Touché Rouché

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Abstract *There seems to be a love-hate relationship between Brouwer's fixed point theorem and the fundamental theorem of algebra; in this note we offer one more tweak at it, and give a version of Rouchés theorem.*

Brouwer's theorem [1],[3],[6], in its simplest form, says that every continuous function on the closed unit disc $\mathbf{D} \subseteq \mathbf{C}$ has a fixed point:

0.1 $$f \in C(\mathbf{D}, \mathbf{D}) \Longrightarrow \exists \lambda \in \mathbf{D}, f(\lambda) = \lambda.$$

The disc $\mathbf{D}$ is an example of a *contractible* space:

1. Definition *Continuous mappings $f : X \to Y$ and $g : X \to Y$ are said to be homotopic if there exists a continuous mapping $(t, x) \mapsto h_t(x) : [0, 1] \times X \to Y$ for which*

1.1 $$h_0 = f \text{ and } h_1 = g.$$

$f : X \to Y$ is said to be contractible if it is homotopic to a constant mapping. A space X is said to be contractible if the identity $I : X \to X$ is a contractible mapping.

It is easily checked that products of contractible mappings are contractible; indeed if $f : X \to Y$ and $g : Y \to Z$ are continuous then

1.2 $$f \text{ contractible or } g \text{ contractible } \Longrightarrow g \circ f \text{ contractible.}$$

Thus contractible mappings form a two-sided ideal in the category of continuous mappings. The reader can easily check that $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{D}$ are each contractible; the status of the circle

1.3 $$\mathbf{S} = \partial \mathbf{D} = e^{2\pi i \mathbf{R}} \cong \mathbf{R}/\mathbf{Z}$$

is not immediately clear. Notice however that if one point is removed then the circle becomes contractible: isomorphism $\mathbf{S} \setminus \{-1\} \cong\,]-\frac{1}{2}, \frac{1}{2}[\, \cong \mathbf{R}$ is given by the mappings

1.4 $$ex_\pi : \mathbf{R} \to \mathbf{S} \ ; \ lg_\pi : \mathbf{S} \setminus \{-1\} \to \mathbf{R}$$

defined by setting

1.5 $$ex_\pi(\theta) = e^{2\pi i \theta} \text{ if } \theta \in \mathbf{R} \ ; \ lg_\pi(e^{2\pi i \theta}) = \theta \text{ if } -\tfrac{1}{2} < \theta < \tfrac{1}{2}.$$

Contractibility on the circle can be tested by extension and by lifting ([5] Theorem 7.10.6; [6] Theorem 1.6, Lemma 3.14):

2. Lemma *If $\varphi \in C(\mathbf{S}, X)$ then necessary and sufficient for φ to be contractible is that*

2.1 $$\varphi \text{ has a continuous extension } \varphi^\wedge : \mathbf{D} \to X.$$

If instead $\varphi \in C(X, \mathbf{S})$ with compact X then necessary and sufficient for φ to be contractible is that

2.2 $$\varphi \text{ has a continuous lift } \varphi^\vee : X \to \mathbf{R}.$$

Proof. Sufficiency is clear in each case from (1.2). For necessity in (2.1) suppose that $(h_t)_{0 \le t \le 1}$ is a homotopy in $C(\mathbf{S}, X)$: we claim

2.3 $$\exists h_0^\wedge \in C(\mathbf{D}, X) \Longrightarrow \exists h_1^\wedge \in C(\mathbf{D}, X).$$

Specifically define for each $\theta \in \mathbf{R}$ and each $r \in [0, 1]$

2.4 $$h_1^\wedge(re^{2\pi i \theta}) = h_0^\wedge(2re^{2\pi i \theta}) \ (0 \le r \le \tfrac{1}{2}) \ , \ = h_{2r-1}(e^{2\pi i \theta}) \ (\tfrac{1}{2} \le r \le 1).$$

Intuitively we construct $h_1^\wedge : \mathbf{D} \to Y \to X$ with $Y = (\mathbf{D} \times \{0\}) \cup (\mathbf{S} \times [0, 1])$, where the embedding of $\mathbf{D}$ in Y is achieved by pasting the interior of the disc across the top of the open cylinder down the sides and across the bottom; klingfilm and a tin of beans would be a mental image.

If instead $(h_t)_{0\le t\le 1}$ is a homotopy in $C(X,\mathbf{S})$ we claim

2.5 $$\exists h_0^\vee \in C(X,\mathbf{R}) \Longrightarrow \exists h_1^\vee \in C(X,\mathbf{R}).$$

By the compactness of $[0,1]$ there is a partition $(t_j)_{j=0}^n$ with $0 = t_0 \le t_1 \le \ldots \le t_n = 1$ for which $\sup_{x\in X}|h_{t_j}(x) - h_{t_{j-1}}(x)| < 2$ for each $j = 1,2,\ldots,n$; if we now define

$$g_j(x) = \frac{h_{t_j}(x)}{h_{t_{j-1}}(x)} \ \textit{for each } x \in X\ ,\ j = 1,2,\ldots,n$$

then $g_j(X) \subseteq \mathbf{S}\setminus\{-1\}$ for each j, while for each $x \in X$ we have $h_1(x) = h_0(x)g_1(x)g_2(x)\ldots g_n(x)$. Thus we can lift h_1 by taking

2.6 $$h_1^\vee(x) = h_0^\vee(x) + \sum_{j=1}^n lg_\pi(g_j(x)) \ \textit{for each } x \in X,$$

where lg_π is given by (1.5) •

Lemma 2 enables us to define the "winding number" or *degree* of a continuous mapping on the circle:

3. Definition *If* $\varphi \in C(\mathbf{S},\mathbf{S})$ *then*

3.1 $$\mathrm{degree}(\varphi) = \varphi_*(1) - \varphi_*(0),$$

where

3.2 $$\varphi_* = \psi^\vee : \mathbf{R} \to \mathbf{R} \ \textit{is a continuous lift for } \psi = \varphi\circ ex_\pi : \mathbf{R} \to \mathbf{S};$$

explicitly

3.3 $$e^{2\pi i\varphi_*(\theta)} = \varphi(e^{2\pi i\theta}) \ \textit{for each } \theta \in \mathbf{R}.$$

The degree is well defined, and an integer, since if X is connected then any two lifts for a continuous function $\varphi : X \to \mathbf{S}$ must differ by a constant. The degree picks out the contractible continuous functions on the circle ([5] Theorem 7.10.7):

4. Theorem *If* $\varphi : \mathbf{S} \to \mathbf{S}$ *is continuous then the following are equivalent:*

4.1 $$\varphi \ \textit{is contractible};$$

4.2 $$\varphi \ \textit{has a continuous extension } \varphi^\wedge : \mathbf{D} \to \mathbf{S};$$

4.3 $$\varphi \ \textit{has a continuous lift } \varphi^\vee : \mathbf{S} \to \mathbf{R};$$

4.4 $$\mathrm{degree}(\varphi) = 0.$$

Proof. The equivalence of the first three conditions is Lemma 2. If $(h_t)_{0\le t\le 1}$ is a homotopy in $C(\mathbf{S},\mathbf{S})$ then we claim

4.5 $$\mathrm{degree}(h_0) = \mathrm{degree}(h_1).$$

This is because the mapping $t \mapsto \mathrm{degree}(h_t)$ is continuous and maps the connected interval $[0,1]$ into the discrete integers $\mathbf{Z}$. Since the winding number of a constant is zero we have proved that (4.1) implies (4.4). Conversely if (4.4) holds then so does (4.3): for we may define $\varphi^\vee$ by setting $\varphi^\vee(e^{2\pi i\theta}) = \varphi_*(\theta)$ if $0 \le \theta \le 1$ •

5. Corollary *The circle* $\mathbf{S}$ *is not contractible.*

Proof. For each $n \in \mathbf{Z}$ we have evidently

5.1 $$\text{degree}(z^n) = n,$$

where $z^n(\lambda) = \lambda^n$ for each $\lambda \in \mathbf{S}$. When $n = 1$ we have the identity map $z = I$, whose winding number is not zero •

It is clear from Theorem 4 that there can be no extension of $z^n : \mathbf{S} \to \mathbf{S}$ to a continuous mapping of the disc into the circle. An alternative way to see this would be to look at "fundamental groups": the fundamental group of the circle turns out to be the integer group $\mathbf{Z}$, while that of the disc (or any contractible space) is the trivial group $\mathbf{O}$. Of course much of the proof that the fundamental group of the circle is $\mathbf{Z}$ is in Theorem 4.

The Brouwer fixed point theorem says that if $f : \mathbf{D} \to \mathbf{D}$ is continuous then the function $f - z : \lambda \mapsto f(\lambda) - \lambda$ vanishes somewhere in $\mathbf{D}$. Here is a "tweaked" version:

6. Theorem *Suppose* $f \in C(\mathbf{D}, \mathbf{D})$ *is continuous, and that* $\varphi \in C(\mathbf{D}, \mathbf{D})$ *is continuous and also satisfies*

6.1 $$\varphi(\mathbf{S}) \subseteq \mathbf{S}.$$

If $\text{degree}(\varphi) \neq 0$ *then there is* $\lambda \in \mathbf{D}$ *with* $f(\lambda) = \varphi(\lambda)$.

Proof. If to the contrary $f - \varphi$ is nonvanishing on $\mathbf{D}$ then we can construct an extension $\varphi^\wedge : \mathbf{D} \to \mathbf{S}$ by taking, for each $\lambda \in \mathbf{D}$, the point $\varphi^\wedge(\lambda)$ to be the point where the line from $f(\lambda)$ through $\varphi(\lambda)$ meets the circle $\mathbf{S}$ •

Theorem 6 applies in particular when $\varphi : \mathbf{D} \to \mathbf{D}$ has the "antipodal property" [6],[7]:

7. Theorem *If* $\varphi : \mathbf{S} \to \mathbf{S}$ *is continuous and contractible then it cannot possibly have the antipodal property,*

7.1 $$\varphi(-z) = -\varphi(z) \text{ on } \mathbf{S},$$

and there must be $\lambda \in \mathbf{S}$ *for which*

7.2 $$\varphi(-\lambda) = \varphi(\lambda).$$

Proof. We claim that the antipodal property (7.1) is incompatible with the lifting property (4.3): for then we would have

7.3 $$\varphi^\vee(-z) = \varphi^\vee(z) + \tfrac{1}{2} + N$$

for some fixed $N \in \mathbf{N}$, which taking $z = 1$ and $z = -1$ gives $2N + 1 = 0$. Now (7.2), the "Borsuk-Ulam lemma" ([6] Corollary 6.29;[7]), follows: for if there were no such λ then $(\varphi(z) - \varphi(-z))/|\varphi(z) - \varphi(-z)|$ - easily checked to be conractible - would have the antipodal property (7.1) •

Theorem 6 applies most famously when $\varphi = z$ is the identity function: this is the "fixed point theorem". If we take more generally $\varphi = z^n$ then we have (*cf* [6] Theorem 3.19) a nice derivation of the "fundamental theorem of algebra":

8. Theorem *If $p = a_n z^n + \ldots + a_1 z + a_0$ is a non constant polynomial, with $n \in \mathbf{N}$ and $a_j \in \mathbf{C}$ with $a_n \neq 0$, then there is $\lambda \in \mathbf{C}$ for which $p(\lambda) = 0$.*

Proof. Put $q(z) = p(kz)/a_n k^n$ with

8.1 $$|a_0| + |a_1|k + \ldots + |a_{n-1}|k^{n-1} < |a_n|k^n :$$

thus $q = b_n z^n + \ldots + b_1 z + b_0$ with

8.2 $$|b_0| + |b_1| + \ldots + |b_{n-1}| < 1 = b_n,$$

and now

8.3 $$f = z^n - q \Longrightarrow f(\mathbf{D}) \subseteq \mathbf{D}.$$

By Theorem 6 there is $\mu \in \mathbf{D}$ for which $q(\mu) = \mu^n - f(\mu) = 0$, and hence $\lambda = k\mu \in \mathbf{C}$ for which $p(\lambda) = 0$ •

The fundamental theorem of algebra is equally valid with the complex conjugate $\overline{z}$ in place of z. We have a curious extension if we notice that, whenever $m \neq n$, the winding number of $z^n \overline{z}^m$ is non-zero: Theorem 8 remains valid with

8.4 $$p = \sum_{j=0}^{n} \sum_{k=0}^{m} a_{jk} z^j \overline{z}^k \text{ with } m \neq n \text{ and } a_{mn} \neq 0.$$

Theorem 6 offers an alternative derivation of a version of Rouchés theorem [8]:

9. Theorem *If $g \in C(\mathbf{D})$ and $h \in A(\mathbf{D})$ satisfy*

9.1 $$|g(\cdot)| \leq |h(\cdot)| \text{ on } \mathbf{S}$$

then

9.2 $$h^{-1}(0) \neq \emptyset \Longrightarrow (g - h)^{-1}(0) \neq \emptyset.$$

Proof. Here $A(\mathbf{D}) \subseteq C(\mathbf{D})$ are the continuous functions on $\mathbf{D}$ which are holomorphic on the interior $\mathbf{D} \setminus \mathbf{S}$. If h vanishes anywhere on $\mathbf{S}$ then by (9.1) g and hence $g - h$ vanish there too: thus we may suppose

$$h^{-1}(0) \cap \mathbf{S} = \emptyset.$$

Define then $\varphi : \mathbf{S} \to \mathbf{S}$ as the normalised restriction of $h : \mathbf{D} \to \mathbf{C}$: for all $\theta \in \mathbf{R}$

$$|h(e^{2\pi i\theta})|\varphi(e^{2\pi i\theta}) = h(e^{2\pi i\theta}).$$

We claim

9.3 $$\text{degree}(\varphi) = 0 \Longleftrightarrow h^{-1}(0) = \emptyset :$$

indeed by the "argument principle" ([4] Theorem 3.7; *cf* [6] Exercise 3.12), for sufficiently large $r < 1$,

9.4 $$\text{degree}(\varphi) = \frac{1}{2\pi i} \int_{r\mathbf{S}} \frac{h'}{h} dz$$

counts with multiplicity the number of zeroes of h in $\mathbf{D} \setminus \mathbf{S}$. To bring Theorem 6 to bear we need to extend φ to $\mathbf{D}$ and normalise g: set for $0 \leq r \leq 1$ and $\theta \in \mathbf{R}$

$$|h(e^{2\pi i\theta})|\varphi(re^{2\pi i\theta}) = \zeta(r)h(re^{2\pi i\theta})$$

and

$$|h(e^{2\pi i\theta})|f(re^{2\pi i\theta}) = \zeta(r)g(re^{2\pi i\theta}),$$

adjusting continuous $\zeta : [0, 1] \to [0, 1]$, with $\zeta(1) = 1$, so that both φ and f take $\mathbf{D}$ into $\mathbf{D}$. Now finally

$$h^{-1}(0) \neq \emptyset \Longrightarrow \text{degree}(\varphi) \neq 0 \Longrightarrow (g - h)^{-1}(0) = (f - \varphi)^{-1}(0) \neq \emptyset \text{ •}$$

Naturally (9.3) need not work for general continuous h: for example $h = |z|$ vanishes at $0 \in \mathbf{D}$ but has restriction $\varphi = 1$ to $\mathbf{S}$.

In higher dimensions the structure of $\mathbf{S}_{n-1} = \partial \mathbf{D}_n \subseteq \mathbf{R}^n$ is more complicated: for example there does not exist a group structure on $\mathbf{S}_2$. However the "special linear group" of orthogonal matrices acts transitively: there is topological isomorphism

9.5 $$SO(n+1)/\begin{pmatrix} SO(n) & 0 \\ 0 & 1 \end{pmatrix} \cong \mathbf{S}_n,$$

with correspondence $\mathbf{T} + \begin{pmatrix} SO(n) & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \xi$ given by

9.6 $$\mathbf{T}\begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} = \xi.$$

There is then a further mapping $\exp : SO(n) \to so(n)$ into a Lie algebra. It is clear from the argument for (2.1) that necessary and sufficient for $\varphi \in C(\mathbf{S}_{n-1}, X)$ to be contractible is that

9.7 $$\varphi \text{ has a continuous extension } \varphi^\wedge : \mathbf{D}_n \to X;$$

it would be nice to adapt the argument of (2.2) to show that it is necessary or sufficient for $\varphi \in C(X, \mathbf{S}_{n-1})$, with compact X, to be contractible that

9.8 $$\varphi \text{ has a continuous lift } \varphi^\vee : X \to so(n).$$

Since the Lie algebra $so(n)$ is a contractible space the condition is certainly sufficient. On the other hand the analogue of "degree(φ)" for continuous mappings $\varphi : \mathbf{S}_{n-1} \to \mathbf{S}_{n-1}$ is [2],[6] notoriously complicated.

References

1. B.H. Arnold, *A topological proof of the fundamental theorem of algebra*, Amer. Math. Monthly **56** (1954) 465-466.
2. B. Booss and D.D. Bleecker, *Topology and analysis: the Atiyah-Singer formula and gauge-theoretic physics*, Springer Verlag 1985.
3. B. Fine and G. Rosenberger, *The fundamental theorem of algebra*, Springer Verlag 1997.
4. J.B. Conway, *Functions of one complex variable*, Springer 1978.
5. R.E. Harte, *Invertibility and singularity*, Dekker 1988.
6. J.J. Rotman, *An introduction to algebraic topology*, Springer Verlag 1988.
7. F.E. Su, *Borsuk-Ulam implies Brouwer, a direct construction*, Notices Amer. Math. Soc. **104** (1997) 855-859.
8. A. Tsarpalias, *A version of Rouchés theorem for continuous functions*, Amer. Math. Monthly **96** (1989) 911-913.