\mathbb{R})$. But this is a dense subset and $\mathcal{A}$ is continuous, so if $\mathcal{F}^{-1}\pi\mathcal{F}$ is continuous they will agree everywhere. It is clear that if $\pi \in L^\infty(\mathbb{R})$, then $\mathcal{F}^{-1}\pi\mathcal{F}$ will be continuous.

It remains to show that $\pi \in L^\infty(\mathbb{R})$. If it were not we could consider $\pi\chi_{\pm[1,2)}$ as an essentially unbounded member of $L^2(\mathbb{R})$ and use Lemma 2.10 to produce $h \in L^2(\mathbb{R})$ so that $h\pi\chi_{\pm[1,2)} \notin L^2(\mathbb{R})$. Then $\check{h} \in L^2(\mathbb{R})$, but $\mathcal{A}\check{h}$ would not be, which is a contradiction.

The converse is a simple matter of algebra and using $\pi(\omega) = \pi(2\omega)$. ■

We can prove a corollary of this which has been proved in different ways in several different contexts (references follow proof).

Corollary 2.13. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which commutes with integer translations and dilations by 2^n. Suppose also that $\mathcal{A}$ preserves inner products. Then $\mathcal{A}$ is of the form:*

$$\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$$

where $\pi(\omega) = \pi(2\omega)$ and $|\pi(\omega)| = 1$ almost everywhere.

Proof. Suppose $|\pi(\omega)| \neq 1$ on some set of positive measure, then at least one of the sets:

$$M_+ = \{\omega \in \pm[1,2) : |\pi(\omega)| > 1\}, M_- = \{\omega \in \pm[1,2) : |\pi(\omega)| < 1\}$$

must have non-zero measure. Suppose that M_+ has positive measure, then we may choose

$\epsilon > 0$ so that:

$$M = \{\omega \in \pm[1,2) : |\pi(\omega)| > 1 + \epsilon\}$$

has positive measure. Consider the function $\mathcal{F}^{-1}\chi_M$. Recall that $\mathcal{F}^{-1}$ just scales the inner product by a constant, so that:

$$(\mathcal{F}^{-1}\chi_M, \mathcal{F}^{-1}\chi_M) = c\ (\chi_M, \chi_M) = c|M|.$$

However, applying $\mathcal{A}$ and taking the inner product:

$$\begin{aligned}
(\mathcal{A}\mathcal{F}^{-1}\chi_M, \mathcal{A}\mathcal{F}^{-1}\chi_M) &= (\mathcal{F}^{-1}\pi\mathcal{F}\mathcal{F}^{-1}\chi_M, \mathcal{F}^{-1}\pi\mathcal{F}\mathcal{F}^{-1}\chi_M) \\
&= c\ (\pi\chi_M, \pi\chi_M) \\
&= c\int_M |\pi(\omega)|^2\, d\omega \\
&\geq c(1+\epsilon)^2|M|,
\end{aligned}$$

which contradicts the hypothesis that $\mathcal{A}$ preserves inner products. We may arrive at a similar contradiction if only M_- has positive measure. ■

This has been proved in different ways by others [5, 36]. A nice way to summarise these operator results and the results in [32] follows. We use $S^\times$ to denote the multiplicative group generated by S.

Theorem 2.14. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which commutes with translation by integers, then:*

1. $\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ *and* $\mathcal{A}(\chi_{[0,1)}) = \chi_{[0,1)}$ *implies* $\mathcal{A} = \mathcal{I}$,
2. $\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ *and* $\mathcal{A}(\chi_{[0,1)}) \in L^1(\mathbb{R})$ *implies* $\mathcal{A} = c\mathcal{I}$,
3. $\mathcal{A}\mathcal{D}_n = \mathcal{D}_n\mathcal{A}$ *for* $n \in S \subset \mathbb{Z} \setminus \{0\}$ *and* $S^\times$ *is dense in* $\mathbb{R}$ *implies* $\mathcal{A} = c\mathcal{I}$,
4. $\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ *implies* $\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$ *where* $\pi \in L^\infty(\mathbb{R})$ *and* $\pi = \mathcal{D}_2\pi$,
5. $\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ *and* $\mathcal{A}$ *is unitary implies* $\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$ *where* $|\pi(\omega)| = 1$ *and* $\pi = \mathcal{D}_2\pi$.

2.4 Working on $\mathbb{R}^n$

Will any of this work on $\mathbb{R}^n$? The first thing to do is to look at how the Fourier transform works in $\mathbb{R}^n$. It is defined by:

$$(\mathcal{F}f)(\vec{\omega}) = \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{x})} f(\vec{x})\, d\vec{x},$$

where $(\cdot,\cdot)$ is the usual inner product on $\mathbb{R}^n$. We want to look at how translation and dilation affect this Fourier transform. First we define $(\mathcal{T}_{\vec{r}}f)(\vec{x}) = f(\vec{x}+\vec{r})$ and $\mathcal{D}_A f(\vec{x}) = f(A\vec{x})$ for $\vec{r} \in \mathbb{R}^n$ and A an n by n invertible matrix. Then for translation:

$$\begin{aligned}
(\mathcal{F}\mathcal{T}_{\vec{r}}f)(\vec{\omega}) &= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{x})} (\mathcal{T}_{\vec{r}}f)(\vec{x})\, d\vec{x} \\
&= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{x})} f(\vec{x}+\vec{r})\, d\vec{x} \\
&= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{y}-\vec{r})} f(\vec{y})\, d\vec{y} \\
&= e^{i(\vec{\omega},\vec{r})} \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{y})} f(\vec{y})\, d\vec{y} \\
&= e^{i(\vec{\omega},\vec{r})} (\mathcal{F}f)(\vec{\omega}),
\end{aligned}$$

using the change of variable $\vec{y} = \vec{x}+\vec{r}$. Similarly for dilation:

$$\begin{aligned}
(\mathcal{F}\mathcal{D}_A f)(\vec{\omega}) &= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{x})} (\mathcal{D}_A f)(\vec{x})\, d\vec{x} \\
&= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},\vec{x})} f(A\vec{x})\, d\vec{x} \\
&= \int_{\mathbb{R}^n} e^{-i(\vec{\omega},A^{-1}\vec{z})} f(\vec{z}) \frac{d\vec{z}}{|\det A|} \\
&= \int_{\mathbb{R}^n} e^{-i(A^{-*}\vec{\omega},\vec{z})} f(\vec{z}) \frac{d\vec{z}}{|\det A|} \\
&= \frac{(\mathcal{F}f)(A^{-*}\vec{\omega})}{|\det A|}
\end{aligned}$$

by using the change of variable $\vec{z} = A\vec{x}$. (Here A^* is used to denote the adjoint of A, and A^{-*} is used to denote the adjoint inverse).

Now we can look at what happens to the analogue of dilation equations. We can look at equations like:

$$f(\vec{x}) = \sum_{\vec{k}} c_{\vec{k}} f(A\vec{x} - \vec{k}).$$

Applying the Fourier transform to both sides and using what we have just derived:

$$\hat{f}(\vec{\omega}) = \sum_{\vec{k}} c_{\vec{k}} e^{-i(A^{-*}\vec{\omega},\vec{k})} \frac{\hat{f}(A^{-*}\vec{\omega})}{|\det A|},$$

or gathering the messy bits into a single trigonometric polynomial p:

$$\hat{f}(\vec{\omega}) = p(A^{-*}\vec{\omega})\hat{f}(A^{-*}\vec{\omega}).$$

So far this closely parallels the one-dimensional case.

2.4.1 What is dilation now?

Now we must decide what sort of matrices to allow for A. In the one-dimensional case we would have considered $\mathcal{D}_{3.4}$ to be a dilation but $\mathcal{D}_{0.5}$ to be an expansion. So we consider a number λ a suitable scale if $|\lambda| > 1$. In $\mathbb{R}^n$ the usual condition which is used for $\mathcal{D}_A$ to be a dilation is that all the eigenvalues of A have norm bigger than 1.

A slightly more intuitive way of phrasing this might be to consider $B = A^{-*}$. The eigenvalues of B will then satisfy $0 < |\lambda| < 1$, so we can consider B in the following way.

Lemma 2.15. *Let B be a matrix. Then the following are equivalent:*

1. *B is a matrix so that all its eigenvalues $\lambda \in \mathbb{C}$ satisfy $0 < |\lambda| < 1$,*
2. *B is invertible and we can find $\alpha < 1$ and C so that $\|B^m\| < C\alpha^m$,*
3. *B is invertible and $\|B^m\vec{x}\| \to 0$ as $m \to \infty$ for any $\vec{x} \in \mathbb{R}^n$.*

Proof. To show that 1 implies 2 we write B in Jordan form:

$$B = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & \\ 0 & & \ddots & \\ 0 & & & J_K \end{pmatrix}.$$

Then we have:

$$B^m = \begin{pmatrix} J_1^m & 0 & \dots & 0 \\ 0 & J_2^m & & \\ 0 & & \ddots & \\ 0 & & & J_K^m \end{pmatrix},$$

so we show that $\|J^m\| \leq C_1\alpha^m$ for any of the blocks J. We can write $J = \lambda I + N$ where λ is an eigenvalue and $N^r = 0$ for some $r < n$. For $m > n > r$:

$$\begin{aligned}
J^m &= (\lambda I + N)^m \\
J^m &= \lambda^m I + m\lambda^{m-1}N + \tfrac{m(m-1)}{2}\lambda^{m-2}N^2 + \ldots + \tfrac{m(m-1)\ldots(m-r+1)}{r!}\lambda^{m-r+1}N^{r-1} \\
\|J^m\| &\leq |\lambda|^{m-r+1}\left(\|I\| + m\|N\| + m^2\|N^2\| + \ldots + m^{r-1}\|N^{r-1}\|\right) \\
\|J^m\| &\leq |\lambda|^{m-r+1}m^n\left(\|I\| + \|N\| + \|N^2\| + \ldots + \|N^{n-1}\|\right) \\
\|J^m\| &\leq |\lambda|^m m^n C_0 \\
\|J^m\| &\leq \alpha^m C_1,
\end{aligned}$$

where we choose $\alpha < 1$ and $\alpha > |\lambda|$ for any of B's eigenvalues. So we see:

$$\|B\| = \left\|\sum J_k\right\| \leq \sum \|J_k\| \leq \sum \alpha^m C_1 = K\alpha^m C_1 \leq KC_1\alpha^m,$$

as required. We also note that as 0 is not an eigenvalue of B this means B must be invertible.

It is obvious that 2 implies 3, so we just have to show that 3 implies 1. To see this suppose B has an eigenvalue λ with norm bigger or equal to 1. Let $\vec{z} = \vec{x} + i\vec{y}$ be a complex eigenvector for this eigenvalue. Then:

$$\|B^m\vec{z}\| = \|\lambda^m\vec{z}\| = \|\lambda|^m\|\vec{z}\| \not\to 0.$$

However B is complex-linear so $B^m\vec{z} = B^m\vec{x} + iB^m\vec{y}$, thus $B^m\vec{x} \not\to 0$ or $B^m\vec{y} \not\to 0$. So all B's eigenvalues must have norm less than 1. We also know that 0 is not an eigenvalue of B as it is invertible. ■

It follows that a possibly more natural definition of matrices which we consider as dilations are matrices A for which $\|B^m\| \to 0$. We focus on these matrices for the moment. Note that $\|B^{-m}\vec{x}\| \to \infty$.

Corollary 2.16. *If $\vec{x} \in \mathbb{R}^n \setminus \{0\}$, then $\|B^{-m}\vec{x}\| \to \infty$ as $m \to \infty$.*

Proof. We know $\|B^m\vec{y}\| \leq C\alpha^m\|\vec{y}\|$ where $0 < \alpha < 1$. Thus, setting $\vec{y} = B^{-m}\vec{x}$ we get $\|B^mB^{-m}\vec{x}\| \leq C\alpha^m\|B^{-m}\vec{x}\|$. Rearranging we get:

$$\|B^{-m}\vec{x}\| \geq \frac{\alpha^{-m}}{C}\|\vec{x}\|,$$

as required. ■

We are going to be interested in the relation:

$$\vec{\omega}_1 \sim \vec{\omega}_2 \qquad \text{iff} \qquad \vec{\omega}_1 = B^m \vec{\omega}_2 \text{ for any } m \in \mathbb{Z}.$$

It is easy to check that this is an equivalence relation. In the one-dimensional case we could easily understand this relation as it amounts to $\omega_1 \sim \omega_2$ if $\omega_1 = 2^m \omega_2$ and we select a set of representatives $\pm[1, 2)$ — that is, for each point in $\mathbb{R} \setminus \{0\}$ there is exactly one point in $\pm[1, 2)$ which is equivalent to that point. We are going to have to choose a similar set of representatives in this more general setting. We could do this with the Axiom of Choice, but it would be nice to see what shape these sets are.

The structure here is quite similar to that in $\mathbb{R}$. In $\mathbb{R}$ the set of points equivalent to ω was $2^{\mathbb{Z}}\omega$, and here the set of points equivalent to $\vec{\omega}$ is $B^{\mathbb{Z}}\vec{\omega}$, the orbit of $\vec{\omega}$ as B acts on $\mathbb{R}^n$. As in $\mathbb{R}$ all the $B^m\vec{\omega}$ are distinct.

Lemma 2.17. *If B has all its eigenvalues satisfying $0 < |\lambda| < 1$, then $B^m\vec{\omega} = B^l\vec{\omega}$ for some non-zero $\vec{\omega}$ only when $m = l$.*

Proof. If we have $B^m\vec{\omega} = B^l\vec{\omega}$, then $B^{m-l}\vec{\omega} = \vec{\omega}$, so 1 is an eigenvalue of B^{m-l}. But the eigenvalues of B^{m-l} are λ^{m-l} where λ is an eigenvalue of B. But this means that $\lambda^{m-l} = 1$ and if $m \neq l$ then we have $|\lambda| = 1$, which is a contradiction. ■

We are going to choose our representatives in the following way. Note that for $\vec{\omega} \in \mathbb{R}^n \setminus \{0\}$ we have $\|B^m\vec{\omega}\| \to 0$ as $m \to \infty$ and $\|B^m\vec{\omega}\| \to \infty$ as $m \to -\infty$. This means we can choose $M \in \mathbb{Z}$ so that $\|B^M\vec{\omega}\| \geq 1$, but for any $m > M$ we have $\|B^m\vec{\omega}\| < 1$. It is the point $B^M\vec{\omega}$ that we take as our representative for the family $B^{\mathbb{Z}}\vec{\omega}$.

Theorem 2.18. *Let $D = \{\vec{x} \in \mathbb{R}^n \ : \ \|\vec{x}\| < 1\}$. Then the set:*

$$R = \left(\bigcap_{m>0} B^{-m} D \right) \setminus D,$$

contains exactly one representative of each coset of $(\mathbb{R}^n \setminus \{0\})/\sim$.

Proof. For each $\vec{\omega} \in \mathbb{R}^n \setminus \{0\}$ we can find a unique $M(\vec{\omega})$ so that $B^{M(\vec{\omega})}\vec{\omega} \notin D$ but $B^m\vec{\omega} \in D$ whenever $m > M(\vec{\omega})$. $M(\vec{\omega})$ is given by:

$$M(\vec{\omega}) = \inf \left\{ m \in \mathbb{Z} \ : B^k\vec{\omega} \in D, \forall k > m \right\}.$$

Note that $M(B^k\vec{\omega}) = M(\vec{\omega}) - k$ as $B^{M(\vec{\omega})-k}(B^k\vec{\omega}) = B^{M(\vec{\omega})}\vec{\omega} \notin D$ and if $m > M(\vec{\omega}) - k$, then $m + k > M(\vec{\omega})$ so $B^m(B^k\vec{\omega}) = B^{m+k}\vec{\omega} \in D$.

We consider the set:

$$R = \{\vec{\omega} \in \mathbb{R}^n \setminus \{0\} : M(\vec{\omega}) = 0\}\,.$$

For any point $\vec{\omega} \in \mathbb{R}^n \setminus \{0\}$ the equivalent point $B^{M(\vec{\omega})}\vec{\omega}$ is in R as $M(B^{M(\vec{\omega})}\vec{\omega}) = M(\vec{\omega}) - M(\vec{\omega}) = 0$. This means that R must contain at least one representative of each point in $\mathbb{R}^n \setminus \{0\}$.

Now suppose that $\vec{\omega}_1 \sim \vec{\omega}_2$ and $M(\vec{\omega}_1) = M(\vec{\omega}_2) = 0$. Then $\vec{\omega}_1 = B^k\vec{\omega}_2$ for some $k \in \mathbb{Z}$. Thus:

$$0 = M(\vec{\omega}_1) = M(B^k\vec{\omega}_2) = M(\vec{\omega}_2) - k = 0 - k,$$

so $k = 0$ and $\vec{\omega}_1 = \vec{\omega}_2$. We conclude that R contains exactly one representative of each point.

Finally we simplify the form of R:

$$\begin{aligned}
R &= \{\vec{\omega} \in \mathbb{R}^n \setminus \{0\} : M(\vec{\omega}) = 0\}\,, \\
&= \{\vec{\omega} \in \mathbb{R}^n \setminus \{0\} : \vec{\omega} \notin D, B^m\vec{\omega} \in D, \forall m > 0\}\,, \\
&= \{\vec{\omega} \in \mathbb{R}^n \setminus \{0\} : B^m\vec{\omega} \in D, \forall m > 0\} \setminus D, \\
&= \{\vec{\omega} : B^m\vec{\omega} \in D, \forall m > 0\} \setminus D, \\
&= \left(\bigcap_{m>0} \{\vec{\omega} : B^m\vec{\omega} \in D\}\right) \setminus D, \\
&= \left(\bigcap_{m>0} \left\{B^{-m}\vec{\omega} : \vec{\omega} \in D\right\}\right) \setminus D, \\
&= \left(\bigcap_{m>0} B^{-m}D\right) \setminus D.
\end{aligned}$$

■

We can now make some observations about the form of R. Firstly, it was not important that we started with the unit disk D. Any bounded set which contained a neighbourhood of the origin would have been suitable.

Secondly, the intersection in the expression for R is actually a finite intersection for any given B. This is because $B^{-1}D$ is bounded, and so $B^{-1}D \subset rD$ for some $0 < r < \infty$. Then choose N so that $\|B^n\| < \frac{1}{r}$ when $n > N$. Thus $B^n rD \subset D$, so $rD \subset B^{-n}D$. We conclude that $B^{-1}D \subset rD \subset B^{-n}D$, and so intersecting with the terms with $m > N$ has no effect.

Third, as the intersection is finite, this means that the set R is a bounded open set intersected with a closed set, and so not just measurable but very well behaved. This fits in well with the case in $\mathbb{R}$ where we used $\pm[1, 2)$ as our set of representatives.

Dilation Matrix	R	$B^{-n}D$
$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda = 2$		
$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\lambda = 1 \pm i$		
$A = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $\lambda = \pm\sqrt{2}i$		
$A = \begin{pmatrix} 0 & .75 \\ 2 & 0 \end{pmatrix}$, $B = -\frac{2}{3}\begin{pmatrix} 0 & -2 \\ -.75 & 0 \end{pmatrix}$, $\lambda = \pm\sqrt{3/2}$		
$A = \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$, $B = \frac{1}{12}\begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}$, $\lambda = 3, 4$		
$A = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix}$, $\lambda = \pm\sqrt{2}i$		

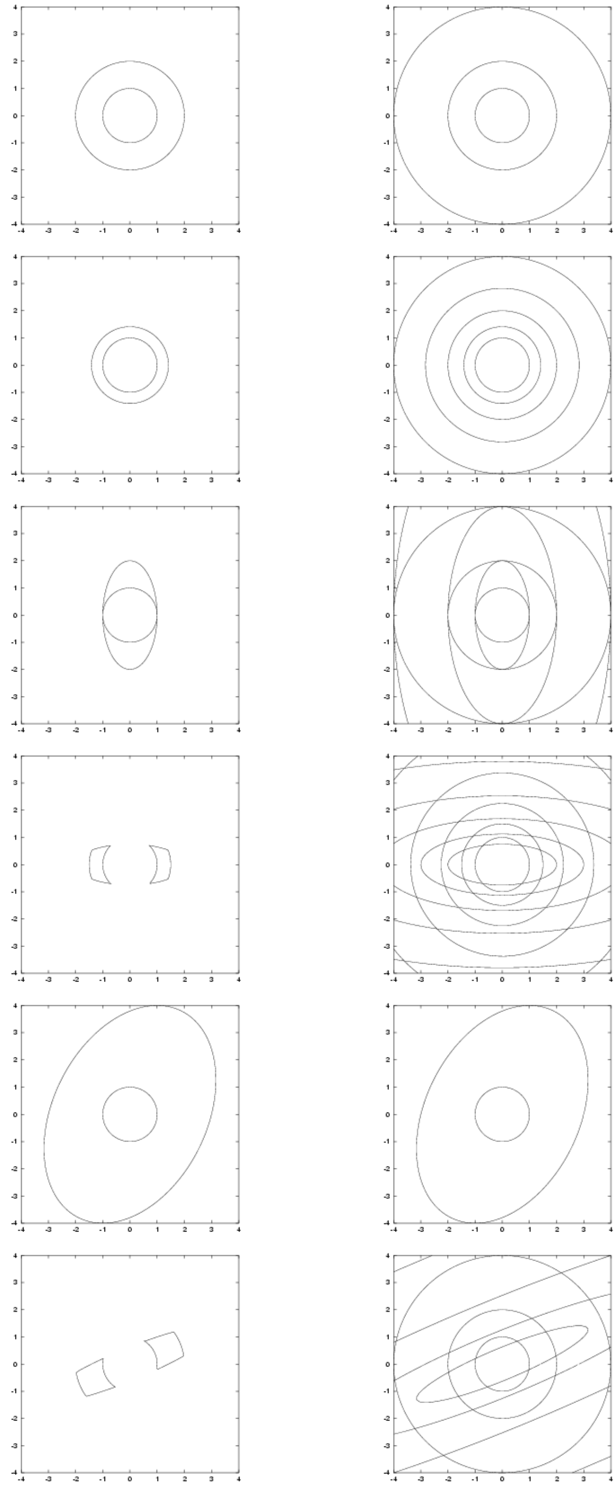


Figure 2.1: For various A the boundary of R and $B^{-n}D$ ($n = 0, 1, \ldots$).

2.4.2 Solutions to the transformed equation

Now we try to use similar definitions and proofs to those we used on $\mathbb{R}$. Again we look at the set of pointwise solutions of the transformed dilation equation with scale A:

$$\Phi_A(p) = \{\phi : \mathbb{R}^n \to \mathbb{C} : \phi(\vec{\omega}) = p(B\vec{\omega})\,\phi(B\vec{\omega})\},$$

where $B = A^{-*}$ for $\|B^k\| \to 0$ as $k \to \infty$. We would like to find $m \in \Phi_A(p)$ so that for any $\phi \in \Phi_A(p)$ we can find $\pi : \mathbb{R}^n \to \mathbb{C}$ so that $\phi = m\pi$ and $\pi(\vec{\omega}) = \pi(B\vec{\omega})$.

Fortunately the generalisations of Lemma 2.5, Lemma 2.6 and Theorem 2.7 are straightforward. Here is the "cut and paste" generalisation of Theorem 2.8.

Theorem 2.19. *Suppose we have $\phi \in \Phi_A(p)$ with the following properties:*

- *if $p(0) = 1$, then $\phi(0)$ is non-zero,*
- *there exists some $\epsilon > 0$ so that ϕ is non-zero on the punctured ball $\{\vec{\omega} \in \mathbb{R}^n : 0 < \|\vec{\omega}\| < \epsilon\}$.*

Then ϕ is maximal in $\Phi_A(p)$.

Proof. Suppose $\psi \in \Phi_A(p)$. The first condition tells us exactly that $\phi(0) = 0 \Rightarrow \psi(0) = 0$. If $\phi(\vec{\omega}) = 0$ for some $\vec{\omega} \neq 0$, then we can choose l so $\phi(\vec{\omega})$ is of the form:

$$\phi(\vec{\omega}) = p\left(B^1\vec{\omega}\right) p\left(B^2\vec{\omega}\right) p\left(B^3\vec{\omega}\right) \dots p\left(B^l\vec{\omega}\right) \phi\left(B^l\vec{\omega}\right),$$

and $B^l\vec{\omega} \in \{\vec{\omega} \in \mathbb{R}^n : 0 < \|\vec{\omega}\| < \epsilon\}$. Now as $\phi\left(B^l\vec{\omega}\right) \neq 0$ we know $p\left(B^k\vec{\omega}\right) = 0$ for some k between 1 and l. But any $\psi(\vec{\omega})$ is also of the form:

$$\psi(\vec{\omega}) = p\left(B^1\vec{\omega}\right) p\left(B^2\vec{\omega}\right) p\left(B^3\vec{\omega}\right) \dots p\left(B^l\vec{\omega}\right) \psi\left(B^l\vec{\omega}\right),$$

and so will have the same 0 factor in it, and accordingly will be zero. ∎

2.5 Applications in $\mathbb{R}^n$

It would be nice to be able to generalise the operator result of Theorem 2.12 from operators on $L^2(\mathbb{R})$ which commute with $\mathcal{D}_2$, to operators on $L^2(\mathbb{R}^n)$ which commute with $\mathcal{D}_A$ for a dilation matrix A. We proved this result for dilation by 2 in $\mathbb{R}$ by focussing on the solutions of the lattice dilation equation:

$$f(x) = f(2x) + f(2x - 1).$$

In $\mathbb{R}$, a lattice dilation equation is one in which the scale a is an integer. This means that it maps any lattice in $\mathbb{R}$ into itself, or $a\mathbb{Z} \subset \mathbb{Z}$. In $\mathbb{R}^n$ a lattice dilation equation is one in which the scale A is a dilation and also $A\Gamma \subset \Gamma$ for some lattice Γ which isn't flat in $\mathbb{R}^n$. Using a change of basis we can arrange that this lattice is $\mathbb{Z}^n$, and so A must have integer entries [15].

Generalising to other lattice dilations in $\mathbb{R}$ is easy, as for scale a we can just use the lattice dilation equation:

$$f(x) = f(ax) + f(ax - 1) + \ldots + f(ax - a + 1),$$

when $a \geq 2$. If $a \leq -2$ we use:

$$f(x) = f(ax + 1) + f(ax + 2) + \ldots + f(ax - a).$$

All these equations[‡] have $\chi_{[0,1)}$ as a well-behaved $L^2(\mathbb{R})$ solution. Armed with these equations and the set of representatives $\pm[1, |a|)$, we can proceed through the proof of Theorem 2.9 with few changes.

Theorem 2.20. *The $L^2(\mathbb{R})$ solutions of the above dilation equations are in a natural one-to-one correspondence with the functions in $L^2(\pm[1, |a|))$.*

Unfortunately the situation is not so easy to deal with in $\mathbb{R}^n$. Let us consider for a moment the important properties which $\chi_{[0,1)}$ has. First, it is a maximal solution of a dilation equation. For Theorem 2.9 we use the fact that its Fourier transform stays away from zero on some nice set of representatives, is bounded, and decays reasonably quickly. We then use the fact that it is a generating function for a multiresolution analysis to prove Theorem 2.12. So, given a lattice dilation A on $\mathbb{R}^n$, we want to find a function g which has all these properties.

2.5.1 Lattice tilings of $\mathbb{R}^n$

In $\mathbb{R}$ our well-behaved generating function was the characteristic function for some set. A possible way to generalise this is to look for is for other suitable characteristic functions. One well studied way (see [15]) of doing this is to look for a compact set G with the following properties (up to measure zero):

1. G has distinct translations, ie. $G \cap (G + \vec{r}) = \emptyset$ for $\vec{r} \in \mathbb{Z}^n \setminus \{0\}$.
2. AG the dilated version of G can be written as a union of its translations, ie. we can

[‡]The second equation is derived from the first using the fact that $\chi_{[0,1)}$ satisfies $f(x) = f(-x + 1)$.

find points $\vec{k}_1, \ldots, \vec{k}_q$ so that:

$$AG = \bigcup_{i=1}^{q} (G + \vec{k}_i).$$

3. G covers $\mathbb{R}^n$ by translation.

$$\mathbb{R}^n = \bigcup_{\vec{r} \in \mathbb{Z}^n} (G + \vec{r}).$$

The first of these conditions tells us that the translates of χ_G are orthogonal. The second tells us that χ_G satisfies a dilation equation and the last tells us that we can get to any part of $\mathbb{R}^n$. In fact the $\vec{k}_1, \ldots, \vec{k}_q$ turn out to be representatives of the equivalence classes of $A\mathbb{Z}^n/\mathbb{Z}^n$, of which there are $q = |\det A|$. Remarkably, a set with these properties will even generate a multiresolution analysis.

The existence of such sets is even a concrete affair. Any candidate for such a set can be shown to be of the form:

$$G = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{j=1}^{\infty} A^{-j} \epsilon_j, \epsilon_j \in \left\{ \vec{k}_1, \ldots, \vec{k}_q \right\} \right\}.$$

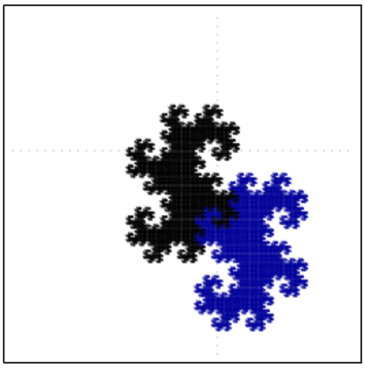
These summations can be thought of as the base A expansion of points in $\mathbb{R}^n$ using the digits $\vec{k}_1, \ldots, \vec{k}_q$ and for this reason the set $\{\vec{k}_1, \ldots, \vec{k}_q\}$ is referred to as the *digit set*. For example, if we take $A = 2$ and $k_1 = 0, k_2 = 1$, then we get:

$$G = \left\{ x \in \mathbb{R} : x = \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}, \epsilon_j = 0 \text{ or } 1 \right\},$$

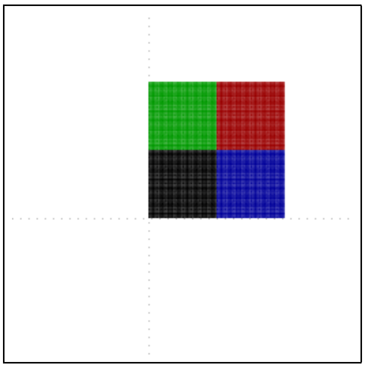
which is the binary expansion of numbers between 0 and 1, so we get $[0, 1)$ back again. These candidate sets have the desired properties iff their measure is 1. Figure 2.2 shows various sets G with their dilations A and digit sets. Note that the same A can produce radically different G if different digit sets are chosen.

The next question is: Given A, when can we select a digit set which will produce a G of measure 1 using the above recipe? In the literature the answer to this question looks rather complicated. To summarise a paragraph of [49], the answer is 'Always' in $\mathbb{R}^n$ for $n = 1, 2, 3$ and 'Always' if $|\det A| > n$; however, the answer is probably 'Sometimes' in general. By the time [28] was published a counterexample in $\mathbb{R}^4$ had been found by Potiopa:

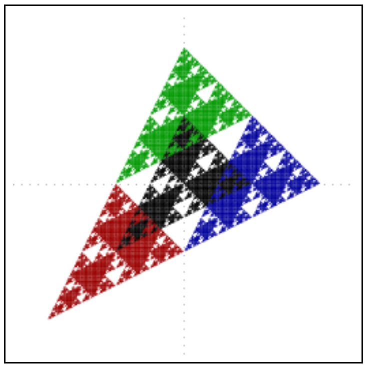
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$



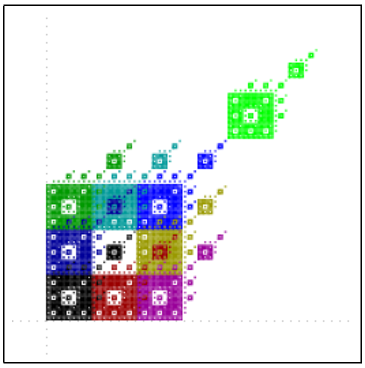
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



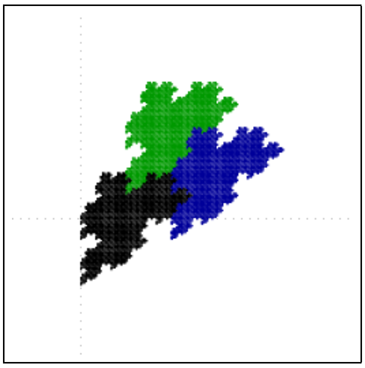
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



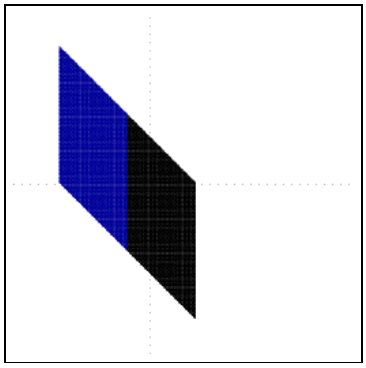
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$



$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$



$$A = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$



$$A = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$$
$$k_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Figure 2.2: Various self-similar-affine tiles.

The reason no suitable digit set can be found for this A relates to the algebra of rings and to fields generated by roots of its characteristic polynomial $x^4 + x^2 + 2$.

2.5.2 Wavelet sets and MSF wavelets

The idea of wavelet sets is dual to that of self-similar-affine tilings. This time, instead of producing characteristic functions which generate MRAs, the aim is to produce wavelets whose Fourier transforms are characteristic functions of sets. These wavelets are referred to as *minimally supported frequency* wavelets, or MSF wavelets. Shannon wavelets, with $\hat{\psi} = \chi_{\pm[\pi,2\pi)}$, are a well-known example of an MSF wavelet in $\mathbb{R}$.

Here existence is not a problem. Dia, Larson and Speegle prove that for any dilation matrix A there always exists a wavelet set in [6]. They do this by producing a set W which is 2π-translation equivalent to $[-\pi, \pi)^n$ and B-dilation equivalent to R. This means that $W = \bigcup E_{\vec{r}}$ where:

$$[-\pi, \pi)^n = \bigcup_{\vec{r}\in\mathbb{Z}^n} E_{\vec{r}} + 2\pi\vec{r},$$

and $W = \bigcup F_m$ where:

$$R = \bigcup_{m\in\mathbb{Z}} B^m F_m,$$

(R as given in Theorem 2.18).

The first of these relations tells us that W is much the same shape as $[-\pi, \pi)^n$ and thus we can produce $L^2(W)$ by using linear combinations of the form:

$$\sum_{\vec{r}\in\mathbb{Z}^n} c_{\vec{r}} e^{(\vec{r},\vec{\omega})} \chi_W.$$

The second relation tells us that W has the same property as R from Theorem 2.18, that is $\mathbb{R}^n = \bigcup B^m W$. This means we can produce $L^2(\mathbb{R}^n)$ as a direct sum of $L^2(B^m W)$:

$$\sum_{\vec{r}\in\mathbb{Z}^n, m\in\mathbb{Z}} c_{\vec{r},m} e^{(\vec{r},\vec{\omega})} \chi_{B^m W}.$$

Taking the inverse transform of this equation we find $L^2(\mathbb{R})$ is the span of:

$$\sum_{\vec{r}\in\mathbb{Z}^n, m\in\mathbb{Z}} c_{\vec{r},m} w(A^m \vec{x} - \vec{r}),$$

where w is a constant multiple of the inverse Fourier transform of χ_W. This makes f a wavelet for scale A.

The construction of W is presented in a quite abstract form in [6], but [7] contains many nice examples and more of a discussion.

This time, the complication is that we are looking for functions which generate MRAs, not wavelets. Examples of wavelets often arise from an MRA, but these MSF wavelets are primary candidates for counterexamples. Note that there is only a need for one MSF wavelet regardless of the value of $\det(A)$, whereas for wavelets arising from an MRA of scale A would require $|\det(A)| - 1$ wavelets.

2.5.3 A traditional proof

Having found no suitable maximal solutions to a dilation equation of scale A, we cannot directly follow the tack we took in $\mathbb{R}$. However, we can prove a generalisation of Theorem 2.12 using more traditional methods.

Lemma 2.21. *Suppose A is a dilation matrix and $\mathcal{A}$ is a bounded linear transform on $L^2(\mathbb{R})$ which commutes with $\mathcal{D}_A$ and $\mathcal{T}_{\vec{r}}$ (for all $\vec{r} \in \mathbb{Z}^n$); then $\mathcal{A}$ commutes with all translations.*

Proof. We note that:

$$\mathcal{T}_{A^m \vec{r}} = \mathcal{D}_{A^{-m}} \mathcal{T}_{\vec{r}} \mathcal{D}_{A^m},$$

so that $\mathcal{A}$ commutes with $\mathcal{T}_{A^m \vec{r}}$. Now we show that $A^m \vec{r}$ is dense in $\mathbb{R}^n$. Let $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$ be given. Note that any point of $\mathbb{R}^n$ is within $\frac{\sqrt{n}}{2}$ of a point in $\mathbb{Z}^n$. Choose m so that $\|A^m\| < \frac{2\epsilon}{\sqrt{n}}$, then $A^{-m}\vec{x}$ must be within $\frac{\sqrt{n}}{2}$ of some $\vec{r}$ in $\mathbb{Z}^n$. Then:

$$\begin{aligned} \|\vec{x} - A^m \vec{r}\| &\leq \|A^m\| \|A^{-m}\vec{x} - \vec{r}\| \\ &< \frac{2\epsilon}{\sqrt{n}} \frac{\sqrt{n}}{2} = \epsilon. \end{aligned}$$

Thus this set is dense in $\mathbb{R}^n$, and so $\mathcal{A}$ commutes with a dense set of translations. By the continuity of $\cdot \mapsto \mathcal{T}_\cdot$ and the continuity of $\mathcal{A}$ we see that $\mathcal{A}$ must commute with all translations. ∎

Theorem 2.22. *Suppose A is a dilation matrix and $\mathcal{A}$ is a bounded linear transform on $L^2(\mathbb{R})$ which commutes with $\mathcal{D}_A$ and $\mathcal{T}_{\vec{r}}$ (for all $\vec{r} \in \mathbb{Z}^n$); then $\mathcal{A}$ is of the form:*

$$\mathcal{A} = \mathcal{F}^{-1} \pi \mathcal{F},$$

where $\pi \in L^\infty(\mathbb{R}^n)$ and $\pi(B\vec{\omega}) = \pi(\vec{\omega})$ for all $\vec{\omega} \in \mathbb{R}^n$.

Proof. By Lemma 2.21 $\mathcal{A}$ commutes with all translations, and so by Theorem 4.1.1 of [29] (page 92) we can find $\rho \in L^\infty(\mathbb{R}^n)$ so that:

$$\mathcal{A} = \mathcal{F}^{-1}\rho\mathcal{F}.$$

That is for any $f \in L^2(\mathbb{R}^n)$:

$$\begin{aligned} \mathcal{F}^{-1}\rho\mathcal{F}f &= \mathcal{A}f, \\ \rho\mathcal{F}f &= \mathcal{F}\mathcal{A}f, \\ \rho(\vec{\omega})\hat{f}(\vec{\omega}) &= (\mathcal{F}\mathcal{A}f)(\vec{\omega}), \end{aligned}$$

for almost every $\vec{\omega} \in \mathbb{R}^n$. Replacing f with $\mathcal{D}_A f$ we get:

$$\begin{aligned} \rho(\vec{\omega})\frac{1}{|\det A|}\hat{f}(B\vec{\omega}) &= \frac{1}{|\det A|}(\mathcal{F}\mathcal{A}f)(B\vec{\omega}), \\ \rho(\vec{\omega})\hat{f}(B\vec{\omega}) &= (\mathcal{F}\mathcal{A}f)(B\vec{\omega}), \\ &= \rho(B\vec{\omega})\hat{f}(B\vec{\omega}), \end{aligned}$$

using the last line of the former derivation to replace the RHS. Thus we choose f so that $\hat{f}$ is never zero[§] and see that $\rho(\vec{\omega}) = \rho(B\vec{\omega})$ for almost every $\vec{\omega}$. We may then adjust ρ on a set of measure zero to get π. ■

2.6 Conclusion

We have concocted the idea of a maximal solution m to a transformed dilation equation and shown that such a solution exists for arbitrary p. While this idea of maximality isn't explicitly stated elsewhere, the idea has certainly been touched upon in the literature (for example Section 8 of [22] or case (c) of Theorem 2.1 in [11]).

There are many possible maximal solutions and we have not invested much time in trying to find m with desirable properties. It is highly likely that by using the properties[¶] of p, better behaved m could be found. We have examined the most likely choice for m, the infinite product, and shown that in the usual cases it will be maximal.

We applied this idea of maximality to the Haar dilation equation. Using an idea from [32], that knowing how an operator affects the Haar MRA tells you lots about the operator, we proved some nice results classifying operators which commute with shifts and dilations.

[§] Say, take $f(x) = e^{-x^2}$

[¶] In the usual case p is analytic and 2π periodic.

It would be interesting to know if our results for an operator $\mathcal{A}$ can be extended from '$\mathcal{A}$ commutes with $\mathcal{D}_2, \mathcal{T}_1$' to '$\mathcal{A}$ sends solutions of scale 2 dilation equations to solutions of the same equation'.

We then generalised these notions to dilation equations on $\mathbb{R}^n$. In our search for a suitable MRA to use within the proofs of our operator results we looked at MSF wavelets and self-affine tiles. Both of these families raise many interesting questions. For a given dilation A an MSF wavelet always exists but a self-affine tile may not. It would be interesting to investigate a hybrid of these ideas, looking for an MRA generated by a function g which has $\hat{g} = c\chi_X$.

It would also be interesting to know if it is possible to generalise the concept of maximal to other situations such as refinable function vectors and distributional solutions to dilation equations.

Chapter 3

Solutions of dilation equations in $L^2(\mathbb{R})$

3.1 Introduction

In Chapter 2 we managed to find the form of all the solutions to a transformed dilation equation, and in particular we nailed down the $L^2(\mathbb{R})$ solutions to $f(x) = f(2x)+f(2x-1)$ exactly. In this chapter we aim to see if these results are suitable for doing calculations.

3.2 Calculating solutions of $f(x) = f(2x) + f(2x - 1)$

Using Theorem 2.9 we will actually calculate some solutions to $f(x) = f(2x) + f(2x - 1)$. We now know that if g is a solution then:

$$\hat{g} = \pi\hat{\chi}_{[0,1)},$$

where $\pi \in L^2(\pm[1, 2))$ and $\pi(\omega) = \pi(2\omega)$. Also, for functions $\check{\psi} \in L^1(\mathbb{R})$ we know that:

$$\mathcal{F}^{-1}(\psi\hat{\chi}_{[0,1)}) = c\check{\psi} * \chi_{[0,1)},$$

where $*$ denotes convolution of two functions and c is a constant depending on the normalisation of the Fourier transform. As $f(x) = f(2x) + f(2x - 1)$ is linear, we ignore this constant.

As π satisfies $\pi(\omega) = \pi(2\omega)$ it will never* be in $L^2(\mathbb{R})$, so we cannot take its inverse Fourier transform in $L^2(\mathbb{R})$. Likewise it will never satisfy $\lim_{\omega\to\infty} |\pi(\omega)| = 0$ and so (by

*Unless trivially $\pi = 0$ almost everywhere.

the Riemann-Lebesgue lemma[†]) cannot be the Fourier transform of an $L^1(\mathbb{R})$ function. If we are to use the convolution result it will have to be in terms of better-behaved functions.

Examining the properties we would expect of $\check{\pi}$, we blindly take the inverse Fourier transform of $\pi(\omega) = \pi(2\omega)$ to get:

$$\check{\pi}(x) = 2\check{\pi}(2x).$$

It is easy to construct a candidate function for $\check{\pi}$. As in the case with π, it looks like we are free to choose the function on $\pm[1, 2)$ and then use to the above relation to determine it (almost) everywhere else.

For this example we will take $\check{\pi}(x)$ to be $\sin(2\pi x)$ on $[1, 2)$ and zero on $-(2, 1]$. There are several reasons for this:

- We will be taking the convolution of $\check{\pi}$ with $\chi_{[0,1)}$. Arranging for each "cycle" of $\check{\pi}$ to have mean zero will simplify this process.
- We will be writing π in terms of sums of:

  $$\mathcal{F}\left(\sin(2\pi x)\chi_{[1,2)}\right).$$

  Using the heuristic "the smoother the function the faster its Fourier transform decays" we arrange that $\sin(2\pi x)\chi_{[1,2)}$ is continuous to make the convergence of the sums easier to determine[‡].
- We leave $\check{\pi}$ identically zero on $\mathbb{R}^-$ as a demonstration of the fact that the two halves are independent.

So we are working with the function:

$$\check{\pi}(x) = \begin{cases} 2^{-n}\sin(2^{-n}2\pi x) & x \in 2^n[1, 2) \\ 0 & \text{otherwise} \end{cases},$$

a sketch of which is shown in Figure 3.1. We could write this as a sum of $\sin(2\pi x)\chi_{[1,2)}(x) = \alpha(x)$ as follows:

$$\check{\pi}(x) = \sum_{n\in\mathbb{Z}} 2^n \sin(2^n 2\pi x)\chi_{[1,2)}(2^n x) = \sum_{n\in\mathbb{Z}} 2^n \alpha(2^n x).$$

If we cut off this sum above and below we get a sequence of bounded compactly-supported functions. These functions will be in $L^1(\mathbb{R})$ and so we will be able to use the convolution

[†]See Theorem 1.2 of [40] for details of the Riemann-Lebesgue lemma.

[‡]Doing the same calculation with cos in place of sin is very slightly harder because $\cos(2\pi x)\chi_{[1,2)}$ is not continuous.

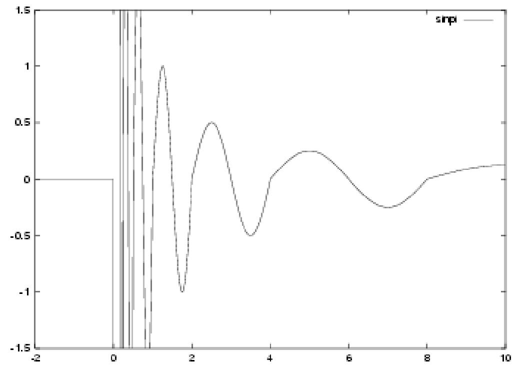


Figure 3.1: $\check{\pi}$ for the example.

result on these. For $m \in \mathbb{N}$ we define:

$$\check{\pi}_m(x) = \sum_{n\in\mathbb{Z}}^{|n|<m} 2^n \alpha(2^n x)$$

We may safely examine the Fourier transform of each of these. We write it in terms of:

$$\hat{\alpha}(x) = \frac{2\pi(e^{-i2\omega} - e^{-i\omega})}{\omega^2 - 4\pi^2},$$

using the dilation property of the Fourier transform to get:

$$\pi_m(\omega) = \sum_{n\in\mathbb{Z}}^{|n|<m} 2^n \left(\frac{1}{2^n}\hat{\alpha}\left(\frac{\omega}{2^n}\right)\right) = \sum_{n\in\mathbb{Z}}^{|n|<m} \hat{\alpha}(2^n\omega) = \sum_{n\in\mathbb{Z}}^{|n|<m} \frac{2\pi(e^{-i22^n\omega} - e^{-i2^n\omega})}{2^{2n}\omega^2 - 4\pi^2}.$$

For fixed ω there are three ways this sum could be troublesome. First as $n \to \infty$. This is not going to be a problem as the numerator is bounded by 4π and the denominator has a factor of 2^{2n}. The second possible issue is what happens when $2^n\omega$ is near $\pm 2\pi$, as the denominator will be small here. This is not a problem as the top also has a zero here, and so $\hat{\alpha}$ is bounded by C, say. Our final possible concern is what happens as $n \to -\infty$. Near zero, however, we know that:

$$\left|e^{-i2\omega} - e^{-i\omega}\right| < k|\omega|,$$

so the contribution to the sum will be less than a geometric sum. In fact, by classifying the terms of the sum into three groups: $|\omega 2^n| < 1$, $1 \leq |\omega 2^n| \leq 8$ and $|\omega 2^n| > 8$, we see that the sum is always less than:

$$\left(\frac{k}{4\pi^2-1}\sum_{n\in\mathbb{Z}}^{|\omega 2^n|<1}|\omega 2^n|\right)+(4C)+\left(\sum_{n\in\mathbb{Z}}^{|\omega 2^n|>8}\frac{4\pi}{2^{2n}\omega^2-4\pi^2}\right)<\frac{2k}{4\pi^2-1}+4C+4\pi,$$

and so the sum is absolutely convergent for any ω. This means that if we define $\pi(\omega) = \lim_{m\to\infty}\pi_m(\omega)$, then $\pi_m \to \pi$ uniformly on compact subsets.

The other implication of this is that π is bounded, and so is certainly in $L^2(\pm[1,2))$. Also any function of the form $\sum_{n\in\mathbb{Z}}\hat{\alpha}(2^n\omega)$ will satisfy $\pi(\omega) = \pi(2\omega)$, so we know that $\pi\hat{\chi}_{[0,1)}$ is the Fourier transform of an $L^2(\mathbb{R})$ solution of $f(x) = f(2x) + f(2x-1)$.

Combining the facts that $\pi_m \to \pi$ uniformly on compact subsets and that π, π_m are all uniformly bounded, it is straightforward to show that $\pi_m f \to \pi f$ for any $f \in L^2(\mathbb{R})$. We apply this to $\hat{\chi}_{[0,1)}$ to get:

$$\begin{aligned}
\mathcal{F}^{-1}(\pi\hat{\chi}_{[0,1)}) &= \mathcal{F}^{-1}(\lim_{m\to\infty}\pi_m\hat{\chi}_{[0,1)}) \\
&= \lim_{m\to\infty}\mathcal{F}^{-1}(\pi_m\hat{\chi}_{[0,1)}) \\
&= \lim_{m\to\infty}\mathcal{F}^{-1}(\mathcal{F}(\check{\pi}_m * \chi_{[0,1)})) \\
&= \lim_{m\to\infty}\check{\pi}_m * \chi_{[0,1)}.
\end{aligned}$$

The final task we are left with is to calculate the convolution.

$$\int_{\mathbb{R}}\check{\pi}_m(y)\chi_{[0,1)}(x-y)\,dy = \int_{x-1}^{x}\check{\pi}_m(y)\,dy.$$

Now we can use the fact that the average of $\check{\pi}_m(y)$ over any interval $[2^k, 2^{k+1})$ is zero, which means only the first and last cycles in $[x-1, x)$ will make a contribution.

$$\check{\pi}_m * \chi_{[0,1)}(x) = \begin{cases}
0 & x \le 2^{-m}. \\
\int_{2^k}^{x} 2^{-k}\sin(2^{-k}2\pi y)\,dy & 2^{-m} < x \le 1 + 2^{-m}, \\
 & \text{where } x \in [2^k, 2^{k+1}). \\
\int_{2^k}^{x} 2^{-k}\sin(2^{-k}2\pi y)\,dy + \int_{x-1}^{2^{l+1}} 2^{-l}\sin(2^{-l}2\pi y)\,dy & 1 + 2^{-m} < x \le 2^{m+1}, \\
 & \text{where } x - 1 \in [2^l, 2^{l+1}). \\
\int_{x-1}^{2^{l+1}} 2^{-l}\sin(2^{-l}2\pi y)\,dy & x - 1 < 2^{m+1} < x. \\
0 & 2^{m+1} \le x - 1.
\end{cases}$$

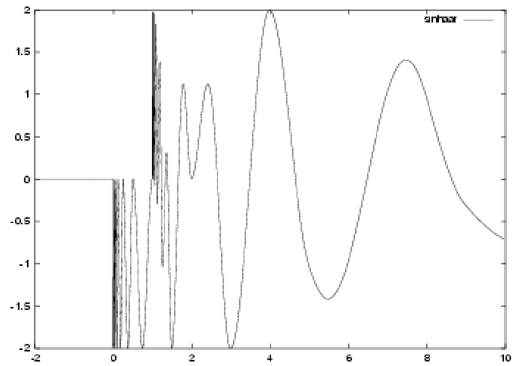


Figure 3.2: $\check{\pi} * \chi_{[0,1)}$ for the example.

This simplifies in the limit $m \to \infty$.

$$\check{\pi} * \chi_{[0,1)}(x) = \begin{cases} 0 & x \leq 0. \\ \int_{2^k}^{x} 2^{-k} \sin(2^{-k} 2\pi y)\, dy & 0 < x \leq 1, \\ & \text{where } x \in [2^k, 2^{k+1}). \\ \int_{2^k}^{x} 2^{-k} \sin(2^{-k} 2\pi y)\, dy + \int_{x-1}^{2^{l+1}} 2^{-l} \sin(2^{-l} 2\pi y)\, dy & 1 < x, \\ & \text{where } x - 1 \in [2^l, 2^{l+1}). \end{cases}$$

This integration is straightforward, and gives (up to a constant) an explicit form for the solution:

$$\check{\pi} * \chi_{[0,1)}(x) = \begin{cases} 0 & x \leq 0. \\ \cos(2^{-k} 2\pi x) - 1 & 0 < x \leq 1, \\ & \text{where } x \in [2^k, 2^{k+1}). \\ \cos(2^{-k} 2\pi x) - \cos(2^{-l} 2\pi (x-1)) & 1 < x, \\ & \text{where } x - 1 \in [2^l, 2^{l+1}). \end{cases}$$

This curiously-shaped solution is shown in Figure 3.2.

3.3 Factoring solutions of $f(x) = f(2x) + f(2x-1)$

By experimenting further it can be seen that our choice of $\check{\pi}$ failed to make it clear exactly what was afoot. Note that $\check{\pi} * \chi_{[0,1)}(x)$ is actually of the form $F(x) - F(x-1)$ where:

$$F(x) = \begin{cases} 1 & x \leq 0 \\ \cos(2^{-k} 2\pi x) & x > 0 \\ & \text{where } x \in [2^k, 2^{k+1}) \end{cases}.$$

Note that $F(x) = F(2x)$. In fact if F is any function such that $F(x) = F(2x) + c$, then by defining $f(x) = F(x) - F(x-1)$ we get a solution of $f(x) = f(2x) + f(2x-1)$. This, of course, gives us a vast selection of solutions as producing a function like F is easy.

By trial and error we can see that we can produce $\chi_{[0,1)}$ and $\log|1 - 1/x|$ by setting F to be $\operatorname{sign}(x)$ and $\log|x|$ respectively. Is it possible to produce any $L^2(\mathbb{R})$ solution this way?

This "factorisation" must be related to the fact that the Fourier transforms of the $L^2(\mathbb{R})$ solutions are of the form:

$$\frac{1 - e^{-i\omega}}{i\omega} \pi(\omega) = \left(1 - e^{-i\omega}\right) \frac{\pi(\omega)}{i\omega}.$$

Letting $\Lambda(\omega) = \frac{\pi(\omega)}{i\omega}$ we observe that the term $1 - e^{-i\omega}$ corresponds to evaluating some function at x and $x - 1$. The Λ term has the property that $\Lambda(\omega) = 2\Lambda(2\omega)$, so (brashly ignoring convergence) we expect its inverse Fourier transform to have a property like $\check{\Lambda}(x) = \check{\Lambda}(2x)$. This $\check{\Lambda}$ roughly corresponds to our F.

We will now show that all f in $L^2(\mathbb{R})$ may be written in this form.

Theorem 3.1. *If f is an $L^2(\mathbb{R})$ solution of $f(x) = f(2x) + f(2x-1)$, then f can be written in the form $f(x) = F(x) - F(x-1)$, where $F(x) = F(2x) + c$, for almost every $x \in \mathbb{R}$.*

Proof. Using Theorem 2.9 we take $\pi \in L^2(\pm[1,2))$ corresponding to f and extend it as usual. Let:

$$\lambda(\omega) = \frac{\pi(\omega)}{i\omega} \chi_{\pm[1,2)}(\omega),$$

then $\lambda \in L^2(\pm[1,2))$. Considered as a function on $\mathbb{R}$, it is a member of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ which has compact support, and so we see its inverse Fourier transform is analytic and decays to zero.

Now we define F_n and f_n as follows:

$$\begin{aligned} F_n(x) &= \left(\sum_{m\in\mathbb{Z}}^{|m|\leq n} \check{\lambda}(2^m x) \right) - n\check{\lambda}(0) \\ f_n(x) &= F_n(x) - F_n(x-1). \end{aligned}$$

The reason for the $n\check{\lambda}(0)$ term is to encourage $F_n(x)$ to converge: As $|2^m x|$ gets large $\check{\lambda}$ goes to zero; however, as $|2^m x|$ goes to zero we almost pick up the value $\check{\lambda}(0)$ for each term, so we must subtract that amount.

Note that $\{f_n\} \subset L^2(\mathbb{R})$, as each f_n is a finite sum of $\check{\lambda}(2^m x)$, and these are all in $L^2(\mathbb{R})$. So we may check what happens to $\hat{f}_n$:

$$\begin{aligned} \hat{f}_n(\omega) &= \left(1 - e^{-i\omega}\right) \sum_{m\in\mathbb{Z}}^{|m|\leq n} 2^m \frac{\pi(2^m\omega)}{i2^m\omega} \chi_{\pm[1,2)}(2^m\omega) \\ &= \left(1 - e^{-i\omega}\right) \sum_{m\in\mathbb{Z}}^{|m|\leq n} \frac{\pi(\omega)}{i\omega} \chi_{\pm[1,2)}(2^m\omega) \\ &= \frac{1 - e^{-i\omega}}{i\omega} \pi(\omega) \chi_{\pm[2^{-n},2^{n+1})}. \end{aligned}$$

Clearly $\hat{f}_n \to \hat{\chi}_{[0,1)}\pi$ in $L^2(\mathbb{R})$ as $n \to \infty$, and so f_n tends to the solution that corresponds to multiplication by π.

What can we say about the F_n? In the limit they have the $F(x) = F(2x) + c$ property:

$$\begin{aligned} F_n(x) - F_n(2x) &= \check{\lambda}(2^{-n}x) - \check{\lambda}(2^{n+1}x) \\ \lim_{n\to\infty} \left(F_n(x) - F_n(2x)\right) &= \lim_{n\to\infty} \left(\check{\lambda}(2^{-n}x) - \check{\lambda}(2^{n+1}x)\right) \\ &= \check{\lambda}(0) - 0. \end{aligned}$$

However, we do not know yet if $F_n(x)$ converges. Rewrite F_n as:

$$F_n(x) = \left(\sum_{m=1}^{n} \check{\lambda}(2^{-m}x) - \check{\lambda}(0) \right) + \sum_{m=0}^{n} \check{\lambda}(2^m x).$$

Examining the first sum and remembering that $\check{\lambda}$ is analytic we write:

$$\check{\lambda}(x) = \check{\lambda}(0) + ax + b(x)x^2,$$

where $b(x)$ is also analytic. Thus the first sum becomes:

$$\sum_{m=1}^{n} 2^{-m}xa + 2^{-2m}x^2b(2^{-m}x) = a(1-2^n)x + x^2\sum_{m=1}^{n} 2^{-2m}b(2^{-m}x)$$

It is easy to see that this sum converges uniformly on compact subsets, as b is bounded on compact subsets. Thus the limit of this first sum is analytic. We can show that the other half of F converges in $L^2(\mathbb{R})$ by showing that it is a Cauchy sequence. Remember $\check{\lambda} \in L^2(\mathbb{R})$, and so:

$$\begin{aligned}
\left\| \sum_{m=n_1}^{n_2} \check{\lambda}(2^m x) \right\|_2 &\leq \sum_{m=n_1}^{n_2} \left\| \check{\lambda}(2^m x) \right\|_2 \\
&\leq \sum_{m=n_1}^{n_2} \frac{1}{\sqrt{2^m}} \left\| \check{\lambda}(x) \right\|_2 \\
&\leq \frac{\sqrt{2}}{\sqrt{2}-1} \frac{1}{\sqrt{2^{n_1}}} \left\| \check{\lambda}(x) \right\|_2 .
\end{aligned}$$

This rate of convergence in $L^2(\mathbb{R})$ is then sufficient to ensure pointwise convergence almost everywhere. Write:

$$F_n^+(x) = \sum_{m=0}^{n} \check{\lambda}(2^m x)$$

and let F^+ be the $L^2(\mathbb{R})$ limit of this sequence. For any $L^2(\mathbb{R})$ function h we have:

$$|\{x : |h(x)| > \delta\}| \leq \frac{\|h\|_2^2}{\delta^2}.$$

Writing this for $F_n^+ - F^+$ we get:

$$\begin{aligned}
\left|\left\{x : |F_n^+(x) - F^+(x)| > \delta\right\}\right| &\leq \frac{\|F_n^+ - F^+\|_2^2}{\delta^2} \\
&= \frac{\left\|\sum_{m>n} \check{\lambda}(2^m x)\right\|_2^2}{\delta^2} \\
&\leq \frac{c}{2^n \delta^2}.
\end{aligned}$$

Accordingly:

$$\begin{aligned}
&\left|\{x : F_n^+(x) \not\to F^+(x)\}\right| \\
= &\left|\left\{x : \exists m \in \mathbb{N}^+ \text{ s.t. } \forall N > 0\ \exists n > N \text{ s.t. } |F_n^+(x) - F^+(x)| \geq \frac{1}{m}\right\}\right| \\
= &\left|\bigcup_{m=1}^{\infty} \left\{x : \forall N > 0\ \exists n > N \text{ s.t. } |F_n^+(x) - F^+(x)| \geq \frac{1}{m}\right\}\right| \\
= &\left|\bigcup_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \left\{x : \exists n > N \text{ s.t. } |F_n^+(x) - F^+(x)| \geq \frac{1}{m}\right\}\right| \\
= &\left|\bigcup_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} \left\{x : |F_n^+(x) - F^+(x)| \geq \frac{1}{m}\right\}\right| \\
\leq &\sum_{m=1}^{\infty} \inf_{N>0} \sum_{n=N+1}^{\infty} \frac{cm^2}{2^n} \\
= &\sum_{m=1}^{\infty} \inf_{N>0} \frac{cm^2}{2^N} \\
= &\sum_{m=1}^{\infty} 0 = 0.
\end{aligned}$$

So we have $F_n \to F$ as the sum of two functions with at least pointwise convergence almost everywhere, as required. ■

3.3.1 A basis for the solutions of $f(x) = f(2x) + f(2x-1)$

We are now in a position to calculate a basis for the solutions of $f(x) = f(2x) + f(2x-1)$ in $L^2(\mathbb{R})$. Firstly, note that as π ranges over $L^2(\pm[1,2))$, $\lambda(\omega) = \frac{\pi(\omega)}{i\omega}$ covers the same space. So we will begin with a basis for the possible λ. Observing that $(e^{\pi i r\omega})_{r\in\mathbb{R}}$ forms an orthogonal basis for $L^2(\pm[1,2))$, we use these as a basis for the λ.

$$\begin{aligned}
\lambda_r(\omega) &= e^{\pi i r\omega}\chi_{\pm[1,2)}(\omega) \\
\check{\lambda}_r(x) &= \frac{1}{2\pi}\int_{\pm[1,2)} e^{\pi i r\omega} e^{i\omega x}\, d\omega \\
&= \frac{-\sin(2x) + (-1)^r \sin(x)}{\pi(\pi r - x)}.
\end{aligned}$$

This function is analytic, decays like $1/x$, and at $x = \pi r$ has value $1/\pi$. The fact that it decays like $1/x$ means that F will converge everywhere, except possibly at $x = 0$. We can then produce a sum for $F(x)$ and sketch $f(x)$ (see Figure 3.3). This basis will not be orthogonal as the relationships between $\hat{f}$, π and λ are not unitary.

3.4 Factoring and other dilation equations

A simple extension of this trick allows us to produce solutions to equations of the form:

$$f(x) = \Delta\left(f(2x) + f(2x-1)\right),$$

for $\Delta \in \mathbb{R}$. If we choose F so that $F(x) = \Delta F(2x) + c$ and, as before, set $f(x) = F(x) - F(x-1)$ then we automatically get a solution to the above equation.

Taking $F(x) = |x|^\beta$ gives $\Delta = |2^{-\beta}|$. We would like to know when the resulting $f(x) = |x|^\beta - |x-1|^\beta$ is in $L^p(\mathbb{R})$. Firstly, for the tails of f (say when $|x| > 2$) we consider:

$$|f(x)| = |x|^\beta \left|1 - \left(1 - \frac{1}{x}\right)^\beta\right| \leq |x|^\beta \frac{C_\beta}{|x|}.$$

Thus the tails are in $L^p(\mathbb{R})$ when $p(\beta-1) < -1$. The other obstacle to being in $L^p(\mathbb{R})$ is that if $\beta < 0$, then f blows up at 0 and 1 like x^β. To keep these peaks in $L^p(\mathbb{R})$ we require $\beta p + 1 > 0$. So in the case[§] where $-\frac{1}{p} < \beta < -\frac{1}{p} + 1$ these functions are in $L^p(\mathbb{R})$. The corresponding range for Δ is $(2^{\frac{1}{p}-1}, 2^{\frac{1}{p}})$ for a given p, or considering any $p \geq 1$ we find $\Delta \in (\frac{1}{2}, 2)$.

It is interesting to note that except for $\Delta = 1$, none of these equations could have a compactly-supported $L^p(\mathbb{R})$ solution. If any equation did, then the solution would also be in $L^1(\mathbb{R})$. However, as shown in [11], compactly-supported $L^1(\mathbb{R})$ solutions only exist when $\Delta = 2^m$ for some nonnegative integer m.

The following gives an indication of how this idea applies to other dilation equations with more complicated forms.

Theorem 3.2. *Suppose that:*

1. $F : \mathbb{R} \to \mathbb{C}$ *satisfies* $F(2x) = F(x) + c$,
2. f_0 *is differentiable,*
3. f_0 *is a solution of a finite dilation equation* $f_0(x) = \sum c_k f_0(2x-k)$,
4. $f_0(x) \to 0$ *as* $|x| \to \infty$.

[§] We haven't excluded this working out for some other values β, but this range will suffice for this example.

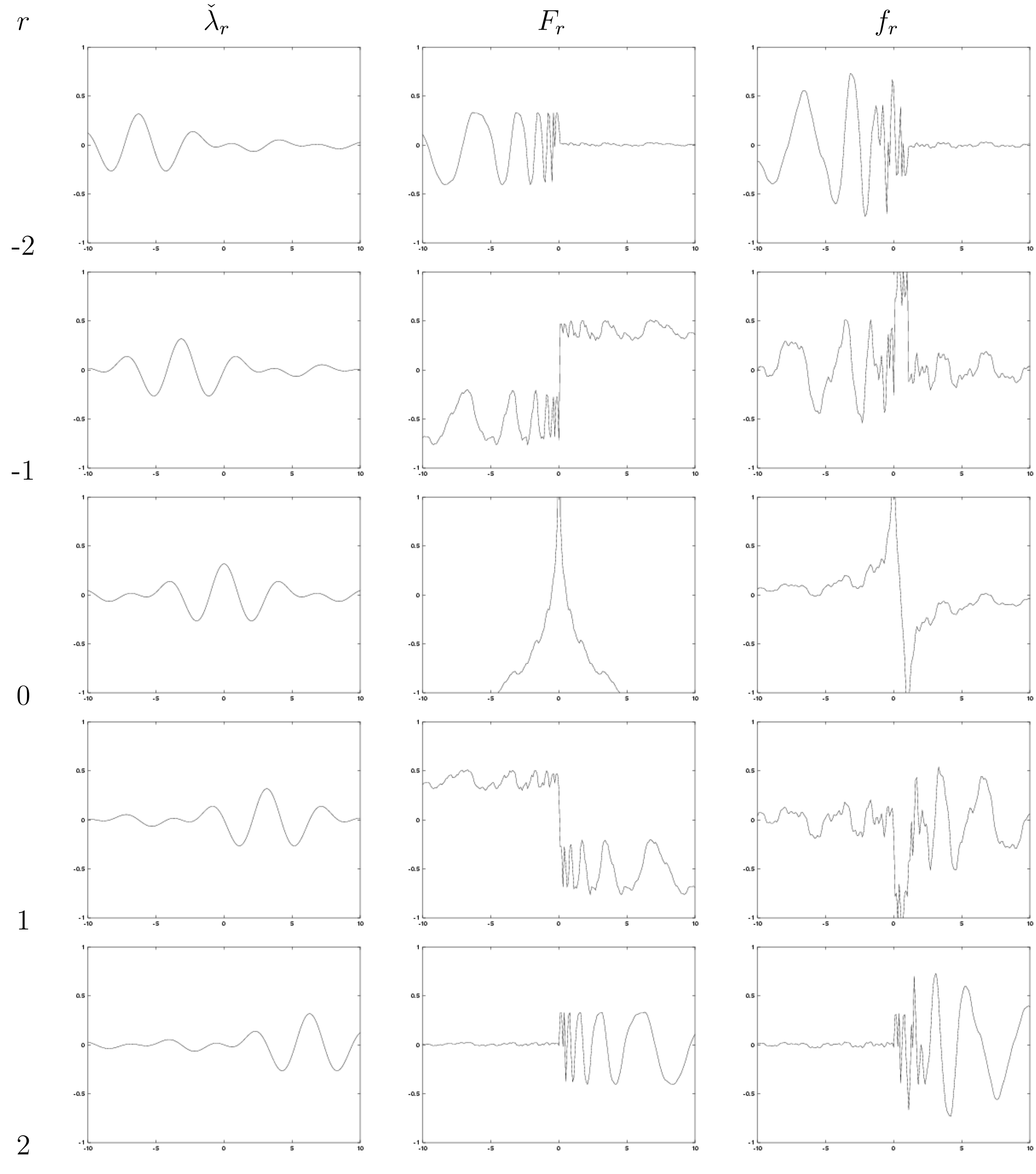


Figure 3.3: Constructing a basis for solutions of the Haar equation.

Then if

$$f(x) = (F * f_0')(x) = \int F(t) f_0'(x-t)\, dt$$

is well-defined, it is also a solution of the same dilation equation.

Proof. First note that f_0' satisfies:

$$\frac{f_0'(x)}{2} = \sum c_k f_0'(2x-k).$$

We plug this into the definition of f.

$$\begin{aligned}
\sum c_k f(2x-k) &= \sum c_k \int F(t) f_0'(2x-k-t)\, dt \\
&= \int F(t) \sum c_k f_0'(2x-k-t)\, dt \\
&= \int F(t) \sum c_k f_0' \left(2\left(x - \frac{t}{2} \right) - k \right) dt \\
&= \int F(t) \frac{f_0'\left(x - \frac{t}{2}\right)}{2}\, dt \\
&= \int F(2t') \frac{f_0'(x-t')}{2}\, 2dt' \\
&= \int (F(t') + c) f_0'(x-t')\, dt' \\
&= f(x) + \int c f_0'(x-t')\, dt' \\
&= f(x) + c \lim_{t\to\infty} (f_0(t) - f_0(-t)) \\
&= f(x)
\end{aligned}$$

■

3.5 $L^2(\mathbb{R})$ solutions of other dilation equations

It would be nice to be able to prove a similar result to Theorem 2.9 for all dilation equations. It is easy to see that if we multiply the Fourier transform of any $L^2(\mathbb{R})$ solution of a dilation equation by $\pi \in L^\infty(\mathbb{R})$ with $\pi(\omega) = \pi(2\omega)$, then we will get another $L^2(\mathbb{R})$ solution of that dilation equation. To apply exactly the same proof to another equation we would need the following:

1. A maximal solution g of the equation in $L^2(\mathbb{R})$.

2. The solution should be bounded away from zero on some $[2^n, 2^{n+1}]$ and $-[2^m, 2^{m+1}]$.
3. The solution must be essentially bounded near the origin.
4. The solution must decay like ω^α where $\alpha < -\frac{1}{2}$.

The obvious candidate to consider for g is the infinite product, $\prod_{r=1}^\infty p(\omega/2^r)$, discussed by Daubechies and Lagarias (see page 14 and [11]). Their main concern is with $L^1(\mathbb{R})$ functions; however, there are dilation equations with only trivial solutions in $L^1(\mathbb{R})$, but interesting solutions in $L^2(\mathbb{R})$. For example, in the case:

$$f(x) = \Delta \left(f(2x) + f(2x-1) \right),$$

they prove that $f = 0$ is the only $L^1(\mathbb{R})$ solution when $|\Delta| < 1$. However, from the discussion in Section 3.4 above we have examples of $L^2(\mathbb{R})$ solutions for $1/\sqrt{2} < \Delta < \sqrt{2}$. The means of dismissing these equations in $L^1(\mathbb{R})$ is to note that the Fourier transform of a solution of such an equation must be unbounded. As we are not interested in restricting the infinite product in this way it may still prove a useful maximal solution.

Let us examine these solutions in the nicest case of the infinite product where $\Delta = 1$. We know we will have a maximal solution bounded at the origin for which we know lots about the zeros¶. Unfortunately the rate of decay of this infinite product seems difficult to get a handle on. The following results indicate why the infinite product may decay or at least why no counterexamples are forthcoming. The first is a variant on a result of Heil and Colella (see Theorem 2 of [21]).

Theorem 3.3. *Let:*

$$\phi(\omega) = \prod_{r=1}^\infty p\left(\frac{\omega}{2^r}\right)$$

where $p(\omega)$ is a trigonometric polynomial with $p(0) = 1$. Suppose that $\phi \in L^2(\mathbb{R})$; then we can find $\alpha \leq -\frac{1}{2}$ such that for all $\delta > 0$ we can find $E \subset [\pi, 2\pi]$ with $|E| > \pi - \delta$ and constants $0 < c, C < \infty$ so that for all $\epsilon > 0$ there exists $N > 0$ so when $n \geq N$:

$$c|2^n\omega|^{\alpha-\epsilon} \leq |\phi(2^n\omega)| \leq C|2^n\omega|^{\alpha+\epsilon}$$

for any $\omega \in E$.

Proof. Define $L(\omega) = \log_2 |p(\omega)|$, noting that this is a 2π periodic integrable function, as p is a trigonometric polynomial and so has zeros of finite degree. Thus we can define α by:

$$\alpha = \frac{1}{2\pi} \int_0^{2\pi} \log_2 |p(\omega)| \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} L(\omega) \, d\omega,$$

¶We know lots about the zeros as the infinite product is analytic.

and α will be a finite number. Now examine:

$$\lim_{n\to\infty} \log_2 \left| \prod_{r=0}^{n-1} p(2^r\omega) \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} L(2^r\omega).$$

Note that the map $\tau : \omega \mapsto 2\omega \mod 2\pi$ is ergodic, so we can apply the Birkhoff Ergodic Theorem[‖] to conclude this limit is α for almost all $\omega \in [0, 2\pi]$. Now by Egoroff's Theorem (page 88 [16]) we may choose a set $E' \subset [\pi, 2\pi]$ on which this is a uniform limit in ω and so that $|E'| > \pi - \delta/2$. Thus for $\epsilon' > 0$ we may choose N' so that when $n \geq N'$ we have:

$$\left| \log_2 \left| \prod_{r=0}^{n-1} p(2^r\omega) \right|^{\frac{1}{n}} - \alpha \right| < \epsilon'$$

for all $\omega \in E'$. Rewriting this:

$$2^{n(\alpha-\epsilon')} < \left| \prod_{r=0}^{n-1} p(2^r\omega) \right| < 2^{n(\alpha+\epsilon')}.$$

Now, by [11], ϕ is entire and so we can choose a subset E of E' with $|E| > \pi - \delta$ on which

[‖] See [48] Theorem 1.14 for details of the Birkhoff Ergodic Theorem.

$|\phi(\omega)| \geq m > 0$. Now:

$$\begin{aligned}
\infty &> \int |\phi(\omega)|^2 \, d\omega \\
&\geq \sum_{n=N}^{\infty} \int_{\pi 2^n}^{2\pi 2^n} |\phi(\omega)|^2 \, d\omega \\
&= \sum_{n=N}^{\infty} 2^n \int_{\pi}^{2\pi} |\phi(2^n\omega)|^2 \, d\omega \\
&= \sum_{n=N}^{\infty} 2^n \int_{\pi}^{2\pi} |\phi(\omega)|^2 \left| \prod_{r=0}^{n-1} p(2^r\omega) \right|^2 \, d\omega \\
&> \sum_{n=N}^{\infty} 2^n \int_E |\phi(\omega)|^2 2^{2n(\alpha-\epsilon')} \, d\omega \\
&> \sum_{n=N}^{\infty} 2^{n(1+2(\alpha-\epsilon'))} \int_E |\phi(\omega)|^2 \, d\omega \\
&\geq \sum_{n=N}^{\infty} 2^{n(1+2(\alpha-\epsilon'))} m^2 |E|
\end{aligned}$$

So for this sum to be finite we must have $1 + 2(\alpha - \epsilon') < 0$, but this is true for any $\epsilon' > 0$, thus $\alpha \leq -\frac{1}{2}$. Now we fix $\epsilon' = \epsilon$ and find N so that for $\omega \in E, n \geq N$:

$$\begin{aligned}
|\phi(2^n\omega)| &= \left| \prod_{r=0}^{n-1} p(2^r\omega) \right| |\phi(\omega)| \\
&\leq 2^{n(\alpha+\epsilon)} M \\
&= \frac{|2^n\omega|^{\alpha+\epsilon}}{|\omega|^{\alpha+\epsilon}} M \\
&\leq |2^n\omega|^{\alpha+\epsilon} \frac{M}{(2\pi)^{\alpha}}.
\end{aligned}$$

where $M = \sup_{\omega \in [\pi, 2\pi]} |\phi(\omega)|$, and:

$$\begin{aligned}
|\phi(2^n\omega)| &= \left| \prod_{r=0}^{n-1} p(2^r\omega) \right| |\phi(\omega)| \\
&\geq 2^{n(\alpha-\epsilon)} m \\
&\geq |2^n\omega|^{\alpha-\epsilon} \frac{m}{\pi^{\alpha}}
\end{aligned}$$

as required. ■

This means we have the required decay except at some proportionally small set of exceptional points. The following deals with the most obvious candidate for an exceptional point: 2π. It shows that if the infinite product is non-zero near 2π, then it will be non-zero around all $2^n 2\pi$.

Theorem 3.4. *Let:*

$$\phi(\omega) = \prod_{r=1}^{\infty} p\left(\frac{\omega}{2^r}\right)$$

where $p(\omega)$ is a trigonometric polynomial with $p(0) = 1$. Suppose that we can find $\delta > 0$ so that $|\phi(\omega)| > m > 0$ on $[2\pi, 2\pi + \delta)$; then we can find $c > 0$ so that $|\phi(\omega)| > c$ on $[2^n 2\pi, 2^n 2\pi + \delta)$ for all $n > 0$.

Proof. First note that as ϕ is continuous and $\phi(0) = 1$ we can find b so that $|\phi(\omega) - 1| < b < 1$ when $|\omega| < \delta$. Now take $\epsilon < \delta$ and look at:

$$\begin{aligned}
\phi(2^n 2\pi + \epsilon) &= \prod_{r=1}^{n-1} p\left(2^r 2\pi + \frac{\epsilon}{2^{n-r}}\right) \phi\left(2\pi + \frac{\epsilon}{2^n}\right) \\
&= \prod_{r=1}^{n-1} p\left(\frac{\epsilon}{2^{n-r}}\right) \phi\left(2\pi + \frac{\epsilon}{2^n}\right) \\
&= \frac{\prod_{r=1}^{\infty} p\left(\frac{\epsilon}{2^{n-r}}\right)}{\prod_{r=n}^{\infty} p\left(\frac{\epsilon}{2^{n-r}}\right)} \phi\left(2\pi + \frac{\epsilon}{2^n}\right) \\
&= \frac{\phi(\epsilon)}{\phi\left(\frac{\epsilon}{2^{n-1}}\right)} \phi\left(2\pi + \frac{\epsilon}{2^n}\right).
\end{aligned}$$

Thus:

$$\begin{aligned}
|\phi(2^n 2\pi + \epsilon)| &= \frac{|\phi(\epsilon)|}{\left|\phi\left(\frac{\epsilon}{2^{n-1}}\right)\right|} \left|\phi\left(2\pi + \frac{\epsilon}{2^n}\right)\right| \\
&> \frac{1-b}{1+b} m,
\end{aligned}$$

as required. ■

Such a ϕ will never be in $L^p(\mathbb{R})$ for $p < \infty$ (nor will it be in $\mathcal{F}(L^1(\mathbb{R}))$ by the Riemann-Lebesgue lemma). So, between these two results we know that an infinite product in $L^2(\mathbb{R})$ has the close to the required decay at many points, and that if it doesn't decay near $2^n 2\pi$, then it couldn't be in $L^2(\mathbb{R})$ anyway.

3.6 Conclusion

In this chapter we actually got our hands dirty and performed calculations using some of our earlier results. In the case of the Haar equation this revealed a factorisation of the $L^2(\mathbb{R})$ solutions which allowed us to form a basis for the space of solutions. It may be possible to improve this basis to an orthonormal basis by looking at the λ we chose as members of a suitably weighted $L^2(\pm[1, 2))$ space. Using some sort of Legendre polynomials might produce a neater basis.

In the folklore of dilation equations the non-uniqueness of $L^2(\mathbb{R})$ solutions could actually be considered surprising. The well-known uniqueness result for $L^1(\mathbb{R})$ in [11] is often quoted for compactly-supported $L^2(\mathbb{R})$, but 'compactly-supported' is frequently implicitly stated. Consequently, people's first reaction is that $L^2(\mathbb{R})$ solutions are unique.

In the Haar case it is also surprising that it is so easy to write down solutions. The equation has been well studied and it seems unlikely that this factorisation has not been noted before.

We then looked to see if these ideas could easily be applied to other dilation equations. We found that a variation of this factorisation trick provided us with some non-compactly-supported solutions to dilation equations of a reasonably neat form. When we moved to the case of arbitrary dilation equations it appeared that the factorisation would work in most cases (particularly in well-studied case where the refinable function is reasonably smooth). The general version of this factorisation involves convolution and so may not be suitable for producing examples suitable for manipulation by hand.

When we tried to generalise Theorem 2.9 to dilation equations which have a compactly-supported $L^2(\mathbb{R})$ solutions, we found evidence to suggest that this also would be successful. However, the final details continue to remain elusive. Perhaps by applying better ergodic techniques and the smoothness of p we may be able to produce the required result.

Considering the general $L^2(\mathbb{R})$ case would be interesting. This would involve examining the possibility that for some dilation equations the infinite product lies, or maybe even all maximal solutions lie, outside $L^2(\mathbb{R})$.

One aspect of this factorisation we did not consider is how to determine F from f. We could follow the argument in Theorem 3.1: take the Fourier transform, divide out to get λ, take the inverse Fourier transform and evaluate the sum; however, is seems that there may be a more natural construction.

Chapter 4

The right end of a dilation equation

4.1 Introduction

Here we will look at one end of a dilation equation in the case where it is compactly-supported. As this is written in English, which is read left to right, it seems most natural to look at the left end — but it could just as easily be the right end.

4.2 Refinable characteristic functions on $\mathbb{R}$

Suppose we wish to find characteristic functions on $\mathbb{R}$ which satisfy dilation equations. What can we say about such a function and the equations which it might satisfy? First we should look at some examples.

1. We know that $\chi_{[0,1)}$ satisfies lots of dilation equations of the form $f(x) = \sum_{k=0}^{n-1} f(nx - k)$.

2. The characteristic function of Cantor's middle third set satisfies the dilation equation $f(x) = f(3x) + f(3x - 2)$. However, it has measure zero, and so is somewhat special.

3. A little experimentation produces examples like $\chi_{\mathbb{R}}(x) = \sum c_k \chi_{\mathbb{R}}(2x - k)$ for any choice of c_k summing to one. This suggests that if we want interesting results we probably need to consider only bounded sets.

4. Experimentation also shows that $\chi_{[0,1)\cup[2,3)}$ will satisfy a dilation equation of scale 2 with coefficients $c_0, c_1, \ldots = 1, 1, -1, -1, 2, 2, -2, -2, 2, 2, -2, -2, \ldots$. This suggests that we need to restrict the coefficients in some way. For the moment we will consider the case with a finite number of non-zero coefficients.

5. Self-affine tiles which generate MRAs provide examples of sets whose characteristic functions satisfy dilation equations whose coefficients are either zero or one. This is

because of the orthogonality requirement. This suggests that characteristic functions may only satisfy dilation equations like this.

Accordingly we consider sets of positive measure which are bounded, which satisfy dilation equations with finitely many non-zero coefficients. We will try to find out what sorts of sets satisfy these equations, and what the possible coefficients are.

The obvious scheme to follow is: look for a translated dilated bit of the set which doesn't overlap with any others — say a leftmost bit. Then, since the dilation equation's only contribution here is the coefficient attached to this translation, it must be 0 or 1 to agree with the original characteristic function. We then aim to somehow remove this bit of the set and repeat until we show all the coefficients are 0 or 1.

The following lemma gives us a starting point for this argument.

Lemma 4.1. *Suppose $\chi_S(x) = \sum d_k \chi_S(2x-k)$, and only finitely many of the d_k are non-zero. Then we can find l so that $\chi_{S+l}(x) = \sum c_k \chi_{S+l}(2x-k)$, $c_0 \neq 0$, $c_k = 0$ when $k < 0$ and $c_k = d_{k-l}$.*

Proof. Take l to be the least k such that $d_k \neq 0$. This obviously gives c_k the properties $c_0 \neq 0$, $c_k = 0$ for $k < 0$. To show that χ_{S+l} satisfies the dilation equation, consider:

$$\begin{aligned}\chi_{S+l}(x) &= \chi_S(x-l)\\ &= \sum d_k \chi_S(2(x-l)-k)\\ &= \sum d_k \chi_S(2x-k-l-l)\\ &= \sum d_{k'-l} \chi_S(2x-k'-l)\\ &= \sum d_{k'-l} \chi_{S+l}(2x-k') = \sum c_{k'} \chi_{S+l}(2x-k').\end{aligned}$$

■

This means that we can always make a translation and have our coefficients start at the origin. Fortunately, this forces our set to lie in the positive real axis.

Lemma 4.2. *If S is bounded and satisfies a dilation equation $\chi_S(x) = \sum c_k \chi_S(2x-k)$, where $c_0 \neq 0$ and $c_k = 0$ when $k < 0$, then $S \cap (-\infty, 0)$ has zero measure.*

Proof. If S has zero measure there is nothing to prove. Let $[a,b]$ be the smallest interval containing all of S's measure. Then $\chi_S(x)$ is supported on $[a,b]$, and $\chi_S(2x-k)$ is supported on $[(a+k)/2, (b+k)/2]$. However, as $\chi_S(x) = \sum c_k \chi_S(2x-k)$ and $[a,b]$ is minimal we have:

$$[a,b] \subset \bigcup_{k>=0} [(a+k)/2, (b+k)/2] = [a/2, \infty].$$

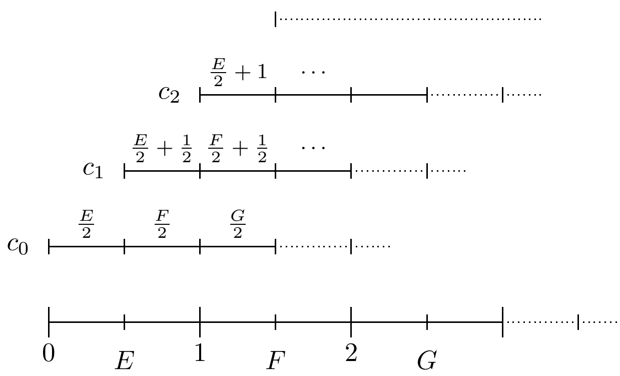


Figure 4.1: The left-hand end of a dilation equation.

So $a/2 \leq a$ which implies $a \geq 0$ as required. ■

Now we can draw a picture. Figure 4.1 shows the situation we are in. The bottom line shows $\mathbb{R}$ where χ_S lives — $E, F, G, \ldots$ are the parts of S contained in $[0,1), [1,2), [2,3), \ldots$. The lines above show where $\chi_S(2x), \chi_S(2x-1), \chi_S(2x-2), \ldots$ live. By summing the values of the functions on the upper lines (scaled by the c_k) we must get something that agrees with the bottom line. This visualisation was suggested to me by Christopher Heil.

The next proof essentially works through the easily extracted information from Figure 4.1.

Theorem 4.3. *If S is bounded and satisifies a dilation equation $\chi_S(x) = \sum c_k \chi_S(2x-k)$ a.e., where $c_0 \neq 0$ and $c_k = 0$ when $k < 0$, then either:*

- *S is of measure zero or,*
- *$c_0 = 1$, the rest of the c_k are integers with $|c_k| \leq 2^k$ and E has non-zero measure in both $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$.*

Proof. First consider the case where E has zero measure. Looking at the interval $[\frac{1}{2}, 1)$ we have:

$$\begin{aligned}\int_{\frac{1}{2}}^{1} \chi_E(x)\,dx &= \int_{\frac{1}{2}}^{1} c_0 \chi_{\frac{F}{2}}(x) + c_1 \chi_{\frac{E}{2}+\frac{1}{2}}(x)\,dx \\ 0 &= c_0 \frac{|F|}{2} + 0,\end{aligned}$$

but we know $c_0 \neq 0$ so $|F| = 0$. Likewise on $[1, \frac{3}{2})$:

$$\begin{aligned}\int_1^{\frac{3}{2}} \chi_F(x)\,dx &= \int_1^{\frac{3}{2}} c_0\chi_{\frac{G}{2}}(x) + c_1\chi_{\frac{F}{2}+\frac{1}{2}}(x) + c_2\chi_{\frac{E}{2}+1}(x)\,dx \\ 0 &= c_0\frac{|G|}{2} + 0 + 0,\end{aligned}$$

so this time $|G| = 0$. Continuing we see that $S = E \cup F \cup G \cup \ldots$ is a countable union of sets of measure 0, and so has measure zero.

On the other hand, if E does not have measure zero, we know that χ_E and $c_0\chi_{\frac{E}{2}}$ agree almost everywhere on $[0, \frac{1}{2})$, and as $c_0 \neq 0$ we know that $c_0\chi_{\frac{E}{2}}$ is non-zero on a set of positive measure. Thus, χ_E is non-zero on a set of positive measure in $[0, \frac{1}{2})$. This means that $c_0\chi_{\frac{E}{2}} = 1$ on some set of positive measure, and so $c_0 = 1$.

Accordingly, $\chi_E(x) = \chi_{\frac{E}{2}}(x)$ for almost every $x \in [0, \frac{1}{2}]$. This means that $x \in E \Leftrightarrow x \in \frac{E}{2}$ for almost every $x \in [0, \frac{1}{2}]$.

Now that we have seen that $E \cap [0, \frac{1}{2})$ has positive measure, what about $E \cap [\frac{1}{2}, 1)$?

$$|E \cap [\frac{1}{2}, 1)| = 0 \Rightarrow |\frac{E}{2} \cap [\frac{1}{4}, \frac{1}{2})| = 0 \Rightarrow |E \cap [\frac{1}{4}, \frac{1}{2})| = 0 \Rightarrow |E \cap [\frac{1}{4}, 1)| = 0.$$

Repeating and taking the limit gives $|E \cap (0,1)| = |E| = 0$, which is a contradiction. Thus $E \cap [\frac{1}{2}, 1)$ has positive measure.

Next, we show by induction that $c_k \in \mathbb{Z}$ and $|c_k| \leq 2^k$. Assume that $c_l \in \mathbb{Z}$ and $|c_l| \leq 2^l$ for $l < k$. Looking at the interval $[\frac{k}{2}, \frac{k+1}{2})$, we see:

$$\chi_X(x) = c_0\chi_Y(x) + \ldots + c_k\chi_{\frac{E}{2}+\frac{k}{2}}(x),$$

for almost every x in the interval. As E has positive measure it must be true for some x in $\frac{E}{2} + \frac{k}{2}$, giving:

$$\chi_X(x) = c_0\chi_X(x) + \ldots + c_k.$$

As all the other terms are 0 or 1 times an integer, c_k must also be an integer. In fact, rearranging for $|c_k|$ we see:

$$|c_k| \leq 1 + |c_0| + |c_1| + \ldots + |c_{k-1}| \leq 1 + 1 + 2 + \ldots + 2^{k-1} = 2^k.$$

This relation trivilally holds for c_k with $k < 0$ as the c_k are all zero. ■

We can also flip the set over and perform the same analysis from the other end. This tells us that we can find N so that $c_N = 1$, $|c_k| \leq 2^{N-k}$ and $c_k = 0$ for $k > N$. The severe limitation of this method is that it is local in nature, and while it makes the coefficients line up at the current point, it doesn't account for the fact that they will have to become zero at some stage in the future.

For example, if we examine c_1 and enumerate the possibilities we find that c_1 must be -1, 0 or 1. However, this method cannot tell us which of these actually arise with a bounded S. This is because aside from $\chi_{[0,1)}$ providing an example for $c_1 = 1$, we can form equations with $c_1 = 0$ by taking $S = \mathbb{R}^+$ and the equation $\chi_S(x) = \chi_S(2x)$. Taking $S = \mathbb{R}$ allows* $\chi_S(x) = \chi_S(2x) - \chi_S(2x-1) + \chi_S(2x-2)$. To pursue this further we need to produce some examples.

4.2.1 Division and very simple functions

Figure 4.1 suggests a possible scheme for producing examples. We make the simplifying assumption that each of $E, F, G, \ldots$ is either the whole unit length interval, or are empty. Suppose we want to find out what dilation equation $\chi_{[0,2)}$ satisfies. Writing out the values of our function on half-intervals we get:

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \end{array}$$

We already know that $c_1 = 1$, so we can fill in the contributions of the row above, leaving blanks for zeros.

$$\begin{array}{ccccccccc} 1 & 1 & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \end{array}$$

Now we know that the next row up is shifted along one, and multiplied by some integer. The upper columns must sum to the bottom one, so by adding up column 2 we see the integer must be zero.

$$\begin{array}{ccccccccc} & 0 & 0 & & & & & & \\ 1 & 1 & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \end{array}$$

Now we can repeat this process to fill in the next lines; first we need a 1 and then a 0:

$$\begin{array}{ccccccccc} & & & 0 & 0 & & & & \\ & & 1 & 1 & & & & & \\ & 0 & 0 & & & & & & \\ 1 & 1 & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \end{array}$$

*Taking $S = \mathbb{R}$ is not really a very good example, as it has no 'left-hand end' and as such our method does not apply to it.

After this we can see that zeros will make all the columns add up. We can read the coefficients of the dilation off as 1, 0, 1. Thus $\chi_{[0,2)}$ satisfies $f(x) = 1f(2x) + 0f(2x-1) + 1f(2x-2) = f(2x) + f(2x-2)$.

This process should seem quite familiar — most of us have been doing it since we were children. Before explaining what is going on, we will take a slightly different approach. Suppose we want to find out whether there is a function (constant on intervals $[n, n+1)$) which satisfies a dilation equation with coefficients 1, 1, -1, -1, 1, 1. A similar process can also answer this question.

We know E has non-zero measure, so our simplifying assumption means that it must be the whole interval. Then we may begin with the following setup:

$$\begin{array}{cccccccccccccc}
 & & & & & 1 & ? & ? & ? & ? & ? & ? & & \\
 & & & & 1 & ? & ? & ? & ? & ? & ? & & & \\
 & & & -1 & ? & ? & ? & ? & ? & ? & & & & \\
 & & -1 & ? & ? & ? & ? & ? & ? & & & & & \\
 & 1 & ? & ? & ? & ? & ? & ? & & & & & & \\
1 & ? & ? & ? & ? & ? & ? & & & & & & & \\
\hline
1 & 1 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ?
\end{array}$$

By summing column 2, we see that the value of the first question mark must be 0, which fills in two gaps in the bottom row and 1 gap in each of the rows above.

$$\begin{array}{cccccccccccccc}
 & & & & & 1 & 0 & ? & ? & ? & ? & ? & & \\
 & & & & 1 & 0 & ? & ? & ? & ? & ? & & & \\
 & & & -1 & 0 & ? & ? & ? & ? & ? & & & & \\
 & & -1 & 0 & ? & ? & ? & ? & ? & & & & & \\
 & 1 & 0 & ? & ? & ? & ? & ? & & & & & & \\
1 & 0 & ? & ? & ? & ? & ? & & & & & & & \\
\hline
1 & 1 & 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ?
\end{array}$$

Now, by summing column 3, the next value can be seen to be one.

$$\begin{array}{cccccccccccccc}
 & & & & & 1 & 0 & 1 & ? & ? & ? & ? & & \\
 & & & & 1 & 0 & 1 & ? & ? & ? & ? & & & \\
 & & & -1 & 0 & -1 & ? & ? & ? & ? & & & & \\
 & & -1 & 0 & -1 & ? & ? & ? & ? & & & & & \\
 & 1 & 0 & 1 & ? & ? & ? & ? & & & & & & \\
1 & 0 & 1 & ? & ? & ? & ? & & & & & & & \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & ? & ? & ? & ? & ? & ? & ? & ?
\end{array}$$

```
int check_it(unsigned long p,int degree)
{
    int i,j;
    int P[DEG],Pd[2∗DEG+1],Q[DEG+1];
    int d;

    /∗ Set up P and Pd(x) = (x+1)P(xˆ2). ∗/
    for( i = 0 ; i ≤ degree ; i++ ) {
        if( p & (1≪i) ) {
            P[i] = 1;
            Pd[2∗i] = 1;
            Pd[2∗i+1] = 1;
        } else {
            P[i] = 0;
            Pd[2∗i] = 0;
            Pd[2∗i+1] = 0;
        }
    }

    /∗ Now see if P(x)|Pd(x). ∗/
    for( i = degree+1 ; i ≥ 0 ; i−− ) {
        Q[i] = d = Pd[degree+i];
        if( d ≠ 0 )
            for( j = degree; j ≥ 0 ; j−− )
                if( P[j] ≠ 0 )
                    Pd[j+i] −= d;
    }

    /∗ Check the remainder. ∗/
    for( i = degree; i ≥ 0 ; i−− )
        if( Pd[i] ≠ 0 )
            return FALSE;

    return TRUE;
}
```

Figure 4.2: Checking a bit pattern to see if is it 2-refinable.

Following this through we end up with the following table, which amazingly all tallies.

					1	0	1	0	1	0	0		
				1	0	1	0	1	0	0			
			−1	0	−1	0	−1	0	0				
		−1	0	−1	0	−1	0	0					
	1	0	1	0	1	0	0						
1	0	1	0	1	0	0							
1	1	0	0	1	1	0	0	1	1	0	0	0	0

This means that the function $\chi_{[0,1)\cup[2,3)\cup[4,5)}$ satisfies the dilation equation $f(x) = f(2x) + f(2x-1) - f(2x-2) - f(2x-3) + f(2x-4) + f(2x-5)$. This equation has -1 as a coefficient, showing that the coefficients don't all have to be 1. Also we see that the set doesn't have to be a single interval.

This process may remind the reader of long division or polynomial division, turned upside down. It turns out that this simplified problem, where the set is just a union of $[n, n+1)$, can be expressed nicely in terms of polynomials. Another nice aspect of this process is that it is easy to implement on a computer (Figure 4.2), making the search for all characteristic functions of less than a certain length a matter of just testing each using this method.

4.3 Polynomials and simple refinable functions

For the duration of this section we will widen our attention slightly, to include all functions which are constant on the intervals $[n, n+1)$.

Theorem 4.4. *Consider the map from functions which are constant on* $[n, n+1)$ *to the polynomials given by:*

$$f(x) = \sum_r a_r \chi_{[r,r+1)}(x) \mapsto \sum_r a_r x^r = P_f(x).$$

Then this map is a linear bijection, transforming the following operations in the following way:

$$\begin{aligned}
(\alpha f + \beta g)(x) &\mapsto \alpha P_f(x) + \beta P_g(x), \\
f\left(\frac{x}{n}\right) &\mapsto \frac{x^n - 1}{x - 1} P_f(x^n), \\
f(x-k) &\mapsto x^k P_f(x), \\
\sum_k c_k f(x-k) &\mapsto P_f(x) Q(x),
\end{aligned}$$

where $Q(x) = \sum c_k x^k$.

Proof. This map is well defined as it is equivalent to:

$$f(x) \mapsto \sum_r f(r) x^r.$$

It is easy to show that it is linear. Examining the dilation property:

$$\begin{aligned} f\left(\frac{x}{n}\right) &= \sum_r a_r \chi_{[r,r+1)}\left(\frac{x}{n}\right) \\ &= \sum_r a_r \chi_{[nr,nr+n)}(x) \\ &= \sum_r a_r \left[\chi_{[nr,nr+1)}(x) + \chi_{[nr+1,nr+2)}(x) + \ldots + \chi_{[nr+n-1,nr+n)}(x)\right] \\ &\mapsto \sum_r a_r \left[x^{nr} + x^{nr+1} + \ldots + x^{nr+n-1}\right] \\ &= \left[1 + x + \ldots + x^{n-1}\right] \sum_r a_r \left(x^n\right)^r \\ &= \frac{x^n - 1}{x - 1} P_f(x^n). \end{aligned}$$

The translation property is similar:

$$\begin{aligned} f(x-k) &= \sum_r a_r \chi_{[r,r+1)}(x-k) \\ &= \sum_r a_r \chi_{[r+k,r+k+1)}(x) \\ &\mapsto \sum_r a_r x^{r+k} \\ &= x^k P_f(x). \end{aligned}$$

Finally, applying the linearity and the translation property:

$$\sum_k c_k f(x-k) \mapsto \sum_k c_k x^k P_f(x) = Q(x) P_f(x).$$

■

Using the above bijection we can look at what happens to dilation equations under this sort of transformation:

$$\begin{aligned} & f(x) = \sum_k c_k f(nx-k) \\ \Leftrightarrow \quad & f\left(\frac{x}{n}\right) = \sum_k c_k f(x-k) \\ \Leftrightarrow \quad & \frac{x^n - 1}{x - 1} P(x^n) = Q(x) P(x). \end{aligned}$$

$Q(x)$	$P(x)$	$Q(x)$	$P(x)$
degree 0		x^3+0x^2+1x+0	x^2+1x+0
$x+1$	1	x^3+2x^2+1x+0	x^2-1x+0
degree 1		x^3+0x^2+0x+1	x^2+1x+1
x^2+1x+0	$x+0$	x^3+ix^2-ix+1	$x^2+(1-i)x-i$
x^2+0x+1	$x+1$	x^3-ix^2+ix+1	$x^2+(1+i)x+i$
x^2+2x+1	$x-1$	x^3+1x^2+1x+1	x^2+0x-1
degree 2		x^3+1x^2+1x+1	x^2+0x-1
x^3+1x^2+0x+0	x^2+0x+0	x^3+3x^2+3x+1	x^2-2x+1
x^3+1x^2+0x+0	x^2+0x+0		

Figure 4.3: Possible values for $Q(x)$ and $P(x)$ from Mathematica.

This is something with which we can actually do calculations. By looking at the points where $x^n = x$ we get some information. At $x = 0$, we find that either $P(0) = 0$ or $Q(0) = 1$. This means that either $f(0) = 0$, which can be avoided by translation, or $c_0 = 1$. Likewise, looking at $x = 1$ tells us that $P(1) = 0$ or $Q(1) = n$, meaning that f had mean zero, or the sum of the coefficients of the dilation equation was the scale.

By examining possible solutions to the relation $P(x)Q(x) = P(x^2)(x + 1)$ using Mathematica (see Figure 4.3) it became clear that looking at the roots can provide more information. For simplicity we stay with scale 2.

Lemma 4.5. *Let R be a polynomial; then $R(x)Q(x) = R(x^2)$ for some polynomial $Q(x)$ iff whenever r is an order-p root of $R(x)$ then r^2 is a root of $R(x)$ of order at least p.*

Proof. If $r = 0$ the result is trivial as $r^2 = r$.

First we show that $R(x)Q(x) = R(x^2)$ implies that if r is an order-p root of $R(x)$, then r^2 is a root of $R(x)$ of order at least p. We differentiate both sides of the equation n times. Using Leibniz's theorem on the left side we get:

$$(RQ)^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} R^{(m)}(x)Q^{(n-m)}(x).$$

Thus a zero of order p of $R(x)$ gives a zero of order p of $R(x)Q(x)$. Differentiating the other side n times is more complicated, we use Faa di Bruno's generalisation† of the chain rule:

$$\frac{d^n}{dx^n}\left(R(x^2)\right) = \sum_{m=0}^{n} R^{(m)}(x^2)H_{n,m}(x),$$

†See [26], Question 1.2.5.21.

where:

$$H_{n,m}(x) = \sum_{\substack{k_1+k_2\ldots k_n=m \\ k_1+2k_2\ldots nk_n=n \\ k_1,k_n\ldots k_n\geq 0}} \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\ldots k_n!(n!)^{k_n}} \left[\frac{d}{dx}(x^2)\right]^{k_1} \left[\frac{d^2}{dx^2}(x^2)\right]^{k_2} \cdots \left[\frac{d^n}{dx^n}(x^2)\right]^{k_n}.$$

Fortunately we are only interested in $H_{n,n}(x)$, which we find is $(2x)^n$. If $R(x)$ has a zero of order p at r, then for $n = 0, 1, \ldots, p-1$:

$$0 = (RQ)^{(n)}(r) = \sum_{m=0}^{n} R^{(m)}(x^2)H_{n,m}(x).$$

Rearranging for $R^{(n)}(r^2)$ we find:

$$R^{(n)}(r^2) = \frac{-\sum_{m=0}^{n-1} R^{(m)}(r^2)H_{n,m}(r)}{(2r)^n},$$

as we know $r \neq 0$. We see that if $R^{(m)}(r^2) = 0$ for $m = 0, 1, \ldots, n-1$ then $R^{(n)}(r^2) = 0$. This is true for $m = 0$ as $R(r^2) = R(r)Q(r) = 0$, so by induction we see $R^{(n)}(r^2) = 0$ for $n = 0, 1, \ldots, p-1$.

Now we prove the converse: if R has the property that whenever r is an order-p root of $R(x)$, r^2 is a root of $R(x)$ of order at least p, then we can find Q so that $R(x)Q(x) = R(x^2)$. We note that it is sufficient to show that $R(x) \mid R(x^2)$.

Let $R(x) = \prod(x - r_m)^{p_m}$. We want to show that each factor of $R(x)$ divides $R(x^2)$. Looking at the factor $(x-r_m)^{p_m}$ we see $R(x)$ has a zero of order p_m at r_m. By hypothesis it has a zero of order at least p_m at r_m^2. Thus $(x-r_m^2)^{p_m}$ is a factor of $R(x)$, and consequently $(x^2-r_m^2)^{p_m}$ is a factor of $R(x^2)$. But this factor is just $(x+r_m)^{p_m}(x-r_m)^p_m$. Thus $(x-r_m)^p_m$ divides $R(x^2)$. Repeating this for each factor of $R(x)$ we see that $R(x) \mid R(x^2)$. ∎

Lemma 4.6.

$$\left\{P(x) : P(x)Q(x) = P(x^2)(x+1)\right\} = \left\{\frac{R(x)}{x-1} : R(x)Q(x) = R(x^2), R(1) = 0\right\}$$

Proof. Suppose $P(x)Q(x) = P(x^2)(x+1)$, then let $R(x) = (x-1)P(x)$. Clearly $R(1) = 0$, and:

$$\frac{R(x^2)}{R(x)} = \frac{(x^2-1)P(x^2)}{(x-1)P(x)} = \frac{(x+1)P(x^2)}{P(x)} = Q(x).$$

Conversely, if $R(1) = 0$, then $(x-1) \mid R(x)$, so $R(x) = P(x)(x-1)$ for some $P(x)$, but then if:

$$\begin{aligned} R(x)Q(x) &= R(x^2) \\ P(x)(x-1)Q(x) &= P(x^2)(x^2-1) \\ P(x)Q(x) &= P(x^2)(x+1). \end{aligned}$$

■

Lemma 4.7. *If R is as in the statement of Lemma 4.5, then the roots of $R(x)$ have norm 0 or 1. Moreover, if a root has norm 1, then it is a root of unity.*

Proof. By Lemma 4.5, if r is a root of $R(x)$, then so is $r^2, r^4, r^8, \ldots$. If $0 < |r| < 1$ or $|r| > 1$, then these will all have distinct norms $\to 0$ or $\to \infty$ respectively. This would mean that $R(x)$ had an infinite number of distinct roots, which is impossible as it is a polynomial.

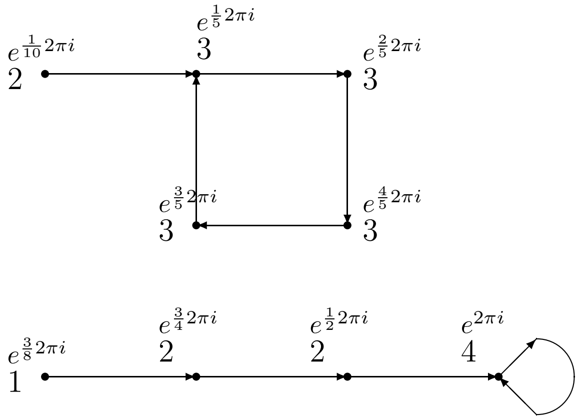
If $|r| = 1$, then the sequence of roots $r, r^2, r^4, \ldots$ must begin to repeat itself, to avoid having an infinite number of roots. Thus $r^{2^k} = r^{2^l}$ for some $k > l$. Thus $r^{(2^k - 2^l)} = 1$, so r is a root of unity. ■

Corollary 4.8. *If P and R are as in the statement of Lemma 4.6, then Lemma 4.7 applies to $P(x)$ as its roots are a subset of those of $R(x)$.*

We now have a method of producing examples of $P(x)$. First "draw" a directed graph where the vertices are the roots of unity. A directed edge goes from one vertex to another if the second is the first squared. Next, choose a vertex on the graph and walk through the graph writing an increasing sequence of integers, one at each vertex. Eventually this process ends up in a loop. At this stage we make all the integers written in the loop the same. Now, we can stop or pick another root of unity and begin the walk again. This provides the roots of $R(x)$, and the largest integer written by each vertex is the order of the root. The roots of $P(x)$ are produced by beginning at 1, if 1 has not already been visited, and finally decrementing the number written at 1. Figure 4.4 shows an example of this procedure.

Lemma 4.9. *If the coefficients of P are real, and P's roots have norm 1, then P is palindromic or anti-palindromic.*

Proof. The polynomial $P\left(\frac{1}{x}\right) x^n$ has coefficients which are those of P but in the opposite



The polynomial $R(x)$ to this graph will be:

$$\begin{aligned} &R(x) \\ &= \left[\left(x - e^{\frac{1}{10}2\pi i}\right)\right]^2 \\ &\quad \left[\left(x - e^{\frac{1}{5}2\pi i}\right)\left(x - e^{\frac{2}{5}2\pi i}\right)\right. \\ &\quad\quad \left.\left(x - e^{\frac{4}{5}2\pi i}\right)\left(x - e^{\frac{3}{5}2\pi i}\right)\right]^3 \\ &\quad \left[\left(x - e^{\frac{3}{8}2\pi i}\right)\right] \\ &\quad \left[(x+i)(x+1)\right]^2 \\ &\quad \left[(x-1)\right]^4 \end{aligned}$$

We have already visited 1, so $P(x) = R(x)/(x-1)$.

Figure 4.4: How to generate $P(x)$.

order (here $n = \deg(P)$). We write $P(x) = \prod (x - r_m)^{p_m}$ where we know that $|r_m| = 1$. So:

$$\begin{aligned} P\left(\frac{1}{x}\right)x^n &= \prod x^{p_m}\left(\frac{1}{x} - r_m\right)^{p_m} \\ &= \prod (1 - r_m x)^{p_m} \\ &= \prod (-r_m)^{p_m}\left(x - \frac{1}{r_m}\right)^{p_m} \\ &= \left(\prod (-r_m)^{p_m}\right)\left(\prod\left(x - \frac{1}{r_m}\right)^{p_m}\right) \\ &= P(0)\prod\left(x - \frac{1}{r_m}\right)^{p_m} \end{aligned}$$

Then taking each factor of $P\left(\frac{1}{x}\right)x^n$:

$$\begin{aligned} \left(x - \frac{1}{r_m}\right)^{p_m} \mid P\left(\frac{1}{x}\right)x^n &\Leftrightarrow (x - r_m)^{p_m} \mid P(x) \\ &\Leftrightarrow (x - \overline{r_m})^{p_m} \mid P(x) \qquad \text{as coefficients of } P \text{ are real,} \\ &\Leftrightarrow \left(x - \frac{1}{r_m}\right)^{p_m} \mid P(x) \qquad \text{as } |r| = 1. \end{aligned}$$

Thus $P(x)$ and $P\left(\frac{1}{x}\right)x^n$ are constant multiples of one another, say $P(x) = kP\left(\frac{1}{x}\right)x^n$.

$$a_0 + a_1x + \dots a_nx^n = k\left(a_0x^n + a_1x^{n-1} + \dots a_n\right)$$

Matching the coefficients of x^0 and x^n we see that $a_0 = ka_n$ and $a_n = ka_0$, giving $a_0 = k^2a_0$. We may cancel the a_0 as all of P's roots are of norm 1, so $k = \pm 1$ as required. ∎

This means that any $P_f(x)$ representing a real-valued solution to a dilation equation will be either palindromic or anti-palindromic. Correspondingly this means that f will be either symmetric or anti-symmetric. This result was spotted by examining all the f which were characteristic functions supported on $[0, 32)$. These were catalogued using a C program (see Figure 4.2) in about a day. Once it was known that they were all palindromic the size of the search space was reduced from $O(2^n)$ to $O(\sqrt{2}^n)$, which significantly speeds any further search.

We note that the examples of $Q(x)$ with complex coefficients in Figure 4.3 provide us with examples of non-palindromic $P(x)$.

4.4 Non-integer refinable characteristic functions

At the beginning of Section 4.3 we assumed that the functions in which we were interested were constant on the intervals $[n, n+1)$. Consequently, the characterisations arrived at in that section cover refinable characteristic functions where the sets are just unions of $[n, n+1)$. Now we consider the question: do refinable characteristic functions have to be of this form?

If we look at scales other than 2, a counterexample can be found in Section V.B of [15]. They show that there is a well-behaved tile whose characteristic function satisfies the dilation equation:

$$f(x) = f(3x) + f(3x-2) + f(3x-4),$$

and is definitely not an interval. Similar counterexamples are available for all scales $n \in \mathbb{Z}, n \geq 3$. The simplest way is to take the translations $0, 1, \dots, n-1$ and shift one of the middle ones by n. This doesn't work when the scale is 2, as moving one of the end points just stretches the tile and continues to produce a union of intervals.

So, does there exist an example of a 2-refinable characteristic function which is not just a union of integer intervals? We are going to examine a recursive procedure which finds candidate dilation equations which might produce such an example.

4.4.1 A recursive search

Again we go back to Figure 4.1, and examine each half interval. At the bottom of the picture we have the intervals $E, F, G, \dots$. Above this we have $\frac{E}{2} + \frac{n}{2}, \frac{F}{2} + \frac{n-1}{2}, \frac{G}{2} + \frac{n-2}{2}, \dots$

and from half-interval to half-interval the alignment of these sets does not change. Now consider the maps:

$$\begin{aligned} V_n &: \left[\frac{n}{2}, \frac{n+1}{2}\right) \to \{0,1\}^{n+1} \\ V_n &= \chi_{\frac{E}{2}+\frac{n}{2}} \times \chi_{\frac{F}{2}+\frac{n-1}{2}} \times \chi_{\frac{G}{2}+\frac{n-2}{2}} \times \ldots. \end{aligned}$$

Note that the first $n+1$ components of V_{n+1} agree with the components of $V_n(x-\frac{1}{2})$. We are going to consider the values which these V_n functions take on.

In general we are interested in being a little lenient and allow almost everywhere solutions, so we consider the list of values which V_n achieves on sets of non-zero measure. This list L_n will be a subset of $\mathcal{P}(\{0,1\}^n)$. We will show how given L_n and $c_0, \ldots, c_n$ we can determine all possible combinations of c_{n+1} and L_{n+1}.

We examine the $n+1^{\text{th}}$ half-interval, where the dilation equation has the form:

$$\chi_{X_{\lfloor \frac{n+1}{2} \rfloor}} = c_0 \chi_{X_{n+1}} + c_1 \chi_{X_n} + \ldots c_n \chi_F + c_{n+1} \chi_E.$$

The possible values of the LHS are given by examining the projection of L_n onto its $\lfloor \frac{n+1}{2} \rfloor^{\text{th}}$ component. Similarly we can determine all the combinations of values taken by $\chi_{X_n}, \ldots, \chi_E$ and we know the values of $c_0, \ldots, c_n$. This means that we can evaluate all the terms on the RHS, bar the first and last term. Also, by Theorem 4.3, we know that χ_E takes the value 1 for at least one combination in L_n.

Thus, remembering that $c_0 = 1$ we can determine possible values for c_{n+1} by:

$$c_{n+1} = \left(\chi_{X_{\lfloor \frac{n+1}{2} \rfloor}} - \chi_{X_{n+1}} \right) - \left(c_1 \chi_{X_n} + \ldots c_n \chi_F \right),$$

where we may have to consider -1, 0 or 1 for the value of the first bracket and the values in the second bracket are simply taken from L_n.

We have now produced a list of candidate values for c_n. For each of these we see what values $\chi_{X_{n+1}}$ takes for every combination of L_n. The new lists of candidate values for each of these c_n are generated by taking each old possibility $l \in L_n$ and adding new possibilities $(l0)$, $(l1)$ or $(l0, l1)$ according to the constraints imposed by the dilation equation.

Thus we have produced a list of values c_n with corresponding possibilities for L_{n+1} and so we can proceed with our recursive search. Unfortunately this process branches rapidly. We may have as many as 3 choices for c_{n+1} and $3^{\#L_n}$ choices for L_{n+1}. This makes the search somewhat impractical by hand, but still possible by computer.

4.4.2 Initial conditions

In the case we are interested in $L_0 = \{(0), (1)\}$: $L_0 = \{(0)\}$ would mean $E \cap [0, \frac{1}{2}) = 0$, contradicting Theorem 4.3. $L_0 = \{(1)\}$ would mean $E \cap [0, \frac{1}{2}) = \frac{1}{2}$, and then E would have to be almost all of $[0, 1)$ meaning that $E \cup F \cup G \cup \ldots$ would have to be a union of integer intervals.

4.4.3 Further checks to reduce branching

It would obviously be nice to cut down on the branching of this procedure as much as possible. One extra check we can perform after calculating c_{n+1} for odd $n+1$ is to make sure that all of the values we expect to be taken by $\chi_{X_{\lfloor \frac{n+1}{2} \rfloor}}$ are actually achieved by the RHS of the dilation equation. By rejecting (c_{n+1}, L_{n+1}) where not all required values are taken we can cut down on branching a little.

We can also impose some conditions based on the close relation between E and F. By integrating over $[\frac{1}{2}, 1)$ we can get the relation $|F| = (1 - c_1)|E|$, where we know c_1 must be 1, 0 or -1. Examining these options:

$c_1 = 1$: In this case $|F| = 0$ and so the only 0 is allowed as a value of χ_F in our list.

$c_1 = 0$: In this case $|F| = |E|$, but we already know $0 < |E| < 1$, so we know that both 0 and 1 must appear in the list of values for χ_F.

$c_1 = -1$: Here we have $|F| = 2|E|$, and so $|F| > 0$ and $|E| \leq \frac{1}{2}$. Suppose $|F| = 1$, then on the next half-interval we have:

$$\chi_F = c_2 \chi_{\frac{E}{2}+1} + c_1 \chi_{\frac{F}{2}+\frac{1}{2}} + c_0 \chi_{\frac{G}{2}}.$$

Plugging in $c_0 = 1$, $c_1 = -1$, $|F| = 1$ we get:

$$1 = c_2 \chi_{\frac{E}{2}+1} - 1 + \chi_{\frac{G}{2}},$$

almost everywhere. But, as E has measure less than 1 we must be able to choose a set of positive measure where $\chi_{\frac{E}{2}+1} = 0$ which would mean:

$$2 = \chi_{\frac{G}{2}},$$

which is a contradiction. We conclude that $0 < |F| < 1$ and so χ_F must take both values.

This lets us cut out a few more possibilities. Attempting to extend this method further leads to a matrix condition like that on page 5.

4.4.4 Examining the results

While following this procedure, whenever we find a $c_{n+1} = 1$ we check to see if it leads to a possibly compactly-supported solution. We do this by appending 0s to the end of our possibilities in L_{n+1} and evaluating the dilation equation on the remainder of the half-intervals. If this check works out we make a note of the set of coefficients, as they possibly lead to a dilation equation with a compactly-supported characteristic function as a solution.

After running[‡] a search for sequences of up to 6 coefficients, the list in Figure 4.5 was obtained. When the suggestions were examined, two easy ways to reject many of them were found.

First, any compactly-supported characteristic function will have a non-zero non-infinite integral. Consequently, we can integrate both sides of the dilation equation to see the coefficients must sum to 2.

Second, we know that there is a unique solution to a dilation equation whose coefficients sum to its scale. This means that we can eliminate any set of coefficients which have solutions which are not of the form in which we are interested. In particular, the division technique in Section 4.2.1 eliminates several sets of coefficients.

To reject the remainder of the solutions we can use the results of [21]. We can calculate α as stated in Theorem 3.3 and then use Theorem 2 of [21] to reject two solutions with $\alpha > -\frac{1}{2}$. The remaining solutions can be rejected using Theorem 1 by observing that $|p\left(\frac{1}{3}2\pi\right)p\left(\frac{2}{3}2\pi\right)| > 1$ and that $p\left(\frac{1}{3.2^n}2\pi\right) \neq 0$.

This method is quite crude, in that it completely ignores the geometry of our problem and considers only the arithmetic of the situation. The fact that it produces suggestions which we already know are unions of integer intervals means it definitely lacks important information about the problem at hand. Nonetheless, even with its overeager suggestions we have still failed to find any counterexamples.

4.5 Functions which are 2- and 3-refinable

When we search for functions which simultaneously satisfy two dilation equations of two unrelated scales, examples seem a bit thin on the ground. The obvious examples all seem to arise from $\chi_{[0,1)}$ and friends via differentiation, integration and convolution.

If we examine this via the Fourier transform we are looking at two dilation equations:

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k),$$

and we end up considering the intersection of the sets $\Phi_2(p)$ and $\Phi_3(q)$, where $p(\omega) =$

[‡]The search was performed with a C program shown in Appendix C. The search for sequences of up to 5 coefficients only takes a few seconds, but due to the high branching factor the search for 6 coefficients takes several days.

$c_0, c_1, c_2, \ldots$						Rejected because:
1	-1	0	1	-1	1	Doesn't sum to 2.
1	-1	1	-1	1		Doesn't sum to 2.
1	-1	1	0	-1	1	Doesn't sum to 2.
1	-1	1	0	0	1	$\|p\left(\frac{1}{3}2\pi\right)p\left(\frac{2}{3}2\pi\right)\| = 1.75 \not< 1$.
1	-1	1	0	1		$\alpha \approx -0.36213 \not\leq -\frac{1}{2}$.
1	-1	1	1	-1	1	Has solution 1 2 3 2 1.
1	-1	2	-1	1		Has solution 1 2 2 1.
1	0	-1	0	1	1	$\|p\left(\frac{1}{3}2\pi\right)p\left(\frac{2}{3}2\pi\right)\| = 2.5 \not< 1$.
1	0	-1	1	1		$\|p\left(\frac{1}{3}2\pi\right)p\left(\frac{2}{3}2\pi\right)\| = 1.75 \not< 1$.
1	0	0	0	0	1	Has solution 1 1 1 1 1.
1	0	0	0	1		Has solution 1 1 1 1.
1	0	0	0	1	1	Doesn't sum to 2.
1	0	0	1			Has solution 1 1 1.
1	0	0	1	-1	1	$\|p\left(\frac{1}{3}2\pi\right)p\left(\frac{2}{3}2\pi\right)\| = 1.75 \not< 1$.
1	0	0	1	0	1	Doesn't sum to 2.
1	0	0	1	1		Doesn't sum to 2.
1	0	0	1	1	1	Doesn't sum to 2.
1	0	1				Has solution 1 1.
1	0	1	-1	1		$\alpha \approx -0.36213 \not\leq -\frac{1}{2}$.
1	0	1	0	0	1	Doesn't sum to 2.
1	0	1	0	1		Doesn't sum to 2.
1	0	1	0	1	1	Doesn't sum to 2.
1	0	1	1			Doesn't sum to 2.
1	0	1	1	0	1	Doesn't sum to 2.
1	0	1	1	1		Doesn't sum to 2.
1	0	1	1	1	1	Doesn't sum to 2.
1	1					Has solution 1.
1	1	-1	-1	1	1	Has solution 1 0 1 0 1.

Figure 4.5: Possible coefficients

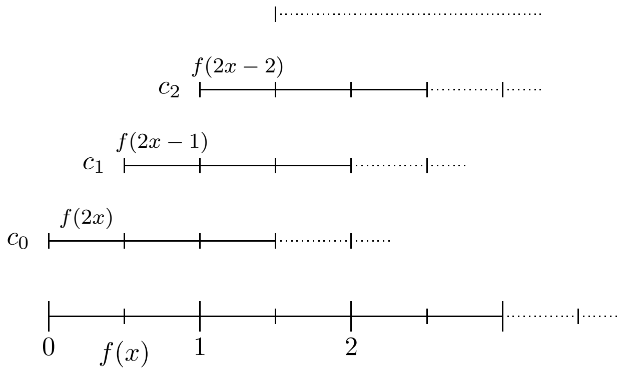


Figure 4.6: The left-hand end of a dilation equation.

$\sum c_k e^{i\omega k}$ and $q(\omega) = \sum d_k e^{i\omega k}$. It seems to be difficult to get a grip on p or q from the hypothesis that this intersection is non-empty.

We redraw Figure 4.1, this time for more complicated functions. We see we can generalise Lemma 4.1 and Lemma 4.2 for arbitrary functions quite easily, allowing us to draw Figure 4.6.

Lemma 4.10. *Suppose $g(x) = \sum d_k g(2x-k)$, and only finitely many of the d_k are non-zero. Then we can find l so that when we translate g by l to get f we find: $f(x) = \sum c_k f(2x-k)$, $c_0 \neq 0$, $c_k = 0$ when $k < 0$ and $c_k = d_{k-l}$.*

Lemma 4.11. *If f is compactly-supported and satisfies a dilation equation $f(x) = \sum c_k f(2x-k)$, where $c_0 \neq 0$ and $c_k = 0$ when $k < 0$, then f is zero almost everywhere in $(-\infty, 0)$.*

Suppose we have f lined up as in the above lemmas, and we have been given the values of f on $[0, \frac{1}{2})$. We may actually determine the values of f on the rest of $\mathbb{R}^+$ using Figure 4.6. Looking at $[0, \frac{1}{2})$ we see that $f(x) = c_0 f(2x)$. Replacing x with $\frac{x}{2}$:

$$f(x) = \frac{f\left(\frac{x}{2}\right)}{c_0} \qquad x \in [0,1).$$

Thus knowing f on $[0, \frac{1}{2})$ determines f on $[0, 1)$. Now looking at $[\frac{1}{2}, 1)$:

$$\begin{aligned} f(x) &= c_1 f(2x-1) + c_0 f(2x) \qquad x \in \left[\frac{1}{2}, 1\right) \\ f(x) &= \frac{f\left(\frac{x}{2}\right) - c_1 f(x-1)}{c_0} \qquad x \in [1,2), \end{aligned}$$

we note that $\frac{x}{2}$ and $x-1$ are in the interval $[0,1)$ on which we have already determined f. Repeating, on interval $[\frac{n}{2}, \frac{n+1}{2})$ we find:

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} c_k f(2x-k) \qquad x \in \left[\frac{n}{2}, \frac{n+1}{2}\right) \\ f(x) &= \frac{f\left(\frac{x}{2}\right) - \sum_{k=1}^{n} c_k f(x-k)}{c_0} \qquad x \in [n, n+1). \end{aligned}$$

We will call this relation the forward substitution formula.

The next issue is, if f satisfies two dilation equations, and it is translated so that $c_0 \neq 0$, $c_k = 0, k < 0$, then do the corresponding conditions also hold for the d_k from the other equation?

Lemma 4.12. *Suppose $f \neq 0$ is 2- and 3-refinable, say:*

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k),$$

and $c_0 \neq 0$ and $c_k = 0$ when $k < 0$, then $d_0 \neq 0$ and $d_k = 0$ for $k < 0$.

Proof. If f were 0 almost everywhere in $[0, \frac{1}{2})$, then f would be zero almost everywhere, by the formula which determines f on $\mathbb{R}^+$ from f on $[0, \frac{1}{2})$.

As f is non-zero on a set of positive measure in $[0, \frac{1}{2})$ we see that $f(x) = c_0 f(2x)$ is non-zero on a set of positive measure in $[0, \frac{1}{4})$. Repeating, we see that f is non-zero on a set of positive measure in $[0, \epsilon)$ for any $\epsilon > 0$. In particular, f is non-zero in $[0, \frac{1}{3})$.

Choose the least l for which $d_l \neq 0$. We aim to show that $l = 0$. Looking at the dilation equation:

$$f(x) = \sum_k d_k f(3x-k),$$

we focus on the leftmost interval, $[\frac{l}{3}, \frac{l+1}{3})$. Because $f(x) = 0$ when $x < 0$ (Lemma 4.11) we may draw a picture like Figure 4.6 for scale 3 around this interval to see that the only contribution when $x \in [\frac{l}{3}, \frac{l+1}{3})$ is:

$$f(x) = d_l f(3x-l).$$

The image of $[\frac{l}{3}, \frac{l+1}{3})$ under $x \mapsto 3x - l$ is $[0,1)$, but f is non-zero on $[0, \epsilon)$, so f is also non-zero on $[\frac{l}{3}, \frac{l+\epsilon}{3})$. Again we use, $f(x) = 0$ when $x < 0$, to see $l \geq 0$.

However, if $d_0 = 0$, then $f(x) = d_0 f(3x) = 0$ on $[0, \frac{1}{3})$ which would be a contradiction, so $d_0 \neq 0$ as required. ■

Theorem 4.13. *Suppose f is 2- and 3-refinable, say:*

$$f(x) = \sum_k c_k f(2x - k) = \sum_k d_k f(3x - k),$$

and $c_0 \neq 0$ and $c_k = 0$ when $k < 0$. Suppose further that f is integrable on some interval $[0, \epsilon]$, then $f(x) = \gamma x^\beta$ on $[0, 1)$ where $\beta = -\log_2 c_0$.

Proof. If $f = 0$, then this is obviously true with $\gamma = 0$. Otherwise, Lemma 4.12 tells us that we are looking at a situation where:

$$\begin{aligned} f(x) &= \sum_{k=0} c_k f(2x - k), \\ f(x) &= \sum_{k=0} d_k f(3x - k), \end{aligned}$$

and $c_0, d_0 \neq 0$. Now, on $[0, \frac{1}{2}) \cap [0, \frac{1}{3})$ we have:

$$\begin{aligned} f(x) &= c_0 f(2x), & f(x) &= d_0 f(3x), \\ f(x) &= c_0^{-1} f(2^{-1}x), & f(x) &= d_0^{-1} f(3^{-1}x), \end{aligned}$$

as long as both sides are evaluated in $[0, \frac{1}{3})$. Iterating:

$$f(x) = c_0^n d_0^m f(2^n 3^m x),$$

for $n, m \in \mathbb{Z}$. Now we integrate on $[0, \epsilon)$ and define:

$$F(x) = \int_0^x f(t)\, dt.$$

By Theorem 8.17 in [38] this function F is continuous and $\frac{d}{dx} F(x) = f(x)$ for almost every $x \in [0, \epsilon)$. We know f is non-zero in $[0, \epsilon)$, so we can choose $\alpha \in [0, \epsilon)$ so that $F(\alpha) \neq 0$. Now consider n, m so that $0 < 2^n 3^m \alpha < \epsilon$.

$$\begin{aligned} F(2^n 3^m \alpha) &= \int_0^{2^n 3^m \alpha} f(t)\,dt \\ &= \int_0^{\alpha} f(2^n 3^m t') 2^n 3^m \,dt' \\ &= \int_0^{\alpha} c_0^{-n} d_0^{-m} f(t') 2^n 3^m \,dt' \\ &= \left(\frac{2}{c_0}\right)^n \left(\frac{3}{d_0}\right)^m F(\alpha) \end{aligned}$$

Using Theorem 185 of [18], we can choose an infinite sequence of pairs of integers n_r, m_r where:

$$\left|\frac{m_r}{n_r} - \log_3 \frac{1}{2}\right| < \frac{1}{n_r^2},$$

and $n_r \to \infty$ as $r \to \infty$. Note that m_r is close to $n_r \log_3 \frac{1}{2}$, in fact:

$$\left|m_r - n_r \log_3 \frac{1}{2}\right| \le \frac{1}{n_r}.$$

Now examining $2^{n_r} 3^{m_r}$:

$$\begin{aligned} 2^{n_r} 3^{m_r} &= 2^{n_r} 3^{n_r \log_3 \frac{1}{2}} \left(3^{m_r - n_r \log_3 \frac{1}{2}}\right) \\ &= 3^{m_r - n_r \log_3 \frac{1}{2}} \le 3^{\frac{1}{n_r}}. \end{aligned}$$

Thus, by choosing r large enough, we may make $|2^{n_r} 3^{m_r} \alpha - \alpha| < \delta$ for any $\delta > 0$. So, $2^{n_r} 3^{m_r} \alpha \to \alpha$ as $r \to \infty$, and since F is continuous:

$$F(2^{n_r} 3^{m_r} \alpha) = \left(\frac{2}{c_0}\right)^{n_r} \left(\frac{3}{d_0}\right)^{m_r} F(\alpha) \to F(\alpha).$$

However, we know $F(\alpha) \neq 0$ so:

$$\lim_{r\to\infty} \left(\frac{2}{c_0}\right)^{n_r} \left(\frac{3}{d_0}\right)^{m_r} = 1.$$

Filling in what we know about n_r, m_r:

$$\begin{aligned}
1 &= \lim_{r\to\infty} \left(\frac{2}{c_0}\right)^{n_r} \left(\frac{3}{d_0}\right)^{m_r} \\
&= \lim_{r\to\infty} \left(\frac{2}{c_0}\right)^{n_r} \left(\frac{3}{d_0}\right)^{n_r \log_3 \frac{1}{2}} \left(\frac{3}{d_0}\right)^{m_r - n_r \log_3 \frac{1}{2}} \\
&= \lim_{r\to\infty} \left(\frac{2.3^{\log_3 \frac{1}{2}}}{c_0 d_0^{\log_3 \frac{1}{2}}}\right)^{n_r} \left(\frac{3}{d_0}\right)^{m_r - n_r \log_3 \frac{1}{2}} \\
&= \lim_{r\to\infty} \left(\frac{1}{c_0 d_0^{\log_3 \frac{1}{2}}}\right)^{n_r} \lim_{r\to\infty} \left(\frac{3}{d_0}\right)^{m_r - n_r \log_3 \frac{1}{2}} \\
&= \lim_{r\to\infty} \left(\frac{1}{c_0 d_0^{\log_3 \frac{1}{2}}}\right)^{n_r} .1
\end{aligned}$$

So $c_0 d_0^{\log_3 \frac{1}{2}} = 1$, or equivalently $\log_2 c_0 = \log_3 d_0 = -\beta$.

Returning to a general pair n, m we can now fill in this β.

$$\begin{aligned}
F(2^n 3^m \alpha) &= \left(\frac{2}{c_0}\right)^{n} \left(\frac{3}{d_0}\right)^{m} F(\alpha) \\
&= \left(\frac{2}{2^{-\beta}}\right)^{n} \left(\frac{3}{3^{-\beta}}\right)^{m} F(\alpha) \\
&= \left(2^{n(\beta+1)} 3^{m(\beta+1)}\right) F(\alpha) \\
&= (2^n 3^m)^{\beta+1} F(\alpha).
\end{aligned}$$

Note that this F agrees with $x^{\beta+1} \frac{F(\alpha)}{\alpha^{\beta+1}}$ on the dense subset $\{2^n 3^m \alpha\} \cap [0, \epsilon)$. As both are continuous they must be the same, and so:

$$f(x) = \frac{d}{dx} F(x) = (\beta+1) x^{\beta} \frac{F(\alpha)}{\alpha^{\beta+1}}.$$

This is easy to extend to $[0, 1)$ using $f(x) = d_0 f(3x)$. ■

It is relatively easy to extend Theorem 4.13 to include the case where an $s \in \mathbb{R}^+$ can be found such that $x^s f(x)$ is integrable on $[0, \epsilon)$.

Corollary 4.14. *If f is as in the statement of Theorem 4.13, then f has form:*

$$f(x) = \sum_{l=0}^{n} a_l (x - l)^{\beta},$$

on $[n, n+1)$, *where* $a_0 = \gamma$ *and for* $l > 0$*:*

$$a_l = \begin{cases} a_{\frac{l}{2}} - \sum_{k=1}^{l} \frac{c_k a_{l-k}}{c_0} & \text{if } l = 0 \mod 2 \\ - \sum_{k=1}^{l} \frac{c_k a_{l-k}}{c_0} & \text{otherwise} \end{cases}.$$

Proof. From Theorem 4.13, we know that $f(x) = \gamma x^\beta$ on $[0, 1)$. Note that the expression given for a_l holds for a_0 as the sum is empty. Now suppose our result is true for $0, 1, 2, \dots n-1$, then applying the forward substitution formula from page 71:

$$\begin{aligned} f(x) &= \frac{f\left(\frac{x}{2}\right) - \sum_{k=1}^{n} c_k f(x-k)}{c_0} \\ &= \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{a_l}{c_0} \left(\frac{x}{2} - l\right)^\beta - \sum_{k=1}^{n} \sum_{l=0}^{n-k} \frac{c_k a_l}{c_0} (x-k-l)^\beta \\ &= \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{a_l}{c_0 2^\beta} (x-2l)^\beta - \sum_{k=1}^{n} \sum_{l'=k}^{n} \frac{c_k a_{l'-k}}{c_0} (x-l')^\beta \qquad \text{using } l' = k + l \\ &= \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} a_l (x-2l)^\beta - \sum_{l'=1}^{n} \sum_{k=1}^{l'} \frac{c_k a_{l'-k}}{c_0} (x-l')^\beta, \end{aligned}$$

using $\beta = -\log_2 c_0$ and reordering the sums.

The largest shift in the first term is $2\left\lfloor \frac{n}{2} \right\rfloor \leq n$, and the largest shift in the second term is clearly n. For any $0 \leq l \leq n$ we look for the coefficient of $(x-l)^\beta$ in the above expression and get:

$$a_{\frac{l}{2}} - \sum_{k=1}^{l} \frac{c_k a_{l-k}}{c_0} = a_l,$$

if l is a multiple of 2, or:

$$- \sum_{k=1}^{l} \frac{c_k a_{l-k}}{c_0} = a_l,$$

otherwise. ■

Corollary 4.14 tells us that solutions to two such dilation equations must be of the form:

$$f(x) = \sum_{l} a_l (x-l)_+^\beta,$$

where x_+ is x if $x > 0$ and zero otherwise. Following much the same argument as Corollary 4.14 we can show that given any reasonable choices of β and a_n we can work backwards and produce a dilation equation which f will satisfy.

Lemma 4.15. *Given a_n and β we define:*

$$f(x) = \sum_{l=0} a_l (x-l)_+^\beta.$$

If $a_0 \neq 0$, then f satisfies the dilation equation $f(x) = \sum_{k=0} c_k f(2x-k)$, where:

$$c_l = \begin{cases} \frac{-c_0 a_{\frac{l}{2}} - \sum_{k=0}^{l-1} c_k a_{l-k}}{a_0} & l = 0 \mod 2 \\ \frac{\sum_{k=0}^{l-1} c_k a_{l-k}}{a_0} & l \neq 0 \mod 2 \end{cases}.$$

Similarly we could produce a set of coefficients for a scale three dilation equation. This means that when f is of this form then f is 2- and 3-refinable and satisfies the conditions of Theorem 4.13. Thus, the form of f given in Corollary 4.14 is both necessary and sufficient for f to be 2- and 3-refinable (with $c_k = 0$ for $k < 0$).

However, in many cases this solution need not be compactly-supported nor need it have only a finite number of non-zero c_k. For example, take f to be a non-palindromic characteristic function. We see from Lemma 4.9 that there must be an infinite number of non-zero c_k. Let us examine what further conditions compact support imposes.

Corollary 4.16. *If $f \neq 0$ is as in the statement of Theorem 4.13 and f is compactly-supported, then $\beta \in \mathbb{N}$.*

Proof. As f is compactly-supported we can find $n > 0$ so that f is zero on $[n, n+1)$. By Corollary 4.14 we know that:

$$\begin{aligned} f(x) &= \sum_{l=0}^{n} a_l (x-l)^\beta, \\ f'(x) &= \sum_{l=0}^{n} \beta a_l (x-l)^{\beta-1}, \\ f^{(m)}(x) &= \sum_{l=0}^{n} \beta(\beta-1)\ldots(\beta-m+1) a_l (x-l)^{\beta-m}. \end{aligned}$$

Each of these must be zero on $(n, n+1)$, so:

$$\begin{pmatrix} x^\beta & (x-1)^\beta & \dots & (x-n)^\beta \\ \beta x^{\beta-1} & \beta(x-1)^{\beta-1} & \dots & \beta(x-n)^{\beta-1} \\ \vdots & & & \vdots \\ \beta\dots(\beta-n+1)x^{\beta-n} & \beta\dots(\beta-n+1)(x-1)^{\beta-n} & \dots & \beta\dots(\beta-n+1)(x-n)^{\beta-n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

As f is non-zero we know $a_0 \neq 0$, so the determinant of this matrix must be zero.

$$\begin{aligned} 0 &= [\beta][\beta(\beta-1)]\dots[\beta\dots(\beta-n+1)] \begin{vmatrix} x^\beta & (x-1)^\beta & \dots & (x-n)^\beta \\ x^{\beta-1} & (x-1)^{\beta-1} & \dots & (x-n)^{\beta-1} \\ \vdots & & & \vdots \\ x^{\beta-n} & (x-1)^{\beta-n} & \dots & (x-n)^{\beta-n} \end{vmatrix} \\ &= \beta^n(\beta-1)^{n-1}\dots(\beta-n+1)^1 x^{\beta-n}(x-1)^{\beta-n}\dots(x-n)^{\beta-n} \begin{vmatrix} x^n & (x-1)^n & \dots & (x-n)^n \\ x^{n-1} & (x-1)^{n-1} & \dots & (x-n)^{n-1} \\ \vdots & & & \vdots \\ x^0 & (x-1)^0 & \dots & (x-n)^0 \end{vmatrix} \\ &= \beta^n(\beta-1)^{n-1}\dots(\beta-n+1)^1 x^{\beta-n}(x-1)^{\beta-n}\dots(x-n)^{\beta-n} \prod_{i<j} \left((x-j)-(x-i)\right). \end{aligned}$$

We conclude that either $\beta - j = 0$ or $x - j = 0$. We can choose $x \in (n, n+1)$ so that the second is not possible. thus $\beta \in \{0, \dots, n\}$. ■

So, in the compactly-supported case we now know that f must be of the form:

$$f(x) = \sum_l a_l (x-l)_+^\beta,$$

where $\beta \in \mathbb{N}$. This means that f is a linear combination of B-splines, which form a basis for piecewise polynomial functions with knots at given positions (see [13] for more details). A very similar result was proved independently in [8], where the authors show that if m and n are independent scales and if f is m- and n-refinable, linearly independent from its translates, and compactly-supported, then it must actually be a B-spline.

Apart from the requirement that β be an integer and the linear constraints which require f to be zero in the long run, the coefficients a_n are fairly arbitrary. Indeed, if we choose $a_0, \dots, a_{L-1}$ arbitrarily, then we may find $a_L, \dots, a_{L+\beta}$ to make f compactly-supported.

Lemma 4.17. *Suppose we have chosen $a_0, \dots, a_{L-1}$ then we may choose $a_L, \dots, a_{L+\beta}$ such* *that*

$$\sum_{l=0}^{L+\beta} a_l(x-l)^\beta = 0.$$

Proof. Note that what we are showing is:

$$\sum_{l=L}^{L+\beta} a_l(x-l)^\beta = -\sum_{l=0}^{L-1} a_l(x-l)^\beta.$$

We will actually show that if the RHS is any $g(x)$, an arbitrary polynomial of degree β, then we may find suitable a_l. Both sides are analytic and vanish when differentiated more than β times, so we only have to show that the first β derivatives agree. We proceed as before.

$$\begin{pmatrix} (x-L)^\beta & (x-L-1)^\beta & \dots & (x-L-\beta)^\beta \\ \beta(x-L)^{\beta-1} & \beta(x-L-1)^{\beta-1} & \dots & \beta(x-L-\beta)^{\beta-1} \\ \vdots & & & \vdots \\ \beta\dots 1(x-L)^0 & \beta\dots 1(x-L-1)^0 & \dots & \beta\dots 1(x-L-\beta)^0 \end{pmatrix} \begin{pmatrix} a_L \\ a_{L+1} \\ \vdots \\ a_{L+\beta} \end{pmatrix} = \begin{pmatrix} g(x) \\ g'(x) \\ \vdots \\ g^{(\beta)}(x) \end{pmatrix}$$

This can be solved as long as the determinant is non-zero. We evaluate it at $x = L + \beta$.

$$\beta^\beta(\beta-1)^{\beta-1}\dots 1^1 \begin{vmatrix} \beta^\beta & (\beta-1)^\beta & \dots & 0^\beta \\ \beta^{\beta-1} & (\beta-1)^{\beta-1} & \dots & 0^{\beta-1} \\ \vdots & & & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix} = \beta^\beta(\beta-1)^{\beta-1}\dots 1^1 \prod_{j<i}(j-i) \neq 0.$$

■

We can extend this result into $L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ to some degree. In the following $\mathcal{H}$ is the Hilbert transform[§].

Theorem 4.18. *Suppose $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ is a non-zero compactly-supported 2- and 3-refinable function. Then the $L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ solutions space of the corresponding scale 2 and 3 equations is the space spanned by f and $\mathcal{H}f$.*

[§] The Hilbert transform $\mathcal{H}$ of a function f can be defined as

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\int \frac{f(t)}{x-t}\,dt.$$

This corresponds to multiplying the Fourier transform by $-i\,\mathrm{sign}(\omega)$. For more details see a book on Fourier analysis such as [40].

Proof. Note that as a spline supported on a compact set f must be bounded, so actually $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. As the Fourier transform of a compactly-supported $L^2(\mathbb{R})$ function $\hat{f}$ will be analytic. Consequently $\hat{f}$ will be a maximal solution of both the scale 2 and scale 3 equations. Thus any solution to both equations can be written:

$$\begin{aligned} \hat{g}(\omega) &= \hat{f}(\omega)\pi_2(\omega), \\ \hat{g}(\omega) &= \hat{f}(\omega)\pi_3(\omega), \end{aligned}$$

where $\pi_2(\omega) = \pi_2(2\omega)$ and $\pi_3(\omega) = \pi_3(3\omega)$. So whenever $\hat{f}(\omega) \neq 0$:

$$\frac{\hat{g}(\omega)}{\hat{f}(\omega)} = \pi_2(\omega) = \pi_3(\omega).$$

Next consider the case where $\hat{f}(0) \neq 0$. Then, as $\hat{f}$ is continuous, there is some interval $[-\epsilon, \epsilon]$ around zero for which $\hat{f}(\omega) \neq 0$. In this region $\pi_2 = \pi_3$ must be in $L^1([-\epsilon, \epsilon])$, so we may integrate. Choosing $0 < \alpha < \epsilon$:

$$\begin{aligned} F(\alpha) &= \int_0^\alpha \pi_2(\omega)\, d\omega \\ &= \int_0^\alpha \pi_2(2^{-m}3^{-n}\omega)\, d\omega \\ &= \int_0^{2^{-m}3^{-n}\alpha} \pi_2(\omega)\, \frac{d\omega}{2^{-m}3^{-n}} \\ &= 2^m 3^n F(2^{-m}3^{-n}\alpha). \end{aligned}$$

As F is continuous and $2^m 3^n$ is dense, this determines $F(x) = \gamma_+ x$ on $[0, \alpha]$ (see Theorem 4.13 for complete details of the same argument). Thus we see that π_2 is constant almost everywhere on $[0, \alpha]$ and so on $\mathbb{R}^+$. Likewise we may show that π_2 is a possibly different constant on $\mathbb{R}^-$. Thus π_2 can be written as a sum of $\chi_{\mathbb{R}}(\omega)$ and $\operatorname{sign}(\omega)$ and so g can be written as a sum of f and $\mathcal{H}f$.

If $\hat{f}(0) = 0$, then we know it is non-zero on some neighbourhood of the origin, since otherwise its analyticity would force it to be zero. We just perform the same argument as above using:

$$\frac{\hat{g}(\omega)}{\omega^{-r}\hat{f}(\omega)},$$

where r is the order of $\hat{f}$'s zero at the origin. ■

4.6 Smoothness and boundedness

Many interesting properties of functions are linear, or perhaps they are interesting because they are linear. Looking back at page 71 we see that our forward substitution formula for a compactly-supported function satisfying a dilation equation produces the function on the next interval by adding the leftmost part of the function to a sum. Thus properties present on the leftmost interval which are preserved by summing will be present on all intervals.

For instance, consider continuity. If f is continuous on $(0, 1)$, then it will be continuous on $(1, 2)$ as f on $(1, 2)$ is given by:

$$f(x) = \frac{f\left(\frac{x}{2}\right) - c_1 f(x-1)}{c_0}.$$

If f is continuous at 0, then $f(0) = 0$, because we have translated f so that $f(x) = 0$ when $x < 0$. Now f continuous on $[0, 1)$ gives f continuous on $[1, 2)$ for a combination of two reasons:

- the term $f(x-1)$ grows continuously from 0 at 1 as $f(x)$ grows continuously from 0 at 0, so the sum on $[0, 1)$ makes a smooth transition to the sum on $[1, 2)$,
- the remaining terms (in this case just $f(x/2)$) are continuous where they are being evaluated (that is to the left of 1 where we know f is continuous).

So the continuity of f on $[0, 1)$ is necessary and sufficient for the continuity of f as a whole.

Similar arguments can be applied to differentiability and other sorts of smoothness.

Given that the dilation equation on $[0, 1)$ has the simple form $f(x) = c_0 f(2x)$ it would seem profitable to determine the implications of this combined with smoothness.

Lemma 4.19. *Suppose we are given compactly-supported f which satisfies a dilation equation with $c_k = 0$ when $k < 0$. Then, if $f \in C^1$ we have $|c_0| < \frac{1}{2}$ or f is zero.*

Proof. We look at $f'(0)$, which must be zero as f' is continuous:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{m \to \infty} \frac{f\left(\frac{x}{2^m}\right)}{\frac{x}{2^m}} = \frac{1}{x} \lim_{m \to \infty} 2^m f\left(\frac{x}{2^m}\right).$$

Now using $f(x) = c_0 f(2x)$:

$$f'(0) = \frac{1}{x} \lim_{m \to \infty} 2^m c_0^m f(x) = \frac{f(x)}{x} \lim_{m \to \infty} 2^m c_0^m.$$

Choosing x so that $f(x)$ is not zero we see that the limit must be 0. Thus, $|2c_0| < 1$ as required. ■

Corollary 4.20. *Suppose we are given compactly-supported f which satisfies a dilation equation with $c_k = 0$ when $k < 0$. Then if $f \in C^n$ we have $|c_0| < 2^{-n}$ or f is zero.*

Proof. Take f and differentiate it $n-1$ times to get g. Then g is in C^1 and satisfies a dilation equation whose coefficients are 2^{n-1} times the c_k. Thus $|2^{n-1}c_0| < \frac{1}{2}$, so $|c_0| < 2^{-n}$. ∎

Corollary 4.21. *There are no non-zero compactly-supported C^∞ solutions to finite dilation equations.*

Proof. If there were, then they would satisfy a dilation equation with $|c_0| < 2^{-n}$ for all n. But this would mean that $c_0 = 0$ which means f must be zero. ∎

Note that this result does not extend to the non-compactly-supported case. For example $f(x) = \sin(\pi x)/\pi x$ is a non-compactly-supported analytic function satisfying $f(x) = \sum c_k f(2x-k)$ where:

$$c_n = \frac{\sin \pi \frac{n}{2}}{\pi \frac{n}{2}} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \quad \text{is even} \\ \frac{2}{\pi n} & n = 1 \mod 4 \\ -\frac{2}{\pi n} & n = 3 \mod 4 \end{cases}.$$

Corollary 4.20 actually extends nicely to C^{n+s} where $n \in \mathbb{N}$ and $0 < s < 1$ is the Hölder exponent of continuity of the n^{th} derivative.

Corollary 4.22. *Suppose we are given compactly-supported f which satisfies a dilation equation with $c_k = 0$ when $k < 0$. Then if $f \in C^{n+s}$ we have $|c_0| \leq 2^{-(n+s)}$ or f is zero.*

Proof. We simply examine the Hölder exponent of continuity of the n^{th} derivative at 0. This means that for any h:

$$|f^{(n)}(h) - f(0)| < k|h-0|^s$$

for some fixed k. Again we examine this at $x2^{-m}$ and get:

$$k > \frac{|f^{(n)}(x2^{-m})|}{(x2^{-m})^s} = \frac{|2^n c_0|^m |f^{(n)}(x)|}{(x2^{-m})^s} = \frac{|f^{(n)}(x)|}{x^s}|2^{n+s}c_0|^m.$$

For this to be bounded as $m \to \infty$ we must have $|2^{n+s}c_0| \leq 1$. ∎

This upper bound on the smoothness, based on $-\log_2 |c_0|$, can be the correct bound. Taking m copies of $\chi_{[0,1)}$ and forming their convolution you get the B-spline of order

	Coefficents				TLoW			SRCfSS		
N	$\|c_0\|$	$-\log_2\|c_0\|$	$\|c_{2N-1}\|$	$-\log_2\|c_{2N-1}\|$	p226	p232	p239	r_{20}	r_∞	UB
2	0.683	0.550	0.183	2.449	0.339	0.500	0.550	0.550	0.550	0.550
3	0.470	1.0878	0.04981	4.3272	0.636	0.915	1.0878	1.0831	1.0878	1.0878
4	0.325	1.6179	0.01498	6.0602	0.913	1.275	1.6179	1.6066	1.6179	1.6179
5	0.2264	2.1429	0.00472	7.7278	1.177	1.596		1.9424		1.9689
6	0.1577	2.6644	0.00152	9.3584	1.432	1.888		2.1637		2.1891
7	0.1109	3.1831	0.00050	10.9651	1.682	2.158		2.4348		2.4604

Various estimates of $n + s$ for the Daubechies's family of extremal phase wavelets (see [10] page 195). The crude upper bounds derived in Corollary 4.22 are shown as $-\log_2|c_0|$ and $-\log_2|c_{2N-1}|$. The values in the TLoW column are lower bounds taken from various tables in [10]. The column labeled SRCfSS is taken from [37] (page 1570) and shows lower bounds (r_{20} and r_∞) and an upper bound.

Figure 4.7: Estimates of smoothness for Daubechies's extremal phase wavelets

$m - 1$. This function is in C^{n+s} for $n + s < m - 1$ and satisfies a dilation equation with $c_0 = 2^{-(m-1)}$. In practice, examining tables of Hölder exponents and coefficients in [10] and [37] shows that this bound on smoothness can be sharp even in more complicated situations (Figure 4.7).

A similar argument can be used to show that if f is bounded on $[0, 1)$, then it will be bounded on any $[0, n)$. Note that for f to be bounded and non-zero on $[0, 1)$ we need $|c_0| \leq 1$. This is because f will behave a bit like x^β, where $\beta = -\log_2|c_0|$, and we need $\beta \geq 0$.

The following gives a bound on how fast f can grow in terms of its coefficients.

Theorem 4.23. *Suppose a compactly-supported f satisfies a dilation equation with $c_k = 0$ when $k < 0$, then if f is bounded on $[0, 1)$ we will find f is bounded by $C(M + 1)^n$ on $[n, n + 1)$ where $M = \max_{k>1}|c_k/c_0|$.*

Proof. First we show that f is bounded by $C_n(M+1)^n$ where C_n is an increasing sequence. Then we show that the C_n are bounded.

Choose $C_0 = \sup_{x\in[0,1)}|f(x)|$. Now we use induction on n, assuming that f is bounded by $C_k(M+1)^k$ for $k < n$ and the C_k are increasing. From our forward substitution formula on page 71 we see:

$$\begin{aligned}
|f(x)| &\leq \frac{\left|f\left(\frac{x}{2}\right)\right| + \sum_{k=1}^{n} |c_k||f(x-k)|}{|c_0|} \\
&\leq \frac{\left|f\left(\frac{x}{2}\right)\right|}{|c_0|} + M \sum_{k=1}^{n} |f(x-k)| \\
&\leq \frac{C_{\lfloor \frac{n}{2} \rfloor}(M+1)^{\lfloor \frac{n}{2} \rfloor}}{|c_0|} + M \sum_{k=1}^{n} C_{n-k}(M+1)^{n-k} \\
&\leq C_{n-1}\left(\frac{(M+1)^{\lfloor \frac{n}{2} \rfloor}}{|c_0|} + M \sum_{k=1}^{n} (M+1)^{n-k}\right) \\
&\leq C_{n-1}\left(\frac{(M+1)^{\frac{n}{2}}}{|c_0|} + M \frac{(M+1)^n - 1}{M}\right) \\
&\leq C_{n-1}\left(\frac{(M+1)^{\frac{n}{2}}}{|c_0|} + (M+1)^n\right).
\end{aligned}$$

Now we define C_n to be:

$$C_n = C_{n-1}\left(\frac{(M+1)^{-\frac{n}{2}}}{|c_0|} + 1\right).$$

and substituting this into the above we get $|f(x)| \leq C_n(M+1)^n$. Clearly C_n is bigger than C_{n-1} as $M \geq 0$.

We must show that these C_n are bounded by some number C. We see that:

$$\begin{aligned}
C_n &= C_0 \prod_{r=1}^{n} 1 + \frac{(M+1)^{-\frac{r}{2}}}{|c_0|} \\
&\leq C_0 \prod_{r=1}^{n} e^{\frac{(M+1)^{-\frac{r}{2}}}{|c_0|}} \\
&\leq C_0 e^{\frac{\sum_{r=1}^{n} (M+1)^{-\frac{r}{2}}}{|c_0|}} \\
&\leq C_0 e^{\frac{\sum_{r=1}^{\infty} (M+1)^{-\frac{r}{2}}}{|c_0|}},
\end{aligned}$$

as required. ■

Examples which get close to this bound are easy to produce. By taking $0 < c_0 \leq 1$ and then taking $c_k = -Mc_0$ for as many $k > 0$ as desired, many of the $\leq$ become $=$ and f achieves exponential behavior.

4.7 Conclusion

This chapter took a look at the ends of a solution to a dilation equation. This has provided us with rather a lot of information about a few different types of dilation equation.

Our first concern was refinable characteristic functions. After extracting some basic results we found that we could represent refinable functions which were constant on the intervals $[n, n+1)$ as polynomials with certain properties. These results generalise in the obvious way to scales other than 2. Moving them to $\mathbb{R}^n$ is more tricky, as polynomial factorisation is not straightforward in more than one variable, though some generalised form of polynomial division such as Groebner bases might make this possible.

For scales other than 2 we saw that refinable characteristic functions which were not a union of intervals like $[n, n+1)$ existed. It seems likely that for scale 2 the only refinable characteristic functions are of this simple form — it would be interesting to show this and understand why scale 2 seems different. It seems plausible that number of 'families' of refinable characteristic function increases with scale and that each family is a union of translates of some basic tile, however we have no evidence to back this up. We have tried analysing refinable characteristic functions in other ways, such as using:

$$f \text{ characteristic function } \Leftrightarrow f = f.f \Leftrightarrow \hat{f} = c\hat{f} * \hat{f},$$

but useful results are not forthcoming.

Analysis of these refinable characteristic functions may prove useful in the study of smooth higher dimensional wavelets. In [3] a tile is chosen and used in the analysis — knowing what families of tiles are available might allow more refined analysis.

The next area this chapter tackled was that of 2- and 3-refinable functions. In the case of a compactly-supported solution we know exactly what form it must have. These results extend to any two independent¶ scales n and m. It would be nice to extend these results to non-compactly-supported functions, where perhaps the only n- and m-refinable functions have Fourier transform of the form $q(\omega)\omega^{-\beta}$ where q is a trigonometric polynomial or 2π periodic. The case of dependent scales is considered in [46].

Finally, the chapter had a go at getting bounds on smoothness and growth rates by examining the ends of the solutions. These bounds are moderately accurate, but probably not that useful, as coefficients are usually chosen with such care as to avoid the cases where these results are sharp. It would be interesting to see the asymptotic behavior of $-\log_2 |c_0|$ for the family of extremal phase wavelets and how far from the estimate of known limit of $C^{0.2N}$ it is.

Overall, this 'look at an end' method seems to be successful. One major problem is that it is unclear how to generalise this to $\mathbb{R}^n$. First, there is not an obvious order on $\mathbb{R}^n$ in the way there is on $\mathbb{R}$ and so the definition of 'an end' is unclear. The likely orders which one might try, lexicographic ordering on some basis, are not preserved by dilation matrices A in the same way multiplying by a positive scalar preserves the order on $\mathbb{R}$.

¶Here independent means the multiplicative group generated by n and m is dense in $\mathbb{R}^+$.

Often in the statement of results in this chapter we have said ‘compactly-supported’ where we could have used ‘supported on a half-line’. In the case where a function is supported on a half-line we know it is safe to use a transform such as the Laplace transform. It is unclear if there is a link here, but it might warrant some investigation.

Chapter 5

Miscellany

5.1 Introduction

This chapter contains a selection of smaller results relating to dilation equations. Section 5.2 arose as a problem which should have a simple solution. It shows how to solve for polynomial solutions of dilation equations.

Section 5.3 discusses ways of combining dilation equations and their solutions. These observations were made while producing a greater catalogue of examples to consider while investigating results.

Figure 2.2 inspired Section 5.4. Here we ask, "When can an expanding matrix A have a simple tile?".

5.2 Polynomial solutions to dilation equations

Suppose we have a dilation equation:

$$f(x) = \sum_k c_k f(\alpha x + \beta_k),$$

and that we are looking for polynomial solutions. Let the degree of some solution be n. Differentiating n times we get:

$$f^{(n)}(x) = \alpha^n \sum_k c_k f^{(n)}(\alpha x + \beta_k).$$

However, $f^{(n)}$ must be a constant, non-zero function, so we can divide through by it to get:

$$1 = \alpha^n \sum_k c_k.$$

This means that providing $|\alpha| \neq 1$ the degree of the polynomial solution is determined by the dilation equation. Indeed it is only worth looking for polynomial solutions when:

$$n = \frac{-\log|\sigma|}{\log|\alpha|}$$

exists and is a positive integer (here σ is $\sum_k c_k$). We can then easily check if $\alpha^n \sigma = 1$.

Now that we have found the degree of this polynomial solution we can proceed to try to find the coefficients. As the solutions to any dilation equation form a vector space we can clearly scale any solution. For this reason we may as well search for monic polynomials as solutions. We write our solution f as:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_n = 1$ and substitute it into the dilation equation, match up the coefficients of x^m for each m and solve the resulting linear equations. However, it is not clear that these equations will have a solution, as they will be quite complicated.

A more sensible approach is to differentiate m times and look at the constant term each time. If we start with $f^{(n-1)}$ the differentiated dilation equation will only have a_n and a_{n-1} in it, and as we know a_n we can easily find a_{n-1}. Then we look at $f^{(n-2)}$ for a_{n-2}, and so on.

It is still not clear that this back substitution will be successful, so let us carry out the calculation to see if it is possible.

First we expand $f^{(m)}$ as a polynomial to get:

$$\sum_{l=0}^{n-m} \frac{(m+l)!}{l!} a_{m+l} x^l.$$

We can fill this into our dilation equation, and then expand using the binomial theorem.

$$\begin{aligned}
f^{(m)}(x) &= \alpha^m \sum_k c_k f^{(m)}(\alpha x + \beta_k) \\
\sum_{l=0}^{n-m} \frac{(m+l)!}{l!} a_{m+l} x^l &= \alpha^m \sum_k c_k \sum_{l=0}^{n-m} \frac{(m+l)!}{l!} a_{m+l} (\alpha x + \beta_k)^l \\
&= \alpha^m \sum_k c_k \sum_{l=0}^{n-m} \frac{(m+l)!}{l!} a_{m+l} \sum_{j=0}^{l} \alpha^j x^j \frac{l!}{j!(l-j)!} \beta_k^{l-j} \\
&= \alpha^m \sum_k c_k \sum_{l=0}^{n-m} \sum_{j=0}^{l} \frac{(m+l)!}{j!(l-j)!} a_{m+l} \alpha^j x^j \beta_k^{l-j}.
\end{aligned}$$

Now we match the coefficients of x^0 to get:

$$m!a_m = \alpha^m \sum_k c_k \sum_{l=0}^{n-m} \frac{(m+l)!}{l!} a_{m+l} \beta_k^l.$$

For $m = n$ we know that we just get the condition $\alpha^n \sigma = 1$. Otherwise we can take the term containing a_m from the right-hand side, and then solve for a_m.

$$\begin{aligned} m!a_m - \alpha^m \sum_k c_k \beta_k^0 m! a_m &= \alpha^m \sum_k c_k \sum_{l=1}^{n-m} \frac{(m+l)!}{l!} a_{m+l} \beta_k^l, \\ a_m &= \frac{\alpha^m \sum_k c_k \sum_{l=1}^{n-m} \frac{(m+l)!}{l!} a_{m+l} \beta_k^l}{m! \left(1 - \alpha^m \sum_k c_k\right)} \end{aligned}$$

If $|\alpha| \neq 1$, then, as α is a solution of $1 - \alpha^n \sum_k c_k = 0$, we cannot have $1 - \alpha^m \sum_k c_k$ being zero for $m \neq n$. This means that the denominator of this fraction is not zero for any $0 \leq m < n$, which in turn means that this back substitution scheme will work whenever $|\alpha| \neq 1$.

The great weakness of this method is that it 'starts' at the highest power of x, and so provides no hint as how to solve for power series solutions of dilation equations.

This method is easily implemented on computer, and testing it highlighted a minor mistake on page 1395 of [11] where:

$$f(x) = \frac{1}{12} f(2x) + \frac{1}{6} \left[f(2x+1) + f(2x-1) \right] - \frac{1}{12} \left[f(2x+2) + f(2x-2) \right]$$

is said to admit $f(x) = x^2$ as a solution. In fact, the solution is $f(x) = x^2 - 4/9$.

5.3 New solutions from old

When you have some dilation equations and some solutions to those equations, there are various ways in which you can produce more solutions to more dilation equations. A simple way to do this is: if you have a collection of solutions which all satisfy a collection of dilation equations, then linear combinations of the solutions will still satisfy the same collection of equations.

Another method which is well known is to differentiate. If f satisfies:

$$f(x) = \sum_k c_k f(2x+k),$$

then f' satisfies:

$$f'(x) = \sum_k 2c_k f'(2x + k).$$

This time the equation has changed as well as the solution. This is actually more interesting in $\mathbb{R}^n$, as the new equation produced is a vector dilation equation:

$$f(\vec{x}) = \sum_{\vec{k}} c_{\vec{k}} f(A\vec{x} + \vec{k}),$$

and on differentiation with respect to all variables becomes:

$$\begin{aligned}
\vec{\nabla} f(\vec{x}) &= \vec{\nabla}_{\vec{x}} f(\vec{x}) \\
&= \vec{\nabla}_{\vec{x}} \sum_{\vec{k}} c_{\vec{k}} f(A\vec{x} + \vec{k}), \\
&= \sum_{\vec{k}} c_{\vec{k}} \vec{\nabla}_{\vec{x}} f(A\vec{x} + \vec{k}), \\
&= \sum_{\vec{k}} c_{\vec{k}} A^T (\vec{\nabla} f)(A\vec{x} + \vec{k}),
\end{aligned}$$

which is a vector refinement equation with coefficients $c_{\vec{k}} A^T$.

A way to produce a new equation without altering the solution is to iterate the dilation equation. For example, if we know f is a solution of:

$$f(x) = f(2x) + f(2x - 1),$$

then we may simply iterate this relation to get:

$$\begin{aligned}
f(x) &= f(2(2x)) + f(2(2x) - 1) + f(2(2x - 1)) + f(2(2x - 1) - 1) \\
&= f(4x) + f(4x - 1) + f(4x - 2) + f(4x - 3).
\end{aligned}$$

Same function, different dilation equation. More generally, what's going on here is:

$$\begin{aligned}
f &= \mathcal{D}_2 P(\mathcal{T}_1) f \\
f &= \mathcal{D}_2 P(\mathcal{T}_1) \mathcal{D}_2 P(\mathcal{T}_1) f \\
f &= \mathcal{D}_2 \mathcal{D}_2 P({\mathcal{T}_1}^2) P(\mathcal{T}_1) f
\end{aligned}$$

where P is some polynomial.

There is a somewhat more interesting way to combine different equations. Suppose we have two dilation equations of the same scale, each with a solution.

$$
\begin{aligned}
f &= \mathcal{D}_2 P(\mathcal{T}_1) f \\
g &= \mathcal{D}_2 Q(\mathcal{T}_1) g.
\end{aligned}
$$

Then by taking the Fourier transform we get:

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{P(e^{\frac{-i\omega}{2}})}{2}\hat{f}(\frac{\omega}{2}) \\
\hat{g}(\omega) &= \frac{Q(e^{\frac{-i\omega}{2}})}{2}\hat{g}(\frac{\omega}{2}).
\end{aligned}
$$

First we multiply these equations and take the inverse Fourier transform:

$$
\begin{aligned}
\hat{f}\hat{g}(\omega) &= \frac{1}{2}\frac{(PQ)(e^{\frac{-i\omega}{2}})}{2}(\hat{f}\hat{g})(\frac{\omega}{2}) \\
(f * g) &= \frac{1}{2}\mathcal{D}_2(PQ)(\mathcal{T}_1)(f * g) \\
(f * g) &= \frac{1}{2}\mathcal{D}_2 P(\mathcal{T}_1) Q(\mathcal{T}_1)(f * g).
\end{aligned}
$$

So the function $f * g$ satisfies a dilation equation of the same scale whose coefficients are a polynomial-style product of the original dilation equation. This is a nice result which has been known for some time (see for example Lemma 2.2 of [34]).

The fact that differentiation and convolution preserve the scale of the equation motivated a suggestion that all 2- and 3-refinable functions might arise from $\chi_{[0,1)}$ via combinations of convolution and differentiation. This is shown to be close to the truth in Section 4.5, as B-splines are produced by taking convolutions of $\chi_{[0,1)}$ with itself.

This method could be used to try to factorise solutions to dilation equations, as the Fundamental Theorem of Algebra will always allow us to factorise the polynomial P from the dilation equation. However, this procedure will usually lead to solutions outside $L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ quite quickly.

A less well known (maybe new?) trick is to produce solutions to an equation of scale $\sqrt[m]{\alpha}$ from a solution of an equation of scale α. For example, suppose we want to produce a $\sqrt{2}$-refinable function. We could begin with $\chi_{[0,1)}$ which satisfies:

$$f(x) = f(2x) + f(2x - 1),$$

or in the frequency domain:

$$\begin{aligned}\hat{f}(\omega) &= \frac{1+e^{i\frac{\omega}{2}}}{2}\hat{f}(\frac{\omega}{2}) \\ &= p(\frac{\omega}{2})\hat{f}(\frac{\omega}{2}) \\ &= \prod_{j=1}^{\infty} p(2^{-j}\omega).\end{aligned}$$

Now consider trying to solve:

$$g(x) = \frac{1}{\sqrt{2}}\left(g(\sqrt{2}x) + g(\sqrt{2}x - 1)\right).$$

This equation is produced from the equation for f above by replacing the scale 2 with $\sqrt{2}$ and then normalising so that the coefficients of the equation sum to the new scale. Again we take the Fourier transform:

$$\begin{aligned}\hat{g}(\omega) &= \frac{1}{\sqrt{2}}\frac{1+e^{\frac{i\omega}{\sqrt{2}}}}{\sqrt{2}}\hat{g}(\frac{\omega}{\sqrt{2}}) \\ &= p(\frac{\omega}{\sqrt{2}})\hat{g}(\frac{\omega}{\sqrt{2}}) \\ &= p(\frac{\omega}{\sqrt{2}})p(\frac{\omega}{2})p(\frac{\omega}{\sqrt{2}^3})p(\frac{\omega}{2^2})\dots \\ &= \left(p(\frac{\omega}{\sqrt{2}})p(\frac{\omega}{2\sqrt{2}})p(\frac{\omega}{4\sqrt{2}})\dots\right)\left(p(\frac{\omega}{2})p(\frac{\omega}{4})\dots\right) \\ &= \hat{f}(\omega)\hat{f}(\sqrt{2}\omega)\end{aligned}$$

Thus $g = f * \mathcal{D}_{\frac{1}{\sqrt{2}}} f = \chi_{[0,1)} * \chi_{[0,\sqrt{2})}$. We can formalise this and avoid reordering infinite products.

Theorem 5.1. *Suppose $f \in L^1(\mathbb{R})$ satisfies a dilation equation:*

$$f(x) = \sum_k c_k f(\alpha x - \beta_k),$$

then for any $m \in \mathbb{N}$ the function g:

$$g(x) = f * \mathcal{D}_{\alpha^{-\frac{1}{m}}} f * \mathcal{D}_{\alpha^{-\frac{2}{m}}} f * \dots * \mathcal{D}_{\alpha^{-\frac{m-1}{m}}} f$$

satisfies the dilation equation:

$$g(x) = \alpha^{-\frac{m-1}{m}} \sum_k c_k g(\alpha^{\frac{1}{m}} x - \beta_k).$$

Proof. Taking $p(\omega) = \frac{1}{\alpha} \sum_k c_k e^{\omega \beta_k}$ to be the symbol of the first dilation equation we then have:

$$\hat{f}(\omega) = p(\frac{\omega}{\alpha})\hat{f}(\frac{\omega}{\alpha}).$$

Note that the symbol for the dilation equation given for g is the same. Looking at the Fourier transform of g:

$$\begin{aligned} \hat{g}(\omega) &= \hat{f}(\omega)\alpha^{\frac{1}{m}}\hat{f}(\alpha^{\frac{1}{m}}\omega)\alpha^{\frac{2}{m}}\hat{f}(\alpha^{\frac{2}{m}}\omega)\dots\alpha^{\frac{m-2}{m}}\hat{f}(\alpha^{\frac{m-2}{m}}\omega)\alpha^{\frac{m-1}{m}}\hat{f}(\alpha^{\frac{m-1}{m}}\omega) \\ &= \hat{f}(\omega)\alpha^{\frac{1}{m}}\hat{f}(\alpha^{\frac{1}{m}}\omega)\alpha^{\frac{2}{m}}\hat{f}(\alpha^{\frac{2}{m}}\omega)\dots\alpha^{\frac{m-2}{m}}\hat{f}(\alpha^{\frac{m-2}{m}}\omega)\alpha^{\frac{m-1}{m}}p(\alpha^{\frac{-1}{m}}\omega)\hat{f}(\alpha^{\frac{-1}{m}}\omega) \\ &= p(\alpha^{\frac{-1}{m}}\omega)\hat{f}(\alpha^{\frac{-1}{m}}\omega)\alpha^{\frac{1}{m}}\hat{f}(\omega)\alpha^{\frac{2}{m}}\hat{f}(\alpha^{\frac{1}{m}}\omega)\dots\alpha^{\frac{m-2}{m}}\hat{f}(\alpha^{\frac{m-3}{m}}\omega)\alpha^{\frac{m-1}{m}}\hat{f}(\alpha^{\frac{m-2}{m}}\omega) \\ &= p(\alpha^{\frac{-1}{m}}\omega)\hat{g}(\alpha^{\frac{-1}{m}}\omega), \end{aligned}$$

as required. ■

Given that $\chi_{[0,1)}$ is n-refinable for $n = 2, 3, 4, \dots$ this means we have an easy way to produce $n^{\frac{1}{m}}$-refinable functions for any $m = 1, 2, 3, \dots$. This collection of scales will be dense in $\{x > 1 : x \in \mathbb{R}\}$. Thus we have a refinable function for each of a dense set of dilations.

5.4 Scales with parallelepipeds as self-affine tiles

Given an expanding matrix A sometimes it is possible to choose a digit set which produces a simple tile which is a parallelepiped. This isn't always possible, however. Numerical experiments, looking at the case where A is a 2×2 matrix and $|\det(A)| = 2$, seemed to indicate that this occurred when $\operatorname{trace}(A) = 0$. Looking at the characteristic polynomial of A this would mean that $A^2 = kI$. (The case $|\det(A)| = 2$ was examined because there is little choice over the digit set; we can always ensure 0 is a digit by translation, and the choice of the other digit would seem to have some kind of affine effect on the set. Numerical experiments on higher determinant matrices are less conclusive as there are more digits to choose, and so less certainty about what a particular digit set tells us.)

In the following, the parallelepipeds are never flat, and so the edges at any corner can be used as a basis of $\mathbb{R}^n$.

Lemma 5.2. *Suppose we have a matrix A and a parallelepiped P such that:*

$$AP = (P + \vec{k}_0) \cup (P + \vec{k}_1) \cup \ldots \cup (P + \vec{k}_{q-1})$$

is a disjoint union (up to sets of measure zero), then each corner of AP is in exactly one of $P + \vec{k}_0, P + \vec{k}_1, \ldots, P + \vec{k}_{q-1}$.

Proof. We may transform the problem so that AP is the unit cube and so that the corner $\vec{c}$ we are about to examine is the bottom-most-left-most corner, in the sense that it is the least corner using the order defined by:

$$\begin{array}{ccc} (x_1, x_2, \ldots, x_n) & \leq & (y_1, y_2, \ldots, y_n) \\ & \Leftrightarrow & \\ x_1 = y_1, \ldots, x_{r-1} = y_{r-1}, & x_r \leq y_r & \text{for some } 1 \leq r \leq n. \end{array}$$

Similarly, we may now choose the least corner $\vec{d}$ of P and the least translation $\vec{k}_i$. Note, that the least point of any parallelepiped will be a corner and thus the least point in AP is $\vec{c}$ and the least point in the union will be $\vec{d} + \vec{k}_i$. So $\vec{c} = \vec{d} + \vec{k}_i$, and $\vec{c}$ is also a member of $P + \vec{k}_i$.

Suppose $\vec{c}$ is also a member of $\vec{k}_j$ with $i \neq j$. Then $\vec{c} = \vec{k}_j + \vec{x}$ for some $\vec{x} \in P$. However, the translates are distinct so $\vec{k}_i < \vec{k}_j$ and $\vec{d} \leq \vec{x}$, which yields:

$$\vec{c} = \vec{d} + \vec{k}_i < \vec{k}_j + \vec{x} = \vec{c},$$

which is a contradiction. ■

Theorem 5.3. *Suppose we have a matrix A and a parallelepiped P such that:*

$$AP = (P + \vec{k}_0) \cup (P + \vec{k}_1) \cup \ldots \cup (P + \vec{k}_{q-1})$$

is a disjoint union (up to sets of measure zero), then A is similar to a weighted permutation matrix.

Proof. Pick any corner $\vec{c}$ of AP, then $\vec{c}$ must be the image under A of some corner $\vec{b}$ of P. Let the edges of P from $\vec{b}$ be $\vec{b} + \vec{e}_1, \ldots, \vec{b} + \vec{e}_n$. Then, by applying A we find the edges at $\vec{c}$ are $A(\vec{b} + \vec{e}_1), \ldots, A(\vec{b} + \vec{e}_n)$ which is just $\vec{c} + A\vec{e}_1, \ldots, \vec{c} + A\vec{e}_n$.

Using Lemma 5.2 we can also write this corner as $\vec{c} = \vec{d} + \vec{k}_r$. The edges of P at $\vec{d}$ must be the same as those at $\vec{b}$, though possibly pointing in the opposite direction, so they are:

$$\vec{d}+s_1\vec{e}_1,\ldots,\vec{d}+s_n\vec{e}_n$$

where $s_n=\pm 1$. Thus the edges of $P+\vec{k}_r\subset AP$ are $\vec{c}+s_1\vec{e}_1,\ldots,\vec{c}+s_n\vec{e}_n$. Remembering that $P+\vec{k}_r$ is the only part of the union forming AP at $\vec{c}$, these edges must be parallel, but may be shorter than the edges $\vec{c}+A\vec{e}_1,\ldots,\vec{c}+A\vec{e}_n$. Thus $\vec{c}+A\vec{e}_i=\vec{c}+\alpha_i\vec{e}_j$ where $|\alpha_j|\geq 1$. We conclude that $A\vec{e}_i=\alpha_i\vec{e}_j$, and so A permutes the edges with some weights. As the edges form a basis for $\mathbb{R}^n$ we see that A is a weighted permutation matrix. ■

Corollary 5.4. *With A as in Theorem 5.3 we have A^m is similar to a diagonal matrix for some m dividing* $\operatorname{lcm}(1,2,\ldots,n)$*. Consequently A is diagonalisable.*

Proof. Take the permutation of the edges and ignore the weights; we get a permutation $\sigma\in S_n$. We may write this as a product of disjoint cycles, all of order between 1 and n. Taking the least common multiple m of the lengths of these cycles gives us the order of σ. Now $A^m\vec{e}_i=\beta_i^{m/l}\vec{e}_i$, where β_i is the product of the weights for the cycle containing i and l is the length of that cycle. Thus A^m is similar to a diagonal matrix with β_j along the diagonal.

Let SA^mS^{-1} be the diagonal matrix. Let λ_j for $j=1..r$ be its distinct diagonal entries, so that:

$$\left(SA^mS^{-1}-\lambda_1 I\right)\cdots\left(SA^mS^{-1}-\lambda_r I\right)=0.$$

We can factorise each term to get:

$$\left(SAS^{-1}-\omega_1\lambda_1^{\frac{1}{m}}I\right)\cdots\left(SAS^{-1}-\omega_m\lambda_1^{\frac{1}{m}}I\right)\cdots\left(SAS^{-1}-\omega_1\lambda_r^{\frac{1}{m}}I\right)\cdots\left(SAS^{-1}-\omega_m\lambda_r^{\frac{1}{m}}I\right)=0,$$

where the ω_i are the m^{th} roots of unity. Each of these factors is distinct from the others, since if two were the same then the λ_j could not be distinct. Thus the minimal polynomial of SAS^{-1} can be factored into distinct linear factors and so SAS^{-1} is diagonalisable. ■

Note, that by relabelling the $\vec{e}_i$ we can ensure that each cycle corresponds to a block of the diagonal matrix. Also, if there is a single cycle with length $m=n$, then we have $A^m\vec{e}_i=\beta\vec{e}_i$ for a basis of vectors in $\mathbb{R}^n$ and so $A^m=\beta I$.

Now consider how the q translates of P pack into AP. We can count how many copies of P are along each edge. There must be a whole number of copies, as they do not overlap and they exactly fill AP. Also, the product of the number along each edge gives how many times bigger AP is than P, so must be the same as $q=|\det(A)|$. If $q=p$, a prime, then the only way to pack them is p translates along one edge and 1 along the rest.

Lemma 5.5. *With A as in Theorem 5.3 we have A is similar to a weighted permutation matrix where the weights are integers.*

Proof. As we observed, the side of AP parallel to $\vec{e}_j$ must be a whole multiple of $\vec{e}_j$, but this side is just:

$$A\vec{e}_{\sigma^{-1}(j)} = \alpha_{\sigma^{-1}(j)}\vec{e}_j$$

so $\alpha_{\sigma^{-1}(j)} \in \mathbb{Z}$. ■

Theorem 5.6. *If A is an expanding matrix with $\det(A) = \pm p$ a prime and P is a parallelepiped such that:*

$$AP = (P + \vec{k}_0) \cup (P + \vec{k}_1) \cup \ldots \cup (P + \vec{k}_{q-1}),$$

then $A^n = \beta I$.

Proof. As observed above, AP is formed by packing p translates of P along one edge of AP. This means that $A\vec{e}_i = \pm p\vec{e}_j$ for some one value of i and $A\vec{e}_i = \pm\vec{e}_j$ for all other edges.

Suppose the permutation produced by A does not consist of a single cycle of length n, then there is some cycle for which all the corresponding α_i are ± 1 and so β_i for this cycle is ± 1. Then, by Corollary 5.4, A^m is similar to a matrix with an eigenvalue of ± 1 and so A has an eigenvalue of modulus 1 by the spectral mapping theorem. This contradicts A being expanding, and so the permutation consists of a single cycle of length n. Thus, as observed above, $A^n = \beta I$. ■

The condition in Lemma 5.5 is actually sufficient for A to have a parallelepiped as a tile.

Theorem 5.7. *Suppose A is similar to a weighted permutation matrix, where the weights are non-zero integers, then there exists a parallelepiped P and vectors $\vec{k}_i$ such that:*

$$AP = (P + \vec{k}_0) \cup (P + \vec{k}_1) \cup \ldots \cup (P + \vec{k}_{q-1}).$$

Proof. Let SAS^{-1} be the weighted permutation matrix with integer weights. Let $\vec{e}_i$ be the

i^{th} column of S^{-1}. Then:

$$A\vec{e}_i = AS^{-1}\begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = S^{-1}SAS^{-1}\begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = S^{-1}\begin{bmatrix} \vdots \\ 0 \\ \alpha_i \\ 0 \\ \vdots \end{bmatrix} = \alpha_i \vec{e}_j.$$

Consider the parallelepiped P with one corner at the origin and $\vec{e}_j$ as the edges from the origin. Then AP is a parallelepiped with edges parallel to P.

Examining the corners of AP, there are corners with edges having all possible orientations given by $\pm\vec{e}_1, \ldots \pm \vec{e}_n$, so it is possible to choose some corner whose orientation is all pluses. We translate the corner of P at the origin to this corner and call it $\vec{k}_0$.

If we compare the lengths of the sides of AP with the lengths of the sides of P, we can clearly stack $|\alpha_i|$ copies of P along edge the edge $\alpha_i\vec{e}_j$ and fill out all of AP exactly as $\alpha_i \in \mathbb{Z}$. Thus:

$$AP = \bigcup \left(P + \vec{k}_0 + \sum_{i=1}^{n} r_i \vec{e}_j \right),$$

where r_i is between $0 \leq r_i < |\alpha_i|$. ■

If we want $\vec{0}$ to be a digit, then it is a simple matter of translating P by $(I - A)\vec{k}_0$ at the end of the calculation. Note that we have not shown we can choose these $\vec{k}_i$ so that their coordinates are integers in the correct basis.

5.5 Conclusion

The production of polynomial solutions of dilation equations was a straightforward exercise. It is unfortunate that it does not shed much light on the production of analytic solutions. The polynomial solutions show that refinable analytic functions are in abundance and there is a more exotic example on page 81. It seems that the geometric nature of dilation equations combined with geometric results like Cauchy's integral formula should have some interesting interaction.

Next we discussed various ways of producing new refinable functions. One possible application of this is to use these results to produce MRAs and multiwavelets. The method of differentiating a function on $\mathbb{R}^n$ to produce a refinable function vector probably carries too much structure to produce independent wavelets.

The technique for producing functions satisfying dilation equations of irrational scales might be useful; some people are interested in constructing wavelets with irrational scaling factors. For example in [14] wavelets are produced by using three or more distinct scales and a set of translations which are generated by digit expansions using the irrational scales in question. In [1] there is some discussion of multiresolution analyses with non-integer scales and wavelets with rational scaling factors.

Our parallelepiped tiles could also be used to produce wavelets. In Chapter 7 of [3] non-separable wavelets are produces from a parallelogram shaped tile. In [44] a method for producing smooth orthonormal wavelets from tiles is presented, however is seems difficult to produce actual numbers from this method.

Theorem 5.7 got as far as showing that a dilation matrix A which was similar to a weighted permutation matrix with integer weights would have a self-similar parallelepiped associated with it. We didn't show that the digit set we chose would necessarily have integer coordinates or that the parallelepiped would have area 1, which are conditions for the tile to generate an MRA. These issues seem to involve similarity of matrices through integer entry matrices, which in turn uses the class number of various rings. Consequently choosing these digits correctly may be quite complicated.

Chapter 6

Further Work

> A number of years ago I heard the following remark: 'When a mathematics student finishes their bachelor's degree they think they know everything. When a mathematics student finishes their master's degree they think they know nothing and when a mathematics student finishes their Ph.D. they realize that nobody else knows anything either'. *(Unattributed in [25])*

Over the course of this work we have hopefully pushed out the edges of what is known about dilation equations a little. Each chapter has answered and asked some questions. I think that the most interesting questions which have arisen are:

- How can the properties of m in $\Phi_2(p)$ be improved given properties of p?
- Do all infinite products in $L^2(\mathbb{R})$ decay? How fast?
- Can we show that E in Figure 4.1 actually has to be an interval for scale 2?
- Can we prove anything about non-compactly supported functions which are both 2- and 3-refinable?
- We have some results for polynomial solutions of dilation equations. They don't obviously generalise to power series solutions, but can they can be made to?

There is other interesting work on dilation equations afoot. Strang and Zhou have produced at least two interesting works examining aspects of refinable functions which seem slightly off the beaten track. They consider the closure of the set of refinable functions in [43] and inhomogeneous refinement equations in [42]. It might be interesting to see if an analogy of Green's functions could be developed for inhomogeneous dilation equations.

To analyse dilation equations Haung in [23] studied the operator which projects of f onto $\overline{\operatorname{span}\{f(2\cdot -k) : k \in \mathbb{Z}\}}$ and considered when iterating this operator converges.

There are also interesting problems in relation to dilation equation equations and subdivision schemes. For example, consider a dilation equation whose coefficients are either positive or zero. The positioning of the positive coefficients seems to have a curious effect on the convergence of the subdivision scheme (eg. see [50]).

In the related area of wavelets, Larson et al. seem to be producing a body of work placing various parts of the theory of wavelets into a more general setting [30, 5, 6, 7]. They have considered such issues as the path-connectedness of the set of wavelets, which could also be considered for refinable functions.

Baggett has some results using the *wavelet multiplicity function* or *wavelet dimension function* which describe when wavelets can arise from an MRA [2]. In [45] Strichartz looks at the shape as opposed to the size of the error when wavelets are used for approximation.

Also in the neighbouring subject of self-similar tiles there is significant work going on, often producing pretty pictures as a byproduct. This includes the geometry of tiles [24], how tiles are linked with their lattice [17] and the rather cutely named *reptiles** where different rotations are allowed for each subtile [35].

There are also other directions in which dilation equations could be taken. One possibility is to replace the sum with a integral and replace the coefficients with some sort of convolution kernel. Another direction would be to look at dilation equations on Abelian groups. Here doubling or tripling make sense, we can take a sub-lattice of the group as a replacement for the integers, the Haar measure allows us to define $L^p(G)$ and we have the theory of group representations to replace the Fourier transform.

*Presumably these are named after the Escher picture of interlocking reptiles.

Appendix A

Glossary of Symbols

Symbol	Usage
a	Interval end points, coefficients.
b	Interval end points, coefficients.
c	Dilation equation coefficients.
d	Second dilation equation coefficients.
e	$\sum 1/n!$, edge vectors.
f	Function satisfying a dilation equation.
g	Function satisfying a dilation equation.
h	Some function, small number.
i	$\sqrt{-1}$.
j	Dummy variable.
k	Translation dummy variable.
l	Dummy variable.
m	Maximal solution to transformed dilation equation, limit on sum.
n	Limit of sum.
o	Too like 0?
p	Symbol of dilation equation, power in $L^p(X)$ space.
q	Symbol of dilation equation.
r	Dummy variable for sums, root of polynomial.
s	Positive exponent.
t	Integration dummy variable.
w	Wavelet.
x	Variable on time side.
y	Second variable on time size.
z	Another time side variable.
A	Dilation matrix.
B	Inverse adjoint of dilation matrix.

Symbol	Usage
C	Arbitrary uninteresting constants.
D	Disk/Ball or radius 1.
E	First part of refinable set.
F	Second part of refinable set.
G	Third part of refinable set.
H	Polynomial valued coefficents.
I	Identity matrix.
J	Jordan form of a matrix.
K	Compact set.
L	List of values in recursive search.
M	Large integer for limits.
N	Large integer for limits, upper limit for sum.
O	Big O order of notation.
P	Polynomial representation of step function.
Q	Polynomial representation of dilation equation.
R	Set of reps of $\mathbb{R}/B$, extra polynomial.
S	A set.
T	Digit matricies.
V	Set of functions in MRA, values taken in recursive search.
W	Wavelet set.
X	Uninteresting set.
Y	Uninteresting set.
α	Miscellaneous functions and constants.
β	Miscellaneous functions and constants.
γ	Miscellaneous constant.
δ	Generic small number, Dirac delta function, Kronecker Delta.
ϵ	Generic small number.
λ	Eigenvalues, miscellaneous functions.
π	3.14159.., multiplicatively periodic function.
ρ	Miscellaneous functions relating to π - usually almost π.
σ	Sum of dilation equation coefficents ($\sum_k c_k$).
τ	Ergodic mapping $x \mapsto 2x \mod L$.
ϕ	Functions satisfying transformed dilation equation.
χ	Characteristic function.
ψ	Functions satisfying transformed dilation equation.
ω	Variable on frequence side.
Γ	Lattice in $\mathbb{R}^n$.
Δ	$p(0)$ where p is the symbol of a dilation equation.

Symbol	Usage
Λ	Miscellaneous function.
Φ	Set of functions satisfying transformed dilation equation.
$\mathcal{A}$	Linear operator.
$\mathcal{D}$	Dilation operator.
$\mathcal{F}$	Fourier transform.
$\mathcal{H}$	Hilbert transform.
$\mathcal{I}$	Identity operator.
$\mathcal{P}$	Power set.
$\mathcal{R}$	Operator which multiplies by complex exponential.
$\mathcal{T}$	Translation operator.
$\mathcal{V}$	Cascade algorithm operator.
$\mathbb{C}$	Complex numbers.
$\mathbb{F}$	Any field.
$\mathbb{N}$	Natural numbers.
$\mathbb{Q}$	Rational numbers.
$\mathbb{R}$	Real numbers.
$\mathbb{Z}$	Integers.

Appendix B

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Appendix C

C program for coefficient searching

```
#include <limits.h>
#include <stdio.h>
#include <stdlib.h>

#define SCALE 2

#ifndef TRUE
#define TRUE (1==1)
#endif
#ifndef FALSE
#define FALSE (1==0)
#endif
#ifndef max
#define max(a,b) (((a)>(b))?(a):(b))
#endif
#ifndef min
#define min(a,b) (((a)>(b))?(b):(a))
#endif

#define MAX_BITS (sizeof(pos_t)*CHAR_BIT)

typedef unsigned long pos_t;
static int c[MAX_BITS], maxn;
static int val0[MAX_BITS], val1[MAX_BITS];

void try_next(int npos,pos_t *list,int n,int osv);

int main(int argc,char **argv)
{
    pos_t startpos[] = {0,1};

    c[0] = 1;
    val0[0] = 0;
    val1[0] = 1;

    if( argc >= 2 )
        maxn = atoi(argv[1]);
    if( maxn <= 0 || maxn >= MAX_BITS )
        maxn = 3;

    /*
     * We start with chi_E taking values 0 and 1.
     * Since we know it takes both these values on
     * [0,0.5) and [0.5,1) we cheat and pretend it
     * took netiher value on the first interval so it
     * is forced to take both values on the second.
     */
    try_next(2,startpos,0,0);

    exit(0);
}

/*
 * Given the list of combinations of values for
 * chi_F, chi_G,... find possible values for c_{n+1}
 * which are consistant with c_0 .. c_n, and recurse
 * with all possible new sets of combinations.
 *
```

```
 * We are passed the number of combinations of values,
 * those combinations (get value for E as list[i]&(1<<n),
 * get value for c_1 term as list[i]&(1<<0) value for c_0
 * term is one of the things we're looking for). We are
 * also passed a list indicating what values the sum came
 * to on the previous parts of this interval.
 */

void try_next(int npos,pos_t *list,int n,int osv)
{
    int i,k,s;
    int *sums,*pos;
    int bigcn,littlecn,gotcn,cn;
    int v0,v1;

    /*
     * Special conditions relating to F.
     */
    if( SCALE == 2 && n+1 == 2 ) {
        if( c[1] == 1 && val1[1] == 1 )
            return;
        if( c[1] == 0 && (val0[1] != 0 || val1[1] != 1) )
            return;
        if( c[1] == -1 && (val0[1] != 0 || val1[1] != 1) )
            return;
    }

    if( (sums = malloc(sizeof(int)*npos)) == NULL ) {
        fprintf(stderr,"Out of memory for sums!\n");
        exit(1);
    }

    if( (pos = malloc(sizeof(int)*npos)) == NULL ) {
        fprintf(stderr,"Out of memory for pos!\n");
        exit(1);
    }

    /*
     * Sum the terms, excluding the c[0] term with chi_E=0.
     * Note that if chi_E were 1 at this possibility then the
     * value for c[n+1] will be -s-1, -s or -s+1, depending on
     * the value of the c[0] term; so
     * we also find range of possible values for c[n+1].
     */
    v0 = val0[(n+1)/SCALE];
    v1 = val1[(n+1)/SCALE];
    gotcn = FALSE;
    for( i = 0; i < npos; i++ ) {
        for( s = 0, k = 1; k <= n; k++ )
            if( list[i] & (1<<(k-1)) )
                s += c[k];
        sums[i] = s;

        if( list[i] & (1<<n) ) {
            if( !gotcn || littlecn < -s-1+v0 )
                littlecn = -s-1+v0;
            if( !gotcn || bigcn > -s+v1 )
                bigcn = -s+v1;
            gotcn = TRUE;
            if( littlecn > bigcn )
                break;
        }
    }

    if( !gotcn ) {
        fprintf(stderr,"No comb had 1 in the c[n] pos!\n");
        abort();
    }

    /*
     * If we're about to stop we're only interested in
     * c[n] = 1, 'cos we know the first and last coeff
     * must be 1.
     */
    if( n+2 >= maxn ) {
        if( littlecn > 1 || bigcn < 1 )
            littlecn = 1, bigcn = 0;
        else
            littlecn = bigcn = 1;
    }

    /*
     * Cycle through possible values of c[n+1].
     */
    for( cn = littlecn; cn <= bigcn; cn++ ) {
        int nnpos,comb,cc,ccc;
```

```
int all_zeros;
int sv; /* bitfield of seen values */
pos_t *nl=NULL,*p;

c[n+1] = cn;
for( i = 0; i < npos; i++ )
    if( list[i] & (1<<n) )
        sums[i] += (cn == littlecn) ? littlecn : 1;

/*
 * For each old possibility, there must be at least one
 * consistant new one. If there are two consistant new
 * ones then we have to try combinations with either
 * and both possibilities present. We also note
 * if all zeros is an option for the c[0] term.
 */
nnpos = 0;
comb = 1;
all_zeros = TRUE;
for( i = 0; i < npos; i++ ) {
    s = sums[i];
    pos[i] = 0;
    if(s == v0 || s == v1)
        pos[i] |= 1, nnpos++;
    else
        all_zeros = FALSE;
    if(s+1 == v0 || s+1 == v1)
        pos[i] |= 2, nnpos++;
    if( pos[i] == 0 )
        break;
    if( pos[i] == 3 )
        comb *= 3;
}
if( nnpos == 0 || i < npos)
    continue;

/*
 * See if this could lead to compactly supported
 * solution by assuming that the undetermined
 * sets all have measure 0 and checking to see
 * if exactly the required values are taken.
 */
if( all_zeros && cn == 1 ) {
```

```
int j,bad;
int vnj0,vnj1;

bad = FALSE;
for( j = n+1; j >= -n; j-- ) {
    if( (j+n) % SCALE == SCALE-1 ) {
        sv = 0;    /* seen no values yet*/
        vnj0 = val0[(j+n)/SCALE];
        vnj1 = val1[(j+n)/SCALE];
    }
    for( i = 0; i < npos; i++ ) {
        for( s = 0, k = max(j,0); k <= min(j+n,n+1);
k++ )

            if( list[i] & (1<<(k-j)) )
                s += c[k];
        if( s != vnj0 && s != vnj1 ) {
            bad = TRUE;
            break;
        }
        sv |= 1<<s; /* mark value as seen */
    }
    if(bad)
        break;
    if( (j+n) % SCALE == 0 && (
        (vnj0 == 0 && !(sv & 1)) ||
        (vnj1 == 1 && !(sv & 2)) ) ) {
            bad = TRUE;
            j++;
            break;
    }
}
if( bad && j <= 0 ) {
    fprintf(stderr,"Passed bad combination.\n");
    abort();
}
if( !bad ) { /* All OK? Output possilbe solution */
    for( i = 0; i <= n+1; i++ )
        printf("%d ", c[i]);
    printf("\n");
    for( i = 0; i < npos ; i++ ) {
        printf("\t");
        for( k = n; k >= 0; k-- )
            printf(list[i]&(1<<k) ? "1":"0");
```

```
                printf(" (%lx)\n", list[i]);
            }
            fflush(stdout);
        }
    }

    /*
     * Don't do the recursion bit unless we have to.
     */
    if( n+2 ≥ maxn )
        continue;

    /*
     * Allocate enough for any of the combinations.
     */
    if( (nl = malloc(sizeof(pos_t)*nnpos)) == NULL) {
        fprintf(stderr,"No mem for new possibilities!\n");
        exit(1);
    }

    /*
     * Go through all combinations, noting the values
     * which the new characteristic function and the
     * sum takes.
     */
    for( cc = 0; cc < comb ; cc++ ) {
        ccc = cc;
        val0[n+1] = val1[n+1] = −1;
        sv = ((n+1) % SCALE == 0) ? 0 : osv;
        for( i = 0, p = nl; i < npos; i++ ) {
            s = sums[i];
#define ADDVAL(v) \
    do { \
        *(p++) = (list[i] ≪ 1) + v; \
        val##v[n+1] = v; \
        switch(s+v) { \
            case 0: sv |= 1; break; \
            case 1: sv |= 2; break; \
            default: fprintf(stderr,"Bad sum!\n"); abort(); \
        } \
    } while(0)
                if( pos[i] == 3 ) {
                    if(ccc % 3 ≠ 2)
                        ADDVAL(0);
                    if(ccc % 3 ≠ 1)
                        ADDVAL(1);
                    ccc / 3;
                } else {
                    if(pos[i] & 1)
                        ADDVAL(0);
                    if(pos[i] & 2)
                        ADDVAL(1);
                }
#undef ADDVAL
            }
            if( val0[n+1] == −1 && val1[n+1] == −1 ) {
                fprintf(stderr,"Function took no values!\n");
                abort();
            }
            if( val0[n+1] == −1 ) val0[n+1] = val1[n+1];
            if( val1[n+1] == −1 ) val1[n+1] = val0[n+1];

            /*
             * If we're at the end of an interval and
             * we haven't taken on all the necessary values
             * then this combination isn't valid.
             */
            if( (n+1) % SCALE == SCALE−1 && (
                (val0[(n+1)/SCALE] == 0 && !(sv & 1)) ||
                (val1[(n+1)/SCALE] == 1 && !(sv & 2)) ) )
                    continue;

            /*
             * Recurse on them.
             */
            try_next(p−nl,nl,n+1,sv);
        }
        free(nl);
    }

    free(pos);
    free(sums);
}
```