COMPACT SEMIGROUPS OF POSITIVE MATRICES

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§1 Introduction

The spectral theory of compact monothetic semigroups of linear operators examined by Kaashoek and West in [1], [2] together with two block matrix theorems where the blocks are either strictly positive or zero are used to give an exposition of Perron-Frobenius theory of positive matrices. The approach in this paper is based on ideas of Smyth and West developed in [4], [5].

We consider a linear operator T in finite dimensions which has a matrix representation [T] relative to a given basis. Where there is no ambiguity we often write the matrix as T. $T \geq 0$ if $[T]_{ij} \geq 0$ ($\forall$ i,j) while $T > 0$ if $[T]_{ij} > 0$ ($\forall$ i,j). The spectrum and spectral radius of T will be denoted by $\sigma(T)$ and r(T) respectively. The trace of T (the sum of its eigenvalues) will be written as tr(T), and the peripheral spectrum will be denoted by $\pi(T) = \{\lambda \in \sigma(T); |\lambda| = r(T)\}$. The i^{th} row and j^{th} column of T relative to the given basis will be written $row_i(T)$ and $col_j(T)$ and the diagonal of T will be denoted diag(T). The spectral projection of T relative to $\pi(T)$ will be written P_π .

Smyth [5] has introduced a hierarchy of subsets of matrices $T \geq 0$.

Definitions. (i) T is *positive* if $T > 0$;

(ii) T is *primitive* if $T^k > 0$ for some positive integer k;

(iii) T is *connected* if $\forall$ i,j $\exists$ a positive integer k such that $[T^k]_{ij} > 0$;

(iv) T is *potent* if $diag(T^k) > 0$ for some positive integer k;

(v) T is *zero-free* if no row or column is zero;

(vi) T has positive spectral radius.

Remarks.

The above sets are strictly ordered by inclusion. T is connected if and only if there exists a positive integer p such that $T + T^2 + ... + T^p > 0$. It is also connected if no basis permutation results in a block representation

$$T = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$$

where U and W are square blocks. If S and T are zero-free then so is ST. It follows that if T is zero-free r(T) > 0. Note also that all these subsets are invariant under a basis permutation, and that if $S \geq T$ and T is contained in any one of these sets then so is S.

The following upgrading lemma will be important.

Lemma 1. *If S, T ≥ 0, T ≠ 0 and S is connected then ST = TS implies T is potent.*

Proof. Observe that by replacing S by $S + S^2 + ... + S^p$ for sufficiently large p we may assume that S > 0. First we show that under these conditions T is zero-free. As $T \neq 0$, $[T]_{ij} > 0$ for some i,j. Then $[ST]_{kj} \geq [S]_{ki}[T]_{ij} > 0$ ($\forall$ k), therefore $[TS]_{kj} > 0$ ($\forall$ k) so $row_k(T)$ is non-zero ($\forall$ k) and taking transposes gives the same result for columns.
We prove that T is potent by induction on the size of the matrix. The result is trivially true for 1 x 1 matrices so assume that it holds for k x k matrices (k = 1,....., n-1).
If T (n x n) is connected the result is trivially true so assume that T is not connected. Then by a basis permutation T has lower triangular block form

$$T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}$$

where T_{11} and T_{22} are square blocks which must be non-zero as T is zero-free. Corresponding to this decomposition

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$ with $S_{ij} > 0$ ($\forall$ i,j).

Since S and T commute we have $T_{11}S_{11} = S_{11}T_{11} + S_{12}T_{21}$. But $tr(T_{11}S_{11}) = tr(S_{11}T_{11})$ so $tr(S_{12}T_{21}) = 0$. By positivity $T_{21} = 0$ but now T_{11} commutes with $S_{11} > 0$ and T_{22} commutes with $S_{22} > 0$. By our induction hypothesis, these blocks are both potent hence so is T. ●

§2 Compact Semigroups

Let T be connected. Then $r(T) > 0$ so, without loss of generality, we take $r(T) = 1$.

Proposition 2. *If T is connected and r(T) = 1 then* $||T^n|| \leq M$ *(n=1,2,......).*

Proof. $\mathcal{S}(T) = cl\{T^n \, ; \, n \geq 1\}$ is a closed monothetic (singly generated) semigroup, further $\mathcal{W} = \mathcal{R}^+\mathcal{S}(T)$ is also a semigroup and $\mathcal{W}_1 = \{W \in \mathcal{W} : ||W|| = 1\}$ is a closed, bounded, non-empty subset of $\mathcal{W}$ which is therefore compact. If $W \in \mathcal{W}_1$ then W is potent by Lemma 1, hence $r(W) > 0$ for each $W \in \mathcal{W}_1$. Further the spectral radius is norm-continuous and therefore attains its minimum μ on the compact set $\mathcal{W}_1$. Then

$$r(W) \geq \mu > 0 \quad (W \in \mathcal{W}_1) \quad \text{and so} \quad r(S)||S||^{-1} \geq \mu \quad (S \in \mathcal{S}(T)).$$

But $r(S) = 1$ for $S \in \mathcal{S}(T)$ hence $||S|| \leq \mu^{-1}$ $(S \in \mathcal{S}(T))$ and the monothetic semigroup $\mathcal{S}(T)$ is closed and bounded, therefore compact. ●

The structure theory for such compact monothetic semigroups [1], [2] now shows that $\mathcal{S}(T)$ contains a unique idempotent which is P_π, that all eigenvalues in $\pi(T)$ are simple and that

$$\mathcal{G}(T) = P_\pi\mathcal{S}(T) = \mathcal{S}(P_\pi T)$$

is a compact monothetic group with unit P_π consisting of all limit points of $\mathcal{S}(T)$. Note that $P_\pi \geq 0$ and that, since P_π commutes with T, P_π is potent hence $diag(P_\pi) > 0$.

Further if $\pi(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ then $\mathcal{G}(T)$ is isomorphic to the compact monothetic subgroup of $\mathcal{C}^k$ $cl\{\kappa_1^n, \kappa_2^n, \ldots\ldots, \kappa_k^n \, ; n \, \mu \, 1\}$. By hypothesis T is potent hence $diag(T^p) > 0$ for some positive integer p. It follows now by [4], Proposition 2 that $\pi(T^p) = \{1\}$ hence the peripheral eigenvalues of T are all p^{th} roots of unity. It follows at once by the above isomorphism for $\mathcal{G}(T)$ that $\mathcal{G}(T)$ is a finite cyclic group generated by T. Put $R = P_\pi T$. Obviously $R \geq 0$ and, since T is connected, $T_p = \Sigma_{n=1}^{p} T^n > 0$ for sufficiently large p. Further

$R_p = \Sigma_{n=1}^{p} R^n = PT_p$. Now since $diag(P) > 0$, $[R_p]_{ij} = [PT_p]_{ij} \geq [P]_{ii}[T_p]_{ij} > 0$ $(\forall \, i, j)$ so $R_p > 0$ and R is connected.

Consider the simple case in which $\pi(T) = \{1\}$. Then by the isomorphism $\mathcal{G}(T)$ consists of one element P_π so $T^n \to P$ $(n \to \infty)$. Thus for a general connected T as we have seen $\pi(T^p) = \{1\}$ for a positive integer p so $\mathcal{G}(T)$ is a finite cyclic group and $(T^p)^n = T^{pn} \to P_\pi$ $(n \to \infty)$.

We now use these results to characterise primitive matrices. Note that if $T^k > 0$ then $T^{k+1} = T^kT$ is the product of a positive with a zero-free matrix which is therefore positive. Hence $T^{k+n} > 0$ for all positive integers n.

Proposition 3. *Let $T \geq 0$ with $r(T) = 1$. Then the following are equivalent :*

(i) T is primitive ;

(ii) T is connected and $\pi(T) = \{1\}$;

(iii) T is connected and $T^n \to P_\pi$;

(iv) T is connected and $\mathcal{G}(T) = \{P_\pi\}$.

Proof. (i) ⇔ (ii). Assume that T is primitive. Then T is connected. Now $T^k > 0$ for some k hence $\mathrm{diag}(T^k) > 0$ therefore $T^{kn} \to P_\pi$ $(n \to \infty)$ so $T^kP_\pi = P_\pi$.

Next we show that $P_\pi > 0$. Suppose not, then $[P_\pi]_{ij} = 0$ for some i, j hence $[T^kP_o]_{ij} = \sum_{m=1}^{n}[T^k]_{im}[P_o]_{mj} = 0$, thus, using positivity, $[P_\pi]_{mj} = 0$ ($\forall$ m), that is $\mathrm{col}_j(P_\pi) = 0$ contradicting the fact that $\mathrm{diag}(P_\pi) > 0$. Hence $P_\pi > 0$ hence $\mathrm{rank}(P_\pi) = 1$ therefore $\pi(T) = \{1\}$. Conversely let T be connected with $\pi(T) = \{1\}$. Then $T^n \to P_\pi$ $(n \to \infty)$ so $T^nP_\pi = P_\pi T^n = P_\pi$ ($\forall$n). Suppose $[P_\pi]_{ij} = 0$ for some i, j. Since T is connected $[T^k]_{ij} > 0$ for some k, therefore $0 = [P_\pi]_{ij} \geq [T^k]_{ij}\,[P_\pi]_{jj}$ so $[P_\pi]_{jj} = 0$ which contradicts the fact that $\mathrm{diag}(P_\pi) > 0$. Hence $P_\pi > 0$ and because $T^n \to P_\pi$ $(n \to \infty)$ it now follows that $T^m > 0$ for some m.

(ii) ⇔ (iii). The preceding remarks show that if $\pi(T) = \{1\}$ then $T^n \to P_\pi$. Conversely if $T^n \to P_\pi$ then $\pi(T^n) \to \pi(P_\pi) = \{1\}$ $(n \to \infty)$ which implies $\pi(T) = \{1\}$.

(ii) ⇔ (iv). It is clear from our remarks prior to Proposition 3 that, if T is connected and $r(T) = 1$, then $\pi(T) = \{1\} \Leftrightarrow \mathcal{G}(T) = \{P_\pi\}$. ●

§3 Block Matrix Representations

The following block representation of a zero-free idempotent $P \geq 0$ is well known ([3], Lemma 5.1.9).

Proposition 4. *Let $P \geq 0$ be a zero-free idempotent matrix of rank h then via a basis permutation P has the block matrix representation*

$$P = \begin{pmatrix} P_{11} & 0 & .. & 0 \\ 0 & P_{22} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & P_{hh} \end{pmatrix}$$

where all the off diagonal blocks are zero, the diagonal blocks are square and $P_{ii} > 0$ is an idempotent of rank 1 (i = 1,......,h).

Proof. If P is connected then $P > 0$, $h = 1$ and the result holds. Assume then that via a basis rearrangement P has the block representation $P = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$ As $P^2 = P > 0$ hence $U^2 = U \geq 0$, and $W^2 = W \geq 0$ and $VU + WV = V$ thus $WVU + WV = WV$, so $WVU = 0$. Now as P is zero-free U has no zero rows and W has no zero columns then $V = 0$. Then U and W are both idempotents ≥ 0. $P = \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$ is zero-free hence so are U and W. The result follows by further reduction until the diagonal blocks are all connected idempotents and therefore > 0. ●

(For a general block representation of ≥ 0 idempotent matrices see [3], Lemma 5.1.9)

Now let $T \geq 0$ be connected with $r(T) = 1$. If $P_\pi T = R$ then $\mathcal{G}(T) = \mathcal{S}(R)$ is a finite cyclic group and we can find $S \in \mathcal{S}(R)$ such that $P_\pi R = RP_\pi = R$, $P_\pi S = SP_\pi = S$, $SR = P_\pi = RS$. Further R, $S \geq 0$ and R is connected, also R, S are zero-free since P_π is, so each block row of the block matrix R (and S) corresponding to the block representation of P will have at least one non-zero block.

Suppose that R_{ij} is a block which is not zero. Then we have s, t such that $[R_{ij}]_{st} > 0$. Now R=PRP and the blocks P_{ii} , $P_{jj} > 0$ ($\forall$ i, j) so, for every compatible pair m, n,
$[R_{ij}]_{mn} \geq [P_{ii}]_{ms}[R_{ij}]_{st}[P_{jj}]_{tn} > 0$, thus the block $R_{ij} > 0$.

We now generalise a well known result for ≥ 0 invertible matrices.

Proposition 5. *Let R, S, P be as above and such that RS = P = SR. Then R (and S) have exactly one block in each row or column which is > 0 and the remaining blocks are zero.*

Proof. Since P is zero-free so are R and S so they both have (at least) one non-zero block in each row or column. Suppose $R_{1k} > 0$, then RS = P so the block $P_{1i} = (RS)_{1i} = \Sigma_{j=1}^{h} R_{1j}S_{ji} = 0$ ($\forall$ i > 1). Taking j = k gives $S_{ki} = 0$ ($\forall$ i > 1), by positivity, that is $\text{blockrow}_k(S)$ has exactly one non-zero block S_{kl} and $S_{kl} > 0$. Reversing the order and taking transposes gives the required result. ●

Replacing each positive block of R with the number one and each zero block with the number zero gives an h x h permutation matrix, which, since R is connected must be a single cycle. A basis permutation then ensures that R has an h x h block representation of the form

$$R = \begin{pmatrix} . & . & . & . & R_{1h} \\ R_{21} & . & . & . & . \\ . & R_{32} & . & . & . \\ . & . & . & . & . \\ . & . & \ldots & R_{h,h-1} & . \end{pmatrix}$$

where R_{1h} and all blocks $R_{i,i-1} > 0$; all others are zero.

Consider the equivalent h x h block representation T. For each i, j block $R_{ij} = (TP)_{ij} \geq T_{ij}P_{jj}$. Since $P_{jj} > 0$ we deduce that $R_{ij} = 0$ implies that $T_{ij} = 0$. Thus the block representation T is subservient to that of R, in the sense that its non-zero blocks can only occur in positions i, j in which $R_{ij} > 0$.

Finally observe that by [1], [2] if $|\lambda| = 1$ and $P(\lambda;R)$ denotes the spectral projection of the point λ associated with the linear operator R that $n^{-1} \sum_{k=1}^{n} \kappa^{-k} R^{k} \; \delta \; P(\kappa; R) \;\; (n \; \delta \; \equiv)$ where $P(\lambda;R) \neq 0$ if and only if $\lambda \in \pi(R)$. But since $R^{h+1} = R$ choosing λ such that $\lambda^h = 1$ gives $h^{-1} \sum_{k=1}^{h} \kappa^{-k} R^{k} = P(\kappa; R)$. To show that every h^{th} root of unity is an eigenvalue of R observe that, from our h x h block representation of R, $diag(R^k) = 0$ $(k = 1, \ldots, h-1)$; but that $diag(R^h) = diag(P) > 0$. Thus $P(\lambda;R) \neq 0$ if, and only if, $\lambda^h = 1$.

With this block matrix representation for T let D be the block diagonal matrix

$$D = diag(e^{i\omega} I_1, e^{2i\omega} I_2, \ldots\ldots\ldots , e^{hi\omega} I_h)$$

where $\omega = 2\pi/h$. Then $DTD^{-1} = e^{i\omega} T$ and the whole spectral theory of T is invariant under rotations by multiples of ω.

Now recall that the trace of T^n is given by $tr(T^n) = \sum_{i=1}^{p} \kappa_i^n$ where $\sigma(T) = \{\lambda_i \, ; 1 \leq i \leq p\}$.

Proposition 6. *If $T \geq 0$ and $r(T) = 1$ then T is primitive $\Leftrightarrow$ T is connected and $tr(T^n) \to 1$ $(n \to \infty)$.*

Proof. Let T be primitive. Then by Proposition 3(iii) $T^n \to P_\pi$ so $tr(T^n) \to tr(P_\pi) = 1$. Conversely let T be connected and, as before, set $P_\pi T = R$. The above discussion shows that the eigenvalues of R are precisely the h^{th} roots of unity for some positive integer h and therefore $tr(R^n) = h$ whenever n is divisible by h, otherwise $tr(R^n) = 0$. However $\pi(T) = \pi(R)$ and as $n \to \infty$ the n^{th} powers of $\sigma(T)\backslash\pi(T)$ go to zero. Hence $tr(T^n) - tr(R^n) \to 0$ as $n \to \infty$ so $tr(T^n)$ is convergent if and only if $h = 1$ and the limit in this particular case is always 1 ●

Corollary. *If $T \geq 0$ and $r(T) = 1$ then T is primitive $\Leftrightarrow$ T is connected and $\{tr(T^n)\}_1^\infty$ is a convergent sequence.*

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