ALGEBRAS OF SYMMETRIC HOLOMORPHIC FUNCTIONS ON ℓ_p

RAYMUNDO ALENCAR, RICHARD ARON, PABLO GALINDO AND ANDRIY ZAGORODNYUK

ABSTRACT. We study the algebra of uniformly continuous holomorphic symmetric functions on the ball of ℓ_p, investigating in particular the spectrum of such algebras. To do so, we examine the algebra of symmetric polynomials on ℓ_p- spaces as well as finitely generated symmetric algebras of holomorphic functions. Such symmetric polynomials determine the points in ℓ_p up to a permutation.

In recent years, algebras of holomorphic functions on the unit ball of standard complex Banach spaces have been considered by a number of authors and the spectrum of such algebras was studied in [1],[2], [7]. For example, properties of $A_u(B_X)$, the algebra of uniformly continuous holomorphic functions on the ball of a complex Banach space X have been studied by Gamelin, et al. Unfortunately, this analogue of the classical disc algebra $A(D)$ has a very complicated, not well understood, spectrum. If X^* has the approximation property, the spectrum of $A_u(B_X)$ coincides with the closed unit ball of the bidual if, and only if, X^* generates a dense subalgebra in $A_u(B_X)$ [5]. In a very real sense, however, the problem is that $A_u(B_{\ell_p})$ is usually too large, admitting far too many functions. For instance, $\ell_\infty \subset A_u(B_{\ell_2})$ isometrically via the mapping $a = (a_j) \rightsquigarrow P_a$, where $P_a(x) \equiv \sum_{j=1}^{\infty} a_j x_j^2$.

This paper addresses this problem, by severely restricting the functions which we admit. Specifically, we limit our attention here to uniformly continuous *symmetric* holomorphic functions on B_{ℓ_p}. By a symmetric function on ℓ_p we mean a function which is invariant under any reordering of the sequence in ℓ_p. Symmetric polynomials in finite dimensional spaces can be studied in [9] or [12]; in the infinite dimensional Hilbert space they already appear in [11]. Throughout this note $\mathcal{P}_s(\ell_p)$ is the space of symmetric polynomials on a complex space ℓ_p, $1 \leq p < \infty$. Such polynomials determine, as we prove, the points in ℓ_p up to a permutation. We will use the notation $A_{us}(B_{\ell_p})$ for the uniform algebra of symmetric holomorphic functions which are uniformly continuous on the open unit ball B_{ℓ_p} of ℓ_p and we also study some particular finitely generated subalgebras. The purpose of this paper is to describe such algebras and their spectra, which we identify with certain subsets of ℓ_∞ and $\mathbb{C}^m$, respectively, and as a result of this we show that $A_{us}(B_{\ell_p})$ is algebraically and topologically isomorphic to a uniform Banach algebra generated by coordinate projections in ℓ_∞. This is done in Section 3, following algebraic preliminaries and a brief examination of the finite dimensional situation in Sections 1 and 2.

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We denote by τ_{pw} the topology of pointwise convergence in ℓ_∞. We follow the usual conventions, denoting by $\mathcal{H}_b(X)$ the Fréchet algebra of $\mathbb{C}-$valued holomorphic functions on a complex Banach space X which are bounded on bounded subsets of X, endowed with the topology of uniform convergence on bounded sets. The subalgebra of *symmetric* functions will be denoted $\mathcal{H}_{bs}(X)$. For any Banach or Fréchet algebra A, we put $\mathcal{M}(A)$ for its spectrum, that is the set of all continuous scalar valued homomorphisms. For background on analytic functions on infinite dimensional Banach spaces we refer the reader to [3].

1. The algebra of symmetric polynomials

Let X be a Banach space and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an *algebraic basis* of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is $q \in \mathcal{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(G_1(x), \ldots, G_n(x))$, in other words, if G is the mapping $x \in X \rightsquigarrow G(x) := (G_1(x), \ldots, G_n(x)) \in \mathbb{C}^n, \;\; P = q \circ G$.

Let $< p >$ be smallest integer number that is greater than or equal to p. In [8] is proved that the polynomials $F_k(\sum a_i e_i) = \sum a_i^k$ for $k =< p >, < p > +1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$. So there are no symmetric polynomials of degree less than $< p >$ in $\mathcal{P}_s(\ell_p)$ and if $< p_1 >=< p_2 >$ then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer p. Throughout we will assume that p is an integer number, $1 \le p < \infty$.

It is well known ([9] XI §52) that for $n < \infty$ any polynomial in $\mathcal{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the *elementary symmetric polynomials* $(R_i)_{i=1}^n, \;\; R_i(x) = \sum_{k_1 < \cdots < k_i} x_{k_1} \ldots x_{k_i}$

Lemma 1.1 *Let $\{G_1, \ldots, G_n\}$ be an algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$, $G_i(y) = \xi_i$, $i = 1, \ldots, n$, then $x = y$ up to a permutation.*

Proof. First we suppose that $G_i = R_i$. Then according to the Vieta formulae [9], the solutions of the equation
$$x^n - \xi_1 x^{n-1} + \ldots (-1)^n \xi_n = 0$$
satisfy the conditions $R_i(x) = \xi_i$ and so $x = (x_1, \ldots, x_n)$ as required. Let now G_i be an arbitrary algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials v_i on $\mathbb{C}^n$. Setting v as the polynomial mapping $x \in \mathbb{C}^n \rightsquigarrow v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \to \mathbb{C}^n$ such that $G = w \circ R$, hence $R = (v \circ w) \circ R$, so $v \circ w = id$. Then v and w are inverse each other since $w \circ v$ coincides with the identity on the open set $Im(w)$. In particular, v is one to one.

Now, the solutions $x_1, \ldots, x_n$ of the equation
$$x^n - v_1(\xi_1, \ldots, \xi_n) x^{n-1} + \cdots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$
satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, hence $\xi = G(x)$. $\square$

Corollary 1.2. *Given* $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ *there is* $x \in \ell_p^{n+p-1}$ *such that*

$$F_p(x) = \xi_1, \dots, F_{p+n-1}(x) = \xi_n.$$

This results shows that any $P \in \mathcal{P}_s(\ell_p)$ has a *"unique" representation* in terms of $\{F_k\}$, in the sense that if $q \in \mathcal{P}(\mathbb{C}^n)$ for some n is such that $P(x) = q(F_p(x), \dots, F_{n+p}(x))$, and if $q' \in \mathcal{P}(\mathbb{C}^m)$ for some m is such that $P(x) = q'(F_p(x), \dots, F_{m+p}(x))$, with, say $n \leq m$, then $q'(\xi_1, \cdots, \xi_m) = q(\xi_1, \cdots, \xi_n)$.

For $x, y \in \ell_p$, we will write $x \sim y$, whenever there is a permutation T of the basis in ℓ_p such that $x = T(y)$. For any point $x \in \ell_p$, δ_x will denote the linear multiplicative functional on $\mathcal{P}_s(\ell_p)$ "evaluation" at x. It is clear that if $x \sim y$ then $\delta_x = \delta_y$.

Theorem 1.3. *Let* $x, y \in \ell_p$ *and* $F_i(x) = F_i(y)$ *for every* $i > p$. *Then* $x \sim y$.

Proof. Call $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$. Without loss of generality, we can assume that $1 = |x_1| = \cdots = |x_k| > |x_{k+1}| \geq \dots$ and $1 \geq |y_1| \geq |y_2| \geq \dots$

If $|y_1| < 1$ then for many big j, $|F_j(x)|$ will be close to k while for all big j, $F_j(y)$ will be close to 0. Thus $|y_1| = 1$. Suppose that $1 = |y_1| = \cdots = |y_m| > |y_{m+1}| \geq \dots$ Claim: $m = k$. Suppose for a contradiction, that $m < k$. Then, for many big j, $|F_j(x)|$ is close to k, while for all big j, $|F_j(y)| < m + 1/2 < k$. This contradiction shows that $m < k$ is false; similarly, $k < m$ is false, and so $m = k$.

Let $\tilde{x} = (x_1, \dots, x_k)$ and $\tilde{y} = (y_1, \dots, y_k)$. Also, for $z = (z_i) \in \ell_p$, let z^j denote the point $(z_1^j, z_2^j, \dots)$. We claim that $\tilde{x} \sim \tilde{y}$, where we associate $\tilde{x} = (x_1, ..., x_k) \in \mathbb{C}^k$, for example, with $(x_1, ..., x_k, 0, 0, ...)$. Consider the function $f : (S^1)^{2k} \to \mathbb{C}$ given by

$$f(\tilde{u}, \tilde{v}) = f(u_1, \dots, u_k, v_1, \dots, v_k) = [u_1 + \cdots + u_k] - [v_1 + \cdots + v_k].$$

Since $F_j(x - \tilde{x})$ and $F_j(y - \tilde{y}) \to 0$ as $j \to \infty$ and since we are assuming that $F_j(x) = F_j(y)$ for all $j \geq p$, it follows that $f(\tilde{x}^j, \tilde{y}^j) \to 0$ as $j \to \infty$. Now, f is obviously a continuous function, and so it follows that for any point $(u, v) \in (S^1)^{2k}$ which is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geq p\}$, $f(u, v) = 0$.

Next, the point $(1, \dots, 1) \in (S^1)^{2k}$ is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geq p\}$. If the net $(\tilde{x}^{j_t}, \tilde{y}^{j_t})_t \to (1, \dots, 1)$, then $(\tilde{x}^{j_t+1}, \tilde{y}^{j_t+1})_t \to (\tilde{x}, \tilde{y})$. Consequently, $f(\tilde{x}, \tilde{y}) = 0$, or in other words $F_1(\tilde{x}) = F_1(\tilde{y})$. Similarly, $F_j(\tilde{x}) = F_j(\tilde{y})$ for all j. From Lemma 1.1 it follows that $\tilde{x} \sim \tilde{y}$. So $F_j(x - \tilde{x}) = F_j(y - \tilde{y})$ for every $j \geq p$ i.e.

$$F_j(0, \dots, 0, x_{k+1}, x_{k+2}, \dots) = F_j(0, \dots, 0, y_{k+1}, y_{k+2}, \dots)$$

for every $j \geq p$. If $|x_{k+1}| = 0$ and $|y_{k+1}| = 0$ then $x_i = 0$ and $y_i = 0$ for $i > k$. Let $|x_{k+1}| = a \neq 0$ then we can repeat the above argument for vectors $x' = (x_{k+1}/a, x_{k+2}/a, \dots)$ and $y' = (y_{k+1}/a, y_{k+2}/a, \dots)$ and by induction we will see that $x \sim y$. □

Corollary 1.4. *Let* $x, y \in \ell_p$. *If for some integer* $m \geq p$, $F_i(x) = F_i(y)$ *for each* $i \geq m$, *then* $x \sim y$.

Proof. Since $m \geq p$ then $x, y \in \ell_m$ and from Theorem 1.3 it follows that $x \sim y$ in ℓ_m. So $x \sim y$ in ℓ_p. □

Proposition 1.5. **(Nullstellensatz)** *Let $P_1, \dots, P_m \in \mathcal{P}_s(\ell_p)$ be such that* $\ker P_1 \cap \cdots \cap \ker P_m = \emptyset$. *Then there are $Q_1, \dots, Q_m \in \mathcal{P}_s(\ell_p)$ such that*

$$\sum_{i=1}^{m} P_i Q_i \equiv 1.$$

Proof. Let $n = \max_i(\deg P_i)$. We may assume that $P_i(x) = g_i(F_p(x), \dots, F_n(x))$ for some $g_i \in \mathcal{P}(\mathbb{C}^{n-p+1})$. Let us suppose that at some point $\xi \in \mathbb{C}^{n-p+1}$, $\xi = (\xi_1, \dots, \xi_{n-p+1})$, $g_i(\xi) = 0$. Then by Corollary 1.2 there is $x_0 \in \ell_p$ such that $F_i(x_0) = \xi_i$. So the common set of zeros of all g_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $q_1, \dots, q_m$ such that $\sum_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(F_p(x), \dots, F_n(x))$. □

2. Finitely generated symmetric algebras

Let us denote by $\mathcal{P}_s^n(\ell_p)$, $n \geq p$ the subalgebra of $\mathcal{P}_s(\ell_p)$ generated by $\{F_p, \dots, F_n\}$. By appealing to Corollary 1.2, one easily verifies that $\mathcal{P}_s^n(\ell_p) \cap \mathcal{P}({}^k\ell_p)$, is a *sup-norm* closed subspace of $\mathcal{P}({}^k\ell_p)$ for every $k \in \mathbb{N}$.

Let $A_{us}^n(B_{\ell_p})$ and $\mathcal{H}_{bs}^n(\ell_p)$ be the closed subalgebras of $A_{us}(B_{\ell_p})$ and $\mathcal{H}_{bs}(\ell_p)$ generated by $\{F_p, \dots, F_n\}$, that is the closure of $\mathcal{P}_s^n(\ell_p)$ in each of the corresponding algebras. Note that for any $f \in \mathcal{H}_{bs}^n(\ell_p)$, with f having Taylor series $f = \sum P_k$ about 0, we have $P_k \in \mathcal{P}_s^n(\ell_p)$. Indeed, if $f \in \mathcal{P}_s^n(\ell_p)$, it is immediate that $P_k \in \mathcal{P}_s^n(\ell_p) \cap \mathcal{P}({}^k\ell_p)$ for all k. Then the same holds for any $f \in \mathcal{H}_{bs}^n(\ell_p)$ by recalling the continuity of the map which assigns to a holomorphic function its k^{th} Taylor polynomial.

By [6] III. 1.4, we may identify the spectrum of $A_{us}^n(B_{\ell_p})$ with the joint spectrum of $\{F_p, \dots, F_n\}$, $\sigma(F_p, \dots, F_n)$. It is well known that $\mathcal{M}(\mathcal{H}(\mathbb{C}^n)) = \mathbb{C}^n$ in the sense that all continuous homomorphisms are evaluations at some point in $\mathbb{C}^n$.

Let us denote by $\mathcal{F}_p^n$ the mapping from ℓ_p to $\mathbb{C}^{n-p+1}$ given by $\mathcal{F}_p^n : x \mapsto (F_p(x), \dots, F_n(x))$. Then $D_p^n := \mathcal{F}_p^n(\overline{B_{\ell_p}})$ is a subset of the closed unit ball of $\mathbb{C}^{n-p+1}$ with the max-norm.

Let K be a bounded set in $\mathbb{C}^n$. Recall that a point x belongs to the *polynomial convex hull* of $K, [K]$, if for every polynomial f , $|f(x)| \leq \sup_{z \in K} |f(z)|$. A set is *polynomially convex* if it coincides with its polynomial convex hull. Recall that the sup norm on K of a polynomial coincides with the sup norm on $[K]$. It is well known (see e.g. [6]) that the spectrum of the uniform Banach algebra $P(K)$ generated by polynomials on the compact set K coincides with the polynomially convex hull of this set. Thus, $[D_p^n]$ denotes the polynomial convex hull of D_p^n.

Theorem 2.1.
(i) *The composition operator* $C_{\mathcal{F}_p^n} : \mathcal{H}(\mathbb{C}^{n+1-p}) \to \mathcal{H}_{bs}^n(\ell_p)$ *given by* $C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n$ *is a topological isomorphism.*
(i') *The composition operator* $C_{\mathcal{F}_p^n} : P([D_p^n]) \to A_{us}^n(B_{\ell_p})$ *given by* $C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n$ *is a topological isomorphism.*
(ii) $\mathcal{M}(\mathcal{H}_{bs}^n(\ell_p)) = \mathbb{C}^{n+1-p}$.
(ii') $\mathcal{M}(A_{us}^n(B_{\ell_p})) = [D_p^n]$.

Proof. Clearly the composition operators are well defined and one to one, so it remains to prove that they are onto.

In (i), let $f \in \mathcal{H}^n_{bs}(\ell_p)$ and $f = \sum P_k$ be the Taylor series expansion of f at 0. Since $P_k \in \mathcal{P}^n_s(\ell_p)$, there is a homogeneous polynomial $g_k \in \mathcal{P}(\mathbb{C}^{n+1-p})$ such that $P_k(x) = g_k(F_p(x), \dots, F_n(x))$. Put $g(\xi_1, \dots, \xi_{n-p+1}) = \sum_{k=1}^{\infty} g_k(\xi_1, \dots, \xi_{n-p+1})$; since g is a convergent power series in each variable, it is separately holomorphic, hence holomorphic. Note that $f = g \circ \mathcal{F}^n_p$.

In (i'), observe that for any $g \in P([D^n_p])$, $||C_{\mathcal{F}^n_p}(g)|| = \sup_{x \in \overline{B_{\ell_p}}} |g \circ \mathcal{F}^n_p(x)| = ||g||_{D^n_p} = ||g||_{[D^n_p]}$. Thus $C_{\mathcal{F}^n_p}$ is an isometry, hence its range is a closed subspace, which moreover contains $\mathcal{P}^n_s(\ell_p)$, therefore $C_{\mathcal{F}^n_p}$ is onto $A^n_{us}(B_{\ell_p})$.

(ii) and (ii') follow from (i), (i'). □

To conclude, we record the following elementary result which will be needed in Section 3.

Lemma 2.2. *If* $(\xi^0_1, \dots, \xi^0_m) \in [D^m_p]$ *and* $n < m$ *then* $(\xi^0_1, \dots, \xi^0_n) \in [D^n_p]$.

Proof. If $(\xi^0_1, \dots, \xi^0_n) \notin [D^n_p]$, there is a polynomial of n variables such that

$$|q(\xi^0_1, \dots, \xi^0_n)| > \sup_{(\xi_1, \dots, \xi_n) \in D^n_p} |q(\xi_1, \dots, \xi_n)|.$$

Consider the polynomial $\tilde{q}$ in m variables given by $\tilde{q}(\xi_1, \dots, \xi_m) = q(\xi_1, \dots, \xi_n)$. Then,

$$\sup_{(\xi_1, \dots, \xi_m) \in D^m_p} |\tilde{q}(\xi_1, \dots, \xi_m)| = \sup_{x \in B_{\ell_p}} |\tilde{q}(F_p(x), \dots, F_{p+m-1}(x))| =$$

$$\sup_{x \in B_{\ell_p}} |q(F_p(x), \dots, F_{p+n-1}(x))| < |q(\xi^0_1, \dots, \xi^0_n)| = |\tilde{q}(\xi^0_1, \dots, \xi^0_m)|.$$

But this means $(\xi^0_1, \dots, \xi^0_m) \notin [D^m_p]$, a contradiction. □

3. Spectrum of $A_{us}(B_{\ell_p})$

In the study of the spectrum of $A_{us}(B_{\ell_p})$ the most decisive feature is that the polynomials $\{F^n_p\}_{n=p}^{\infty}$ generate a dense subalgebra. Actually for every $f \in A_{us}(B_{\ell_p})$ its Taylor polynomials are easily seen to be symmetric, using the fact (see, e.g., [3]) each such polynomial can be calculated by integrating f.

Note that there are symmetric holomorphic functions on B_{ℓ_p} which are not in $A_{us}(B_{\ell_p})$. One such example is $f = \sum_{k=p}^{\infty} F_k$. To see that f is holomorphic on the *open* ball B_{ℓ_p}, let $x \in B_{\ell_p}$ be arbitrary and choose $\rho < 1$ such that $||x|| < \rho$. Then, $\sum_{k=p}^{\infty} |F_k(x)|$ converges since the sequence $(F_k(\frac{x}{\rho})) = (\frac{F_k(x)}{\rho^k})$ is null. On the other hand, $f \notin A_{us}(B_{\ell_p})$ since $f(te_1) = \frac{t^p}{1-t^p} \to \infty$ as $t \uparrow 1$.

First we will show that the spectrum of the uniform algebra of symmetric holomorphic functions on B_{ℓ_p} does not coincide with equivalence classes of point evaluation functionals. The example also shows that D^n_p is not polynomially convex.

Example 3.1. For every n put $v_n = \frac{1}{n^{1/p}}(e_1 + ... + e_n) \in \overline{B_{\ell_p}}$. Then $\delta_{v_n}(F_p) = 1$ and $\delta_{v_n}(F_j) \to 0$ as $n \to \infty$ for every $j > p$. By compactness of $\mathcal{M}(A_{us}(B_{\ell_p}))$ there is an accumulation point ϕ of the sequence $\{\delta_{v_n}\}$. Then $\phi(F_p) = 1$ and $\phi(F_j) = 0$ for all $j > p$. From Corollary 1.4 it follows that there is no point z in ℓ_p such that $\delta_z = \phi$. Another, more geometric, way of looking at this example is to fix $k \in \mathbb{N}$ and consider $D_p^{p+k} \subset \mathbb{C}^{k+1}$. It is straightforward that $(1, 0, ..., 0) \notin D_p^{p+k}$, although this point is a limit of the sequence $(\mathcal{F}_p^{p+k}(v_n)) = (1, \frac{1}{n^{1/p}}, ..., \frac{1}{n^{(k-1)/p}})$. Intuitively, the accumulation point ϕ corresponds to the point $(1, 0, ...0, ...) \in \overline{B_{\ell_\infty}}$.

Let us denote by $\Sigma_p := \{(a_i)_{i=p}^\infty \in \ell_\infty : (a_i)_{i=p}^n \in [D_p^n]$ for every $n\}$. As a consequence of Lemma 2.2, Σ_p is the *limit of the inverse sequence* ([4] 2.5) $\{[D_p^n], \pi_n^m, \mathbb{N}\}$ where $\pi_n^m : \mathbb{C}^m \to \mathbb{C}^n$ is the projection onto the first n coordinates. When Σ_p is endowed with the product topology, that is the topology of coordinatewise convergence, it is a non- empty compact Hausdorff space by ([4] 3.2.13). Σ_p is a weak-star compact subset of the closed unit ball ℓ_∞ since the weak star topology and the pointwise convergence topology coincide on the closed unit ball of ℓ_∞.

Now we describe the spectrum of $A_{us}(B_{\ell_p})$. It is immediate that it is a connected set; it suffices to recall Shilov's idempotent theorem ([6], III.6.5) and notice that there are no idempotent elements in $A_{us}(B_{\ell_p})$.

Theorem 3.2. *Σ_p is homeomorphic to the spectrum of $A_{us}(B_{\ell_p})$.*

Proof. (cf ([10], 8.3)) First of all, observe that any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$ is completely determined by the sequence of values $\{\Psi(F_n)\}$ since Ψ is determined by its behaviour on $\mathcal{P}_s(\ell_p)$, the algebra generated by $\{F_n\}$, which in turn is dense in $A_{us}(B_{\ell_p})$.

We construct an embedding

$$j : (a_i)_{i=p}^\infty \in \Sigma_p \rightsquigarrow \Phi \in \mathcal{M}(A_{us}(B_{\ell_p})),$$

and prove that it is a homeomorphism. Given $(a_i)_{i=p}^\infty \in \Sigma_p$ a homomorphism $j[(a_i)_{i=p}^\infty] := \Phi$ on $A_{us}(B_{\ell_p})$ is defined in the following way: Every polynomial $P \in \mathcal{P}_s(\ell_p)$ may be written as $g \circ \mathcal{F}_p^n$ for some $n \in \mathbb{N}$ and some polynomial g in $n - p + 1$ variables. Thus we may define $\Phi(P) := g(a_p, \ldots, a_n)$. Certainly $\Phi(P)$ is well defined since if $P = h \circ \mathcal{F}_p^m$ for some other polynomial h, and, say, $m > n$, then by Corollary 1.2, $h = \tilde{g}$, where $\tilde{g}$ has the same meaning as in Lemma 2.2. Hence $g(a_p, \ldots, a_n) = \tilde{g}(a_p, \ldots, a_n, \ldots, a_m) = h(a_p, \ldots, a_n, \ldots, a_m)$. It is easy now to see that Φ is linear and multiplicative on the subalgebra of symmetric polynomials. Also $|\Phi(P)| = |g(a_p, \ldots, a_n)| \leq ||g||_{[D_p^n]} = ||g||_{D_p^n} \leq ||P||$. Therefore Φ is uniformly continuous on $\mathcal{P}_s(\ell_p)$, and hence it has a continuous linear and multiplicative extension to the closure of $\mathcal{P}_s(\ell_p)$ that is, to $A_{us}(B_{\ell_p})$. We still denote this extension by Φ.

Obviously, j is one to one. Moreover j is also an onto mapping: Indeed, for any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$, the sequence $\{\Psi(F_n)\} \in \Sigma_p$ because $\{\Psi(F_n)_{n=p}^m\}$ is an element of the joint spectrum of $\mathcal{M}(A_{us}^m(B_{\ell_p}))$ (obtained just by taking the restriction of Ψ to $A_{us}^n(B_{\ell_p})$) which we know to be $[D_p^m]$. Of course, $j[\{\Psi(F_n)\}] = \Psi$ since they coincide on each F_n.

Next, this embedding is continuous. To see this, observe first that the spectrum $\mathcal{M}(A_{us}(B_{\ell_p}))$ is an equicontinuous subset of the dual space $(A_{us}(B_{\ell_p}))^*$. Therefore, the weak-star topology coincides on it with the topology of pointwise convergence on the elements of the dense set of all symmetric polynomials, and hence on the generating system $\{F_n\}_{n=p}^\infty$.

Finally j is a homeomorphism as the continuous bijection between two compact Hausdorff spaces. □

We can view Σ_p as *"the joint spectrum"* of the sequence $\{F_n\}_{n=p}^{\infty}$, since $\Phi(F_n) = a_n$.

We denote by $\mathcal{F}_p$ the mapping $x \in \overline{B_{\ell_p}} \rightsquigarrow (F_p^n(x)) \in \mathbb{C}^{\mathbb{N}}$. Note that $\mathcal{F}_p(\overline{B_{\ell_p}}) \subset \Sigma_p$. So we may remark that the set $D_p = \mathcal{F}_p(\overline{B_{\ell_p}}) \subset \Sigma_p$ corresponds to the set of point evaluation multiplicative functionals on $A_{us}(B_{\ell_p})$. Actually, we have that $D_p \subset B_{c_0} \cup \{(e^{pi\theta}, \cdots, e^{ni\theta}, \cdots) | \, \theta \in [0, 2\pi]\}$. To see this, we first let $x \in \overline{B_{\ell_p}}$ be such that $|x_m| < 1$ for all $m \in \mathbb{N}$. Then, as we observed in the proof of Theorem 1.3, the sequence $(F_n(x))_{n=p}^{\infty}$ converges to 0. In case $x \in \overline{B_{\ell_p}}$ is such that $|x_{m'}| = 1$ for some $m' \in \mathbb{N}$, then m' is unique, $x_{m'} = e^{i\theta}$ and further, $x_m = 0$ if $m \neq m'$. Thus $F_n(x) = e^{ni\theta}$.

It is clear that $\overline{D_p^n} \subset [D_p^n]$ but we do not know whether this embedding is proper. This is related to a corona type theorem for $A_{us}(B_{\ell_p})$ since D_p is dense in Σ_p if $\overline{D_p^n} = [D_p^n]$ for all $n \in \mathbb{N}$.

Note that if $q > p$ then $D_p \subset D_q$ and the inclusion is strict. Indeed, let $x \in B_{\ell_q}$ so that $x \notin \ell_p$. If $\mathcal{F}_q(y) = \mathcal{F}_p(x)$ for some $y \in \ell_q$ then $x \sim y$ in ℓ_q and so $x \sim y$ in ℓ_p, which is a contradiction.

Proposition 3.3. *$\Sigma_p \subset \ell_\infty$ is polynomially convex and coincides with the polynomial convex hull of $D_p \subset (\ell_\infty, \tau_{pw})$.*

Proof. Let $(a_i)_{i=p}^{\infty} \in \ell_\infty$ be such that $|P((a_i))| \leq ||P||_{\Sigma_p}$ for all polynomials $P \in \mathcal{P}(\ell_\infty)$. For any $n \geq p$ and any $g \in \mathcal{P}(\mathbb{C}^{n+1-p})$, the mapping Q given by $(x_i)_{i=p}^{\infty} \in \ell_\infty \rightsquigarrow g(x_p, \ldots, x_n)$ is a polynomial on ℓ_∞. Hence

$$|g(a_p, \ldots, a_n)| = |Q((a_i))| \leq ||Q||_{\Sigma_p} \leq ||g||_{[D_p^n]}.$$

Therefore $(a_p, \ldots, a_n) \in [D_p^n]$, as we want and Σ_p is polynomially convex. So to finish, it is enough to check that Σ_p is contained in the polynomial convex hull of D_p. To do this, let $(a_i)_{i=p}^{\infty} \in \Sigma_p$ and $P \in \mathcal{P}((\ell_\infty, \tau_{pw}))$. As P is pointwise continuous, it depends on a finite number of variables, say $x_p, \ldots, x_n$. Thus the mapping q given by $(x_p, \ldots, x_n) \rightsquigarrow P(x_p, \ldots, x_n, 0, \ldots, 0, \ldots)$ is a polynomial on $\mathbb{C}^{n+1-p}$. Since $(a_p, \ldots, a_n) \in [D_p^n]$,

$$|P((a_i))| = |P(a_p, \ldots, a_n, 0, \ldots, 0, \ldots)| = |q(a_p, \ldots, a_n)|$$
$$\leq ||q||_{[D_p^n]} = ||q||_{D_p^n} \leq ||P||_{D_p},$$

it follows that $(a_i)_{i=p}^{\infty}$ belongs to the polynomial convex hull of D_p. □

Theorem 3.4. *There is an algebraic and topological isomorphism between $A_{us}(B_{\ell_p})$ and the uniform Banach algebra on Σ_p generated by the $w^*(\ell_\infty, \ell_1)$ continuous coordinate functionals $\{\pi_k\}_{k=p}^{\infty}$.*

Proof. For every $f \in A_{us}(B_{\ell_p})$ and $\Phi \in \mathcal{M}(A_{us}(B_{\ell_p}))$ denote by $\hat{f}(\Phi) = \Phi(f)$ the standard Gelfand transform which is known to be an algebraic isometry into $C(\Sigma_p)$. Recall that the range of the Gelfand transform is a closed subalgebra which, as we are going to see, will coincide with A_p, the uniform Banach subalgebra of $C(\Sigma_p)$ generated by the coordinate functionals $\{\pi_k\}_{k=p}^{\infty}$.

Since $\hat{F}_k(\xi) = \xi_k$ for $\xi = (\xi_i)_i \in \Sigma_p$, it follows that the Gelfand transform of F_k is the k^{th} coordinate functional on ℓ_∞. As $A_{us}(B_{\ell_p})$ is the closure of the algebra generated by $\{F_k : k \geq p\}$, it follows that $\hat{f} \in A_p$ for every $f \in A_{us}(B_{\ell_p})$. Therefore A_p is precisely the range of the Gelfand transform. □

Proposition 3.5. *The mapping $S : f \in A(D) \to F \in A_{us}(B_{\ell_p})$ defined by $F((x_i)) = \sum_{i=1}^\infty x_i^p f(x_i)$ is an isometry onto the closed subspace $\mathcal{F}$ of $A_{us}(B_{\ell_p})$ generated by $\{F_{k+p}\}_{k=0}^\infty$.*

Proof. Let $f(z) = \sum_{k=0}^\infty c_k z^k$ be the Taylor series expansion. For each $(x_i) \in B_{\ell_p}$, put

$$F((x_i)) := \sum_{k=0}^\infty c_k F_{k+p}((x_i)) = \sum_{k=0}^\infty \sum_{i=1}^\infty c_k x_i^{p+k}.$$

Since $|F_{k+p}((x_i))| \leq ||(x_i)||^{p+k}$ and the series $\sum_{k=0}^\infty c_k t^k$ is absolutely convergent in the open unit disc,

$$\sum_{k=0}^\infty \sum_{i=1}^\infty |c_k x_i^{p+k}| = \sum_{k=0}^\infty |c_k| \sum_{i=1}^\infty |x_i^{p+k}| =$$

$$\sum_{k=0}^\infty |c_k| F_{k+p}((|x_i|)) \leq \sum_{k=0}^\infty |c_k|(||(x_i)||^{p+k}) = ||(x_i)||^p \sum_{k=0}^\infty |c_k|(||(x_i)||^k) < \infty.$$

So $F((x_i))$ is well defined and $F((x_i)) = \sum_{i=1}^\infty \sum_{k=0}^\infty c_k x_i^{p+k} = \sum_{i=1}^\infty x_i^p f(x_i)$.

Also $|F((x_i))| = |\sum_{i=1}^\infty x_i^p f(x_i)| \leq \sum_{i=1}^\infty |x_i^p||f(x_i)| \leq ||f||_D ||(x_i)||^p$, and hence $||F||_{B_{\ell_p}} \leq ||f||_D$. On the other hand, if $a \in D$ and $x_0 = (a, 0, \dots, 0, \dots)$, we have $x_0 \in B_{\ell_p}$ and $|F(x_0)| = |a|^p |f(a)|$. By the maximum principle, it follows that $||F||_{B_{\ell_p}} \geq ||f||_D$. Consequently, $||F||_{B_{\ell_p}} = ||f||_D$.

Now we check that $F \in A_{us}(B_{\ell_p})$ and then that actually, $F \in \mathcal{F}$. To do this, let $s_m(t) = \sum_{k=0}^m c_k t^k$ be the partial sums of the Taylor series of f and let $\psi_n = \frac{1}{n}(s_0 + s_1 + \cdots + s_n)$ be the Cesáro means. Put $S_m((x_i)) = \sum_{k=0}^m c_k F_{k+p}((x_i)) = \sum_{i=1}^\infty x_i^p s_m(x_i)$. Then

$$\Psi_n((x_i)) = \frac{1}{n}(S_0((x_i)) + S_1((x_i)) + \cdots + S_n((x_i))) =$$

$$\frac{1}{n} \sum_{i=1}^\infty x_i^p (s_0(x_i) + s_1(x_i) + \cdots + s_n(x_i)) = \sum_{i=1}^\infty x_i^p \psi_n(x_i)$$

are the Cesáro means partial sums of $\sum_{k=0} c_k F_{k+p}$.

Since

$$|\Psi_n((x_i)) - F((x_i))| = |\sum_{i=1}^\infty x_i^p (\psi_n(x_i) - f(x_i)| \leq ||\psi_n - f|| \cdot ||(x_i)||,$$

the uniform convergence of ψ_n to f on D implies the uniform convergence of Ψ_n to F on B_{ℓ_p}. So $F \in A_{us}(B_{\ell_p})$ and moreover $F \in \mathcal{F}$ since every Ψ_n is obviously in $\mathcal{F}$.

The mapping S being an isometry, its range is a closed subspace of $A_{us}(B_{\ell_p})$. Therefore, its range is onto $\mathcal{F}$ since F_{k+p} is the image of z^k. □

Proposition 3.6. *$\Sigma_p \neq \bar{B}_{\ell_\infty}$ for every positive integer p.*

Proof. We show that no point of the form $(e^{i\theta}, \pm 1, 0, \dots, 0, \dots)$ is in Σ_p. This will follow from Proposition 3.5 applied to every linear fractional transformation $f(z) = \frac{z-a}{1-\bar{a}z}$, $|a| < 1$, whose Taylor series $f(z) = -a + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1-|a|^2)z^n$ has radius of convergence bigger than 1. Its image F by the mapping S in 3.5 is $F = -aF_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1-|a|^2)F_{n+p}$, Moreover the convergence of this series is uniform on B_{ℓ_p}, and therefore the Gelfand transform of F is $\hat{F} = -a\pi_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1-|a|^2)\pi_{n+p}$. Pick θ such that $-ae^{i\theta} = |a|$ and assume that the point $(e^{i\theta}, 1, 0, \ldots, 0, \ldots)$ is in Σ_p. Then $|\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| \leq ||F|| = ||f|| = 1$. However, $|\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| = |(-a\pi_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1-|a|^2)\pi_{n+p})(e^{i\theta}, 1, 0, \ldots, 0, \ldots))| = |-ae^{i\theta} + 1 - |a|^2| = |a| + 1 - |a|^2 > 1$, which is a contradiction. □

We remark that arguments similar to those in Theorem 1.3 enable us to show that no point of the form $(1, -1, -1, z_4, z_5, ...) \in \overline{B_{\ell_\infty}}$ can be in Σ_p.

Our final result describes the class of functionals on ℓ_∞ which belong to the range of of $A_{us}(B_{\ell_p})$ under the Gelfand transform, thereby completing a circle of connections between $A_{us}(B_{\ell_p}), A(D), C(\Sigma_p)$, and certain functionals on ℓ_∞. Recall that such Gelfand transforms are weak-star continuous on Σ_p.

Proposition 3.7. *Let ϕ be a linear functional on ℓ_∞ weak-star continuous on Σ_p. Then ϕ is the Gelfand transform of some $F \in A_{us}(B_{\ell_p})$ and, furthermore, there is $f \in A(D)$ with $||\phi||_{\Sigma_p} = ||f||_D$ and such that*

$$\phi(\mathcal{F}_p(x)) = \sum_{i=1}^{\infty} a_i^p f(a_i) \quad x = (a_i) \in B_{\ell_p}.$$

Proof. Every $(a_i)_{i=p}^{\infty} \in \Sigma_p$ is the $w(\ell_\infty, \ell_1)$ convergent series $\Sigma_{i=p}^{\infty} a_i e_i$. Therefore, $\phi((a_i)) = \Sigma_{i=p}^{\infty} a_i \phi(e_i)$ and, setting $c_i = \phi(e_i)$, we have that the series $\Sigma_{i=p}^{\infty} c_i \pi_i$ is pointwise convergent in Σ_p to ϕ. Moreover, the partial sums of this series are uniformly bounded on Σ_p since

$$\begin{aligned} |\Sigma_{j=p}^{l} c_j \pi_j((a_i))| &= |\Sigma_{j=p}^{l} c_j a_j| = |\Sigma_{j=p}^{l} \phi(e_j) a_j| \\ &= |\phi(a_p, \cdots, a_l, 0, \cdots, 0, \cdots)| \leq ||\phi||_{\ell_\infty}. \end{aligned}$$

Thus ϕ is the weak limit in $C(\Sigma_p)$ of the series $\Sigma_{i=p}^{\infty} c_i \pi_i$. Since each of the terms in the series belongs to the range of the Gelfand transform, it follows that there is $F \in A_{us}(B_{\ell_p})$ such that $\hat{F} = \phi$ and also that the series $F = \sum_{i=p}^{\infty} c_i F_i$ converges weakly in $A_{us}(B_{\ell_p})$.

Note that $||\phi||_{\Sigma_p} = ||F||_{B_{\ell_p}}$, and also that F belongs to the weakly closed subspace $\mathcal{F}$ generated by $\{F_{k+p}\}_{k=0}^{\infty}$. Thus by Proposition 3.5 there is $f \in A(D)$ such that $F(x) = F(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i=1}^{\infty} x_i^p f(x_i)$. Therefore, $\phi(\mathcal{F}_p(x)) = \hat{F}(\mathcal{F}_p(x)) = F(x)$ as we wanted. □.

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IMECC, Universidade Estadual de Campinas, C.P. 6065, Campinas, SP 13081, Brazil
E-mail address: alencar@ime.unicamp.br

Department of Mathematics, Kent State University, Kent, OH 44242, USA. Current address: School of Mathematics, Trinity College, Dublin 2, Ireland
E-mail address: aron@mcs.kent.edu

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain
E-mail address: Pablo.Galindo@uv.es

Inst. for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, 3 b, Naukova str., Lviv, Ukraine, 290601
E-mail address: sirand@mebm.lviv.ua