SETS OF WEAK SEQUENTIAL CONTINUITY FOR POLYNOMIALS

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ABSTRACT. Let $P : E \to \mathbb{K}$ be an $N-$homogeneous polynomial, where E is a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We study properties of the set $C_P = \{x \in E : P$ is weakly sequentially continuous at $x\}$.

INTRODUCTION

Our interest in set of points of weak sequential continuity of a polynomial arises from the following simple observations. If P is any $2-$homogeneous scalar valued polynomial on E which is weakly sequentially continuous at 0, then P is weakly sequentially continuous at every point of E. However, the analogous result for $3-$homogeneous polynomials is false. (We shall recall the simple details for these observations, as well as the necessary background material, below.) Given an $N-$homogeneous polynomial $P : E \to \mathbb{K}$, we let $C_P = \{x \in E : P$ is weakly sequentially continuous at $x\}$. Our aim in this paper is to study C_P. In Section 1, we examine general properties of this set, obtaining for example a formula for $C_{P.Q}$. This formula will enable us to obtain information about non-reducibility of polynomials, and our techniques will also yield information about, for example, 3 and $4-$homogeneous polynomials on ℓ_2. Later in this section, we raise and given partial answers to the following questions:
(a). Given $P \in \mathcal{P}(^N E)$, does there exist $Q \in \mathcal{P}(^{N+1}E)$ such that $C_P = C_Q$?
(b). Given P and $Q \in \mathcal{P}(^N E)$, does there exist a polynomial R such that $C_R = C_P \cap C_Q$, or such that $C_R = C_P \cup C_Q$? In Section 2, we focus our attention on properties of C_P when the underlying space E is separable, or has an unconditional finite dimensional decomposition.

Our methods shed light on the structure of certain spaces of polynomials and, at several places in the text we have inserted examples to illustrate this. Our examples will be restricted to ℓ_p-spaces; note that for spaces with the Dunford Pettis property, every polynomial is weakly sequentially continuous, and so there are no examples of interest for these spaces. The same occurs with T' (the dual of Tsirelson's original space), and in fact there are Banach spaces E without the Dunford-Pettis property such that both $\mathcal{P}(^n E) = \mathcal{P}_{wsc}(^n E)$ and $\mathcal{P}(^n E') = \mathcal{P}_{wsc}(^n E')$ for every $n \in \mathbb{N}$ ([C-G-G], Theorem 5.4).

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As usual, $\mathcal{P}(^N E)$ denotes the continuous $N-$homogeneous scalar valued polynomials P on E, that is those functions P to which there is a necessarily unique continuous symmetric $N-$linear mapping $A : E \times ... \times E \to \mathbb{K}$ such that $P(x) = A(x, ..., x)$ for all $x \in E$. We recall that given P and the associated A as above, the notation $A(x^j, y^{N-j})$ means $A(\overbrace{x, ..., x}^{j}, \overbrace{y, ..., y}^{N-j})$. We refer to the recent book by S. Dineen [Di] for background material.

We will be interested in the subspace $\mathcal{P}_{wsc}(^N E) \subset \mathcal{P}(^N E)$ consisting of polynomials P which are weakly sequentially continuous at *every* $x \in E$. We will also make use of the space $\mathcal{P}_{wsc0}(^N E)$ of those polynomials which are weakly sequentially continuous at $0 \in E$. A related paper by C. Boyd and R. Ryan is worth mentioning. In [B-R], the authors study those polynomials P which are weakly continuous on bounded sets at the origin.

As remarked above, if $P : E \to \mathbb{K}$ is any $2-$homogeneous polynomial which is weakly sequentially continuous at 0 and if $(x_n) \to x_0$ weakly, then $P(x_n) - P(x_0) = A(x_n, x_n - x_0) + A(x_0, x_n - x_0) = P(x_n - x_0) + 2A(x_0, x_n - x_0)$, where A is the symmetric continuous bilinear form associated to P. Now $P(x_n - x_0) \to 0$ by hypothesis and $A(x_0, x_n - x_0) \to 0$ since $A(x_0, \cdot)$ is a continuous linear form, and so the assertion is proved. Moreover, for $P \in \mathcal{P}(^3 \ell_2)$, $P(x) \equiv x_1 \sum_{n=1}^{\infty} x_n^2$, it is easy to verify that $C_P = \{x \in \ell_2 \ : \ x_1 = 0\}$.

§1. General properties of C_P

We begin with several basic properties of the set C_P.

Proposition 1. Let $P \in \mathcal{P}(^N E)$.

(1). C_P is a closed subset of E.

(2). (cf: Cor 2, [B-R]) If $x \in C_P$, then $\lambda x \in C_P$ for every $\lambda \in \mathbb{K}$. In particular, if $C_P \neq \emptyset$, then $0 \in C_P$.

(3). $C_P = \cap_{j=0}^{N-2} \{x \in E \ : \ $ the $N-j$-homogeneous polynomial $\Phi_j(x) : y \to A(x^j, y^{N-j})$ is in $\mathcal{P}_{wsc0}(^{N-j} E)\}$.

Proof. (1). Let $(y_j) \subset C_P$ converge in norm to $x \in E$. To show that $x \in C_P$, let (x_n) be a sequence which converges weakly to x. Given $\epsilon > 0$, first choose j so that $||y_j - x|| < \epsilon$ and then choose n_0 such that for all $n \geq n_0$, $|P(y_j + [x_n - x]) - P(y_j)| < \epsilon$. Therefore, for all $n \geq n_0$, $|P(x_n) - P(x)| \leq |P(x_n) - P(y_j + [x_n - x])| + |P(y_j + [x_n - x]) - P(y_j)| + |P(y_j) - P(x)|$. Since (x_n) is bounded and P is uniformly continuous on bounded sets, the first and third terms above are $\leq C\epsilon$ for some absolute constant C. The middle term is dominated by ϵ, and so the proof of (1) is complete.

(2). If $x \in C_P$, it is straightforward that $\frac{1}{\lambda} x \in C_P$ for every $\lambda \in \mathbb{K}$, $\lambda \neq 0$. By part (1), $0 = \lim_n \frac{1}{n} x \in C_P$.

(3). If $x \in C_P$, then an application of the polarization formula (see, e.g., [Di], p. 8) shows that for every $j = 0, 1, ..., N, \Phi_j(x)$ is weakly sequentially continuous at x. By part (2), each $\Phi_j(x) \in \mathcal{P}_{wsc0}(^{N-j}E)$. Conversely, suppose that each $\Phi_j(x) : y \rightsquigarrow A(x^j, y^{N-j})$ is weakly sequentially continuous at 0, and let (x_n) be a sequence in E which converges weakly to x. The result follows immediately from the Taylor series development of P about x, $P(x_n) - P(x) = \sum_{j=0}^{N-1} \begin{pmatrix} N \\ j \end{pmatrix} \Phi_j(x)(x_n - x)$, and the fact that $\Phi_{N-1}(x)$ is linear and hence automatically weakly sequentially continuous. □

As a consequence of part (3) of the above proposition, we have the following.

Corollary 2. Suppose that $\mathcal{P}(^lE) = \mathcal{P}_{wsc}(^lE)$ for each $l = 1, ..., r-1$ and that $\mathcal{P}(^rE) \neq \mathcal{P}_{wsc}(^rE)$. Then the following hold:

(i). For every $P \in \mathcal{P}(^rE)$, either $C_P = \emptyset$ or $C_P = E$, and

(ii). For every $P \in \mathcal{P}(^{r+1}E)$, either $C_P = \emptyset$ or C_P is a subspace of E.

Note that (i) above strengthens the example given in the introduction. We will show in Theorem 11 that a further strengthening of Corollary 2 holds if we assume that E is separable. Let us now discuss a sufficient condition for C_P to be the entire space E, that is for P to be weakly sequentially continuous everywhere.

The proof of the next proposition depends on the following result.

Lemma 3. Let $P \in \mathcal{P}(^rE, F)$ be an $r-$homogeneous polynomial between Banach spaces E and F. If $\{\gamma_1, ..., \gamma_{r+1}\}$ is a linearly independent subset of E' and if $S \subset F$ is a subspace such that $P(\gamma_j^{-1}(0)) \subset S$ for each $j = 1, ..., r+1$, then $P(E) \subset S$.

Proof. When $r = 1$, the result is trivial, and we'll proceed by induction. Assume the result for $k = 1, ..., r-1$. Let $x \in E$ be an arbitrary point, so that x can be expressed as $x = x_1 + e$ where $\gamma_1(x_1) = 0$ and $e \in \cap_{i=2}^{r+1} \gamma_i^{-1}(0)$. Now, $P(x) = \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} A(x_1^j, e^{r-j}) = P(x_1) + \sum_{j=1}^{r-1} \begin{pmatrix} r \\ j \end{pmatrix} A(x_1^j, e^{r-j}) + P(e)$. Note that both the first and last terms in the previous sum belong to S.

For each fixed e, $x_1 \rightsquigarrow A(x_1^j, e^{r-j})$ is a $j-$homogeneous polynomial, with $1 \leq j \leq r-1$, which takes each of the r hyperplanes $\gamma_2^{-1}(0), ..., \gamma_{r+1}^{-1}(0)$ into S. Indeed, by the polarization formula, $A(x_1^j, e^{r-j})$ can be expressed as a finite linear combination of vectors of the form $P(sx_1 + te)$, $s, t \in \mathbb{Z}$. So, if $x_1 \in \gamma_k^{-1}(0)$ for some $k = 2, ..., r+1$, it follows that $sx_1 + te$ is also in $\gamma_k^{-1}(0)$, and hence $P(\gamma_k^{-1}(0)) \in S$. Therefore $A(x_1^j, e^{r-j}) \in S$.

By the induction hypothesis, the image of the polynomial $x_1 \rightsquigarrow A(x_1^j, e^{r-j})$ lies in S for each $j = 1, ..., r-1$, and so every $P(x) \in S$. □

Proposition 4. Let $P \in \mathcal{P}(^N E)$ be such that for $N-1$ linearly independent functionals $\gamma_i,\ 1 \le i \le N-1$, we have $\gamma_i^{-1}(0) \subset C_P$. Then $C_P = E$.

Proof. For each $j = 0, 1, ..., N$, let $\Phi_j \in \mathcal{P}(^j E, \mathcal{P}(^{N-j}E))$ be given by $\Phi_j(x)(y) \equiv A(x^j, y^{N-j})$. By Proposition 1 (3), $C_P = \cap_{j=0}^{N-2}\{x \in E\ :\ \Phi_j(x) \in \mathcal{P}_{wsc0}(^{N-j}E)\}$. Hence, for each fixed i and j, since $\gamma_i^{-1}(0) \subset C_P$ it follows that $\Phi_j(\gamma_i^{-1}(0)) \subset \mathcal{P}_{wsc0}(^{N-j}E)$. By Lemma 3, it follows that for every $x \in E,\ \Phi_j(x) \in \mathcal{P}_{wsc0}(^{N-j}E)$. Hence $C_P = E$, as required. □

The same method of proof shows that if $\mathcal{P}(^j E) = \mathcal{P}_{wsc}(^j E)$ for $j = 1, ..., s$, and $P \in \mathcal{P}(^N E)$ is such that C_P contains $N - s$ hyperplanes, then in fact $C_P = E$.

As an example of the use of this result, consider the polynomial $P \in \mathcal{P}(^5\ell_2),\ P(x) = x_1x_2x_3\sum_{j=1}^{\infty} x_j^2$. It is easy that $C_P = \{x \in \ell_2\ :\ x_1 = 0 \text{ or } x_2 = 0 \text{ or } x_3 = 0\}$. Proposition 4 implies that there is no $Q \in \mathcal{P}(^4\ell_2)$ such that $C_P = C_Q$. Similarly, since $\mathcal{P}(^2\ell_3) = \mathcal{P}_{wsc}(^2\ell_3)$, the polynomial $P(x) \in \mathcal{P}(^6\ell_3),\ P(x) = x_1x_2x_3\sum_{j=1}^{\infty} x_j^3$, is such that $C_P \ne C_Q$ for any $5-$homogeneous polynomial Q.

We now turn to relations between C_P and C_Q. Although the proof of the theorem below is not difficult, we will see that the result is useful for much of what follows.

Theorem 5. If $P \in \mathcal{P}(^N E)$ and $Q \in \mathcal{P}(^M E)$, then

$$C_{P\cdot Q} = (C_P \cap C_Q) \cup (C_P \cap P^{-1}(0)) \cup (C_Q \cap Q^{-1}(0)).$$

Proof. We will only prove that $C_{P\cdot Q} \subseteq (C_P \cap C_Q) \cup (C_P \cap P^{-1}(0)) \cup (C_Q \cap Q^{-1}(0))$, the reverse inclusion being quite easy. Let $x \in C_{P\cdot Q}$ and let $(x_n) \to x$ weakly in E. Since $x + t(x_n - x) \xrightarrow{w} x$, we have that $P(x + t(x_n - x))Q(x + t(x_n - x)) \to P(x)Q(x)$ as $n \to \infty$ for all $t \in \mathbb{K}$. Applying the polarization formula to both P and Q and the associated symmetric multilinear forms A and B, respectively, we obtain

$$P(x + (t(x_n - x)) = \sum_{j=0}^{N} \begin{pmatrix} N \\ j \end{pmatrix} A(x^j, (x_n - x)^{N-j})t^{N-j},$$

$$Q(x + (t(x_n - x)) = \sum_{l=0}^{M} \begin{pmatrix} M \\ l \end{pmatrix} B(x^l, (x_n - x)^{M-l})t^{M-l}.$$

Since $\{\begin{pmatrix} N \\ j \end{pmatrix} A(x^j, (x_n - x)^{N-j})\}$ and $\{\begin{pmatrix} M \\ l \end{pmatrix} B(x^l, (x_n - x)^{M-l})\}$ are bounded sequences for each j and l, by passing to a subsequence we may assume that there are $\alpha_j,\ j = 0, ..., N$ and $\beta_l,\ l = 0, ..., M$, such that $\{\begin{pmatrix} N \\ j \end{pmatrix} A(x^j, (x_{n_k} - x)^{N-j})\} \to \alpha_j$ and $\begin{pmatrix} M \\ l \end{pmatrix} B(x^l, (x_{n_k} - x)^{M-l}) \to \beta_l$. Note, in particular, that $\alpha_N = P(x)$ and $\beta_M = Q(x)$.

Consequently,

$$P(x+t(x_{n_k}-x))\cdot Q(x+t(x_{n_k}-x))\to (\sum_{j=0}^{N}\alpha_j t^{N-j})\cdot(\sum_{l=0}^{M}\beta_l t^{M-l}),$$

and so

$$(\sum_{j=0}^{N}\alpha_j t^{N-j})\cdot(\sum_{l=0}^{M}\beta_l t^{M-l})=\alpha_N\cdot\beta_M.$$

Now, in order for the product of two polynomials to be constant, either both polynomials must be constant or one of them should be identically zero. Thus, we have three possibilities:
(1). $\alpha_0=\cdots\alpha_{N-1}=\beta_0=\cdots\beta_{M-1}=0$, and so $P(x_{n_k})\to P(x)$ and $Q(x_{n_k})\to Q(x)$.
(2). $\alpha_0=\cdots\alpha_N=0$, so that $P(x_{n_k})\to P(x)=0$.
(3). $\beta_0=\cdots\beta_M=0$, so that $Q(x_{n_k})\to Q(x)=0$.

Summarizing, for *each* sequence $(x_n)\in E$ which converges weakly to x, we have a subsequence such that (1), (2), or (3) holds. We need to prove that the same possibility holds for *every subsequence.* First, observe that if (2) holds for some sequences and (1) holds for some other sequences, then (2) holds for every sequence. A similar remark obviously holds with (3) and (1).

For the remaining case, let us suppose that there are two sequences (x_n) and (y_n), both weakly convergent to $x\in E$, such that (2) holds while (3) fails for (x_n), and that (3) holds while (2) fails for (y_n). In other words, by passing to a subsequence we may suppose that $P(x_n)\to P(x)=0$ while $Q(x_n)\to\beta\neq Q(x)$, and that $Q(y_n)\to Q(x)=0$ although $P(y_n)\to\alpha\neq P(x)$. Since $x\in C_{P\cdot Q}$, $(t+1)x\in C_{P\cdot Q}$ for every $t\in\mathbb{K}$. Therefore, $P(tx_n+y_n)\cdot Q(tx_n+y_n)\to P((t+1)x)\cdot Q((t+1)x)$. As before, passing to a subsequence we have that

$$P(tx_{n_k}+y_{n_k})\cdot Q(tx_{n_k}+y_{n_k})=(\sum_{j=0}^{N}\begin{pmatrix}N\\ j\end{pmatrix}A(x_{n_k}^j,y_{n_k}^{N-j})t^j)\cdot(\sum_{l=0}^{N}\begin{pmatrix}M\\ l\end{pmatrix}B(x_{n_k}^l,y_{n_k}^{M-l})t^l)$$

converges to the product of polynomials

$$(\sum_{j=0}^{N}a_j t^j)\cdot(\sum_{l=0}^{M}b_l t^l),$$

and this product should be 0. But, for this to occur, one of the polynomials must be identically 0. However, $a_0=\alpha\neq 0$ and $b_M=\beta\neq 0$. Thus, we have a contradiction, and the theorem is proved. □

Since $C_\gamma=E$ for $\gamma\in E'$, we get the following.

Corollary 6. If $P\in\mathcal{P}(^NE)$ and $\gamma\in E'$, then $C_{\gamma\cdot P}=C_P\cup\gamma^{-1}(0)$.

Example 7. We now apply this result in several settings.
1. First, we show that the polynomial $P\in\mathcal{P}(^3\ell_2)$, $P(x)=x_1\sum_j x_{2j}^2+x_2\sum_j x_{2j+1}^2$ is

irreducible in the space of polynomials. Indeed, suppose that $P = Q \cdot R$ where Q and R are non-trivial homogeneous polynomials; necessarily one of the factors, say R, is in E'. By Corollary 6, $C_P = C_Q \cup R^{-1}(0)$. Now, since $C_P = \{x \in \ell_2 \ :\ x_1 = x_2 = 0\}$ and since C_Q is either $\emptyset$ or ℓ_2, we have a contradiction.

2. Also, let $1 < p < \infty$ and let N be the smallest integer $\geq p$. If $P \in \mathcal{P}({}^N\ell_p)$ is given by $P(x) = \sum_{i\in J} x_i^N$ where $J \subset \mathbb{N}$ is an arbitrary infinite set, then $C_P = \emptyset$. Therefore if $Q \in \mathcal{P}({}^{N+1}\ell_p)$, $Q(x) = x_1 \sum_i x_{2i}^N + x_2 \sum_i x_{2i+1}^N$, then $C_Q = \{x \in \ell_p \ :\ x_1 = x_2 = 0\}$, which implies by Corollary 6 that Q is irreducible.

3. Similarly, let $P : \mathcal{L}^p[0,1] \to \mathbb{R}$ $(1 < p < \infty)$, $P(f) = \int_a^b f(t)^2 dt$, where $0 \leq a < b \leq 1$. Then $C_p = \emptyset$, since the Haar basis $(f_n) \overset{w}{\to} 0$ although $P(f_n) \not\to 0$. From this, it follows that for the 3−homogeneous polynomial $Q : \mathcal{L}^p[0,1] \to \mathbb{R}$, $Q(f) = \int_0^{1/2} f(t)dt \int_0^{1/2} f^2(t)dt + \int_{1/2}^1 f(t)dt \int_{1/2}^1 f^2(t)dt$, C_Q is the intersection of two hyperplanes, $C_Q = \{f \in \mathcal{L}^p[0,1] \ :\ \int_0^{1/2} f(t)dt = \int_{1/2}^1 f(t)dt = 0\}$. Once again, C_Q is irreducible.

4. As our final example, fix $p \in (1,\infty)$ and let N be the smallest integer $\geq p$. Define $R \in \mathcal{P}({}^{N+2}\ell_p)$ by

$$R(x) = \sum_{i=1}^{\infty} \left(\frac{x_i^2}{2^i}\right) \sum_{j\in F_i} x_j^N ,$$

where the sets $\{F_i\}$ form a partition of $\mathbb{N}$ into infinite sets. We claim that R is irreducible. We first prove that this is the case when the underlying field is $\mathbb{C}$. A straightforward argument shows that $C_R = \{0\}$. For every $j < N$, every $P \in \mathcal{P}({}^j\ell_p)$ is approximable by finite polynomials, and hence every such P belongs to $\mathcal{P}_{wsc}({}^j\ell_p)$. Therefore, if R could be written as $R = R_1 \cdot R_2$, then we would have three possibilities:

(i). Both R_1 and R_2 have degree $< N$. In this case, C_R would be equal to ℓ_p.

(ii). $R_1 \in \ell_p'$, in which case $C_R = R_1^{-1}(0) \cup C_{R_2}$.

(iii). $R_1 \in \mathcal{P}({}^2\ell_p)$. Suppose first that $N > 2$. By Theorem 5, $C_R \supset C_{R_1} \cap R_1^{-1}(0) = R_1^{-1}(0)$, which is always an unbounded set. Now, suppose that $N = 2$, in which case our remarks in the introduction imply that the only possibilities for both C_{R_1} and C_{R_2} are either $\emptyset$ or ℓ_p.

Therefore, in none of the three cases is it possible to have $C_{R_1\cdot R_2} = \{0\}$, and the argument in the complex case is complete. As for the real case, it suffices to recall [[B-S], Theorem 3] that the complexification of a real polynomial is unique. Therefore, if R could be factored as $R = P \cdot Q$ where P, Q, and R are real polynomials, then the product of the complexifications of P and Q would have to be the polynomial R considered on complex ℓ_p. But, this would contradict our work which showed that in the complex case, R cannot be factored in such a way.

We now turn our attention to three 'permanence' questions. We will make several general comments about them here, returning to obtain more complete answers in Section 2 in the context of E being separable or having an unconditional finite dimensional decomposition.

Question 1. Given $P \in \mathcal{P}(^N E)$, does there always exist $Q \in \mathcal{P}(^{N+1}E)$ such that $C_P = C_Q$?

Note that the answer is obviously yes if $C_P = E$. Note that the converse question, of whether there is $R \in \mathcal{P}(^{N-1}E)$ such that $C_P = C_R$, has a trivial negative answer in case $N = 2$ and $C_P \neq E$; a less trivial negative answer is given in the example following the proof of Proposition 4. We remark that we know of no negative example to Question 1 or to Question 2, which follows.

Question 2. Given $P, Q \in \mathcal{P}(^N E)$, does there exist $R \in \mathcal{P}(^M E)$ for some M such that $C_R = C_P \cap C_Q$?

In §2, dealing with spaces E with unconditional finite dimensional decomposition, we show that under certain conditions Question 2 has an affirmative answer with, moreover, $M = N$. When E is an arbitrary *real* Banach space, Question 2 has a simple positive answer:

Proposition 8. If $P, Q \in \mathcal{P}(^N E)$ for a real Banach space E, then the polynomial $R \equiv P^2 + Q^2 \in \mathcal{P}(^{2N}E)$ is such that $C_R = C_P \cap C_Q$.

Proof. For the non-trivial implication, let $x \in C_R$ and let $x_n \to x$ weakly. Writing the Taylor series of P and Q as functions of $t \in \mathbb{R}$, we obtain $P(x + t(x_n - x)) = \sum_{j=0}^N \begin{pmatrix} N \\ j \end{pmatrix} A(x^j, (x_n - x)^{N-j})t^{N-j}$ and $Q(x + t(x_n - x)) = \sum_{j=0}^N \begin{pmatrix} N \\ j \end{pmatrix} B(x^j, (x_n - x)^{N-j})t^{N-j}$; here A (resp. B) denotes the symmetric $N-$linear form associated to P (resp. Q).

Choose a subsequence $(x_{n_k})_k$ such that $P(x + t(x_{n_k} - x)) \to \sum_{j=0}^N \alpha_j t^{N-j}$ and $Q(x + t(x_{n_k} - x)) \to \sum_{j=0}^N \beta_j t^{N-j}$, where $\alpha_N = P(x)$ and $\beta_N = Q(x)$. Since $x \in C_R$ and $(x + t(x_{n_k} - x)) \xrightarrow{w} x$, $P(x + t(x_{n_k} - x))^2 + Q(x + t(x_{n_k} - x))^2 \to P(x)^2 + Q(x)^2$. Therefore, for all $t \in \mathbb{R}$, $(\sum_{j=0}^N \alpha_j t^{N-j})^2 + (\sum_{j=0}^N \beta_j t^{N-j})^2 = \alpha_N^2 + \beta_N^2$. Since the coefficients are real, it follows that $\alpha_0 = ... = \alpha_{N-1} = \beta_0 = ... = \beta_{N-1} = 0$, and this means that $P(x_{n_k}) \to P(x)$ and $Q(x_{n_k}) \to Q(x)$. Summarizing, whenever we have a sequence (x_n) which tends weakly to x, we can always find a subsequence (x_{n_k}) such that $P(x_{n_k}) \to P(x)$ and $Q(x_{n_k}) \to Q(x)$, and this implies that $x \in C_P \cap C_Q$. □

Our final question complements Question 2:

Question 3. Given $P, Q \in \mathcal{P}(^N E)$, does there exist $R \in \mathcal{P}(^M E)$ for some M such that $C_R = C_P \cup C_Q$?

Question 3 has a negative answer, at least if we require $M = N$. For instance, if $P, Q \in \mathcal{P}(^4\ell_2)$, $P(x) = x_1 x_2 \sum_{j=1}^\infty x_j^2$, and $Q(x) = x_2 x_3 \sum_{j=1}^\infty x_j^2$, then $C_P = \{x \in \ell_2 \ : \ x_1$ or $x_2 =$

$0\}$ and $C_Q = \{x \in \ell_2 \,:\, x_2 \text{ or } x_3 = 0\}$, so that $C_P \cup C_Q = \{x \in \ell_2 \,:\, x_1 \text{ or } x_2 \text{ or } x_3 = 0\}$. However, by the example following Proposition 4, there is no $R \in \mathcal{P}(^4\ell_2)$ such that $C_R = C_P \cup C_Q$.

Our final result in this section will give a general situation in which Question 3 has an affirmative answer. We will need the following consequence of Theorem 5, from which the proposition below follows.

Lemma 9. Suppose that $P, Q \in \mathcal{P}(^N E)$.

(a). If $C_P \subset P^{-1}(0)$ and $C_Q \subset Q^{-1}(0)$, then the polynomial $P \cdot Q \in \mathcal{P}(^{2N} E)$ is such that $C_{P \cdot Q} = C_P \cup C_Q$.

(b). If $Q \in \mathcal{P}_{wsc}(^N E)$, then $C_{P+Q} = C_P$.

Proposition 10. Suppose that $P, Q \in \mathcal{P}(^N E)$. If there exist $P_1, Q_1 \in \mathcal{P}_{wsc}(^N E)$ such that $C_P \subset (P+P_1)^{-1}(0)$ and $C_Q \subset (Q+Q_1)^{-1}(0)$, then $R \equiv (P+P_1)\cdot(Q+Q_1) \in \mathcal{P}(^{2N} E)$ satisfies $C_R = C_P \cup C_Q$.

In particular, if C_P and C_Q are complemented subspaces of E with associated projections Π_P and Π_Q, then $R \equiv (P - P \circ \Pi_P) \cdot (Q - Q \circ \Pi_Q)$ satisfies $C_R = C_P \cup C_Q$.

§2. C_P for special Banach spaces

In this section, we study the sets of weak sequential continuity C_P when P is an $n-$homogeneous polynomial on a *separable* Banach space or, at times, on a Banach space with unconditional *finite dimensional decomposition* (FDD). Not surprisingly, our results are considerably sharper with these added hypotheses.

Our first result characterizes C_P for separable E.

Theorem 11. Let E be a separable Banach space and let $P \in \mathcal{P}(^N E)$ such that $C_P \neq \emptyset$. Then there is a sequence $(P_i)_{i=1}^\infty$, each $P_i \in \mathcal{P}(^{n_i} E)$ with $n_i \in \{1, 2, ..., N-2\}$, such that

$$C_P = \cap_{i=1}^\infty P_i^{-1}(0).$$

The proof of Theorem 11 will be presented after the following Proposition.

Proposition 12. Let E be a separable Banach space.

(a). If $S \subset E$ is a closed subspace of E, then there is a sequence $(\phi_i)_{i=1}^\infty \subset E'$ such that $S = \cap_{i=1}^\infty \phi_i^{-1}(0)$.

(b). If $S \subset F$ is a closed subspace of an arbitrary Banach space F and if $P \in (^r E, F)$, then there is a sequence $(P_i)_{i=1}^\infty \subset \mathcal{P}(^r E)$ such that $P^{-1}(S) = \cap_{i=1}^\infty P_i^{-1}(0)$.

Proof. (a). Let T be an algebraic complement of S; that is, T is a (not necessarily closed) subspace of E such that every element $x \in E$ can be written uniquely as $x = y + z$ where $y \in S$ and $z \in T$. Let $\{z_i\}_{i=1}^{\infty}$ be a dense subset of $\{z \in T \ : \ ||z|| = 1\}$, and for each i, let $\phi_i \in E'$ be a functional of norm 1 such that $\phi_i|_S \equiv 0$ and $\phi_i(z_i) = 1$. It is clear that $S \subset \cap_{i=1}^{\infty}\phi_i^{-1}(0)$. For the converse, let $x \in \cap_{i=1}^{\infty}\phi_i^{-1}(0),\ x = y+z$ where $y \in S$ and $z \in T$. If $z \neq 0$, then $\frac{z}{||z||}$ can be approximated by some z_j. Consequently, $\phi_j(\frac{z}{||z||})$ is close to 1, which contradicts the fact that $\phi_j(x) = \phi_j(z) = 0$. Therefore $x = y \in S$.
(b). We begin by applying [F, Ry], to obtain a factorization of P as $P = \tilde{P} \circ \Phi_r$, where $\Phi_r : E \to \hat{\bigotimes}_{\pi,s}^{r} E$ is the canonical mapping $x \rightsquigarrow x \otimes ... \otimes x$ and $\tilde{P} : \hat{\bigotimes}_{\pi,s}^{r} E \to F$ is the canonical linear mapping associated to P. We apply part (a), obtaining that $\tilde{P}^{-1}(S) = \cap_{i=1}^{\infty}\phi_i^{-1}(0)$ for some collection $\{\phi_i \ : \ i = 1, 2, ...\} \subset (\hat{\bigotimes}_{\pi,s}^{r} E)'$. Therefore, $P^{-1}(S) = \cap_{i=1}^{\infty} P_i^{-1}(0)$ where $P_i = \phi_i \circ \Phi_r$, which completes the proof. □

Proof of Theorem 11. Using Proposition 1 and the fact that $C_P \neq \emptyset$, we see that

$$C_P = \cap_{j=1}^{N-2}\Phi_j^{-1}(\mathcal{P}_{wsc0}(^{N-j}E)).$$

Since each $\Phi_j \in \mathcal{P}(^jE, \mathcal{P}(^{N-j}E))$, we may apply part (b) of Proposition 12 to conclude that each $\Phi_j^{-1}(\mathcal{P}_{wsc0}(^{N-j}E))$ is an intersection of kernels of $j-$homogeneous polynomials on E. The result follows by taking as the required sequence (P_i) all the $j-$homogeneous polynomials, $j = 1, ..., N-2$, so obtained. □

It is worth noting that the same arguments show that for separable Banach spaces E such that $\mathcal{P}(^lE) = \mathcal{P}_{wsc}(^lE)$ for $l = 1, ..., r-1$, then for every N and every $P \in \mathcal{P}(^NE)$, either $C_P = \emptyset$ or $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$, each P_i being an n_i-homogeneous polynomial with $n_i \in \{1, 2, ..., N-r\}$.

After Proposition 15, we will indicate how Proposition 12 shows how every closed subspace S of ℓ_p is equal to C_P for some $([p]+1)-$homogeneous polynomial P. Perhaps more interesting is the fact that we do not know if the same holds for $\ell_2(I)$ for uncountable index set I. In particular, we do not know if there exists $P \in \mathcal{P}(^N\ell_2(I))$ such that C_P is non-empty and separable.

For the remainder of this paper, we will only consider Banach spaces E with unconditional FDD. We first recall some notation which will be needed in what follows. Let E have unconditional finite dimensional decomposition $\{E_n\}_{n\in\mathbb{N}}$ with associated projections $\{\Pi_n\}_{n\in\mathbb{N}}$. If $J = \{n_j\}_{j\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, let $\sigma_j = \Pi_{n_j} - \Pi_{n_{j-1}}$, and define the *block diagonal $N-$homogeneous polynomials with respect to J* to be

$$\mathcal{D}_J(^NE) \equiv \{P \in \mathcal{P}(^NE) \ : \ P \text{ has the representation } P(x) = \sum_{j=1}^{\infty} P(\sigma_j(x))\}.$$

We use the simpler notation $\mathcal{D}(^N E)$ for $\mathcal{D}_{\mathbb{N}}(^N E)$.

The following result, concerning polynomials which are diagonal with respect to an increasing subsequence of natural numbers, is motivated by the observation (see, e.g., [Theorem 10, [S]]) that if $P : \ell_p \to \mathbb{K}$ is of the form $P(x) = \sum_{n=1}^{\infty} a_n x_n^k$ where, necessarily, $(a_n) \in \ell_\infty$, then either $C_P = \emptyset$ or $C_P = \ell_p$ (when $(a_n) \in c_0$). Although the result holds in the context of Banach spaces E with unconditional FDD, the simple analogue of this argument is false. Indeed, one can find such an E and a $P \in \mathcal{P}(^2 E)$ such that $(||P \circ \sigma_n||)_{n\in\mathbb{N}} \in c_0$, although $P \notin \mathcal{P}_{wsc}(^2 E)$ [See 2.4, [D-G]].

Proposition 13. Let E be a Banach space with unconditional FDD and let $J \subset \mathbb{N}$. If $P \in \mathcal{D}_J(^N E)$, then either $C_P = \emptyset$ or E.

Proof. It suffices to show that if $x \notin C_P$ for some $x \in E$, then $0 \notin C_P$. Let (x_n) be a sequence which converges weakly to x such that for some $\epsilon > 0$, $|P(x_n) - P(x)| > \epsilon$ for every n. By passing to subsequences, we see that there is a block sequence (u_n) relative to some subset $J_1 \subset J$ which tends weakly to 0 and such that $||u_n - (x_n - x)|| \to 0$. Note that for all $n \geq$ some n_0, $|P(u_n + x) - P(x)| \geq \frac{\epsilon}{2}$. To show that $0 \notin C_P$, it suffices to show that $(P(u_n)) \not\to 0$.

Let (σ_j) be the family of projections associated to J_1, so that $P \in \mathcal{D}_{J_1}(^N E)$ and $P(x) = \sum_{j=1}^{\infty} P(\sigma_j(x))$. Since

$$\sigma_j(u_n + x) = \begin{cases} \sigma_j(x) & \text{if } j \neq n \\ u_n + \sigma_n(x) & \text{if } j = n, \end{cases}$$

$P(u_n + x) = \sum_{j=1}^{\infty} P(\sigma_j(u_n + x)) = \sum_{j=1}^{\infty} P(\sigma_j(x)) + [P(u_n + \sigma_n(x)) - P(\sigma_n(x))] =$ $= P(x) + [P(u_n + \sigma_n(x)) - P(\sigma_n(x))]$, and so $|P(u_n + x) - P(x)| = |P(u_n + \sigma_n(x)) - P(\sigma_n(x))|$. Therefore, $\frac{\epsilon}{2} < |P(u_n + \sigma_n(x))) - P(\sigma_n(x))| = |\sum_{j=1}^{N} \begin{pmatrix} N \\ j \end{pmatrix} A(\sigma_n(x)^{N-j}, u_n^j)| =$ $|\sum_{j=1}^{N-1} \begin{pmatrix} N \\ j \end{pmatrix} A(\sigma_n(x)^{N-j}, u_n^j) + P(u_n)|$. Now, since $||\sigma_n(x)|| \to 0$ and the sequence (u_n) is bounded, each $A(\sigma_n(x)^{N-j}, u_n^j) \to 0$, and so we conclude that $P(u_n) \not\to 0$. $\square$

By [D-G] we know that in spaces with unconditional FDD, there is $n \in \mathbb{N} \cup \{\infty\}$ such that every $k-$homogeneous polynomial on E of degree $k \leq n$ is weakly sequentially continuous. Moreover, if there exists a $k-$homogeneous polynomial which is not weakly sequentially continuous, then for every $l \geq k$, there is an $l-$homogeneous polynomial which is not weakly sequentially continuous anywhere. Consequently, we have the following improvement of Theorem 11.

Proposition 14. Let E be a Banach space with unconditional FDD. Suppose that $r-1 \equiv \sup\{n \in \mathbb{N} \;:\; \mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)\} \in \mathbb{N}$.
(i). If $P \in \mathcal{P}(^rE)$, then either $C_P = \emptyset$ or $C_P = E$.
(ii). If $P \in \mathcal{P}(^NE)$ with $N \geq r+1$, then either $C_P = \emptyset$ or $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$ where each $P_i \in \mathcal{P}(^{n_i}E)$ with $n_i \in \{1, ..., N-r\}$.

In Proposition 14 (ii), note that if $N-r < r$, then all the polynomials P_i are weakly sequentially continuous. We have a sort of 'reciprocal' of this observation:

Proposition 15. Let E be a Banach space with unconditional FDD. Suppose that $r-1 \equiv \sup\{n \in \mathbb{N} \;:\; \mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)\} \in \mathbb{N}$. Let $(P_i)_{i=1}^{\infty}$ be a sequence of *weakly sequentially continuous* polynomials, where each $P_i \in \mathcal{P}(^{n_i}E)$ satisfies $n_i \in \{1, ..., N-r\}$ for some fixed N. Then there exists a polynomial $P \in \mathcal{P}(^NE)$ such that $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$.
Proof. We begin by recalling ([D-G], Prop. 1.8) that with the given hypotheses on E and r, there exist a subsequence $J \subset \mathbb{N}$, a block-diagonal polynomial $\Phi \in \mathcal{D}_J(^rE)$, and a normalized block sequence (u_j) with respect to J such that $\Phi(u_j) = 1$ for every $j \in \mathbb{N}$. In particular, $\Phi \notin \mathcal{P}_{wsc}(^rE)$.

Now, let

$$P(x) = \sum_{i=1}^{\infty} \frac{P_i(x)}{2^i} \sum_{j \in F_i} \Phi(\sigma_j(x)) u_j^*(\sigma_j(x))^{N-r-n_i}.$$

Here, $\{F_i\}_{i=1}^{\infty}$ is a partition of $\mathbb{N}$ into infinite subsets, (u_j^*) is the biorthogonal sequence associated to the block sequence (u_j), and we are assuming without loss of generality that $||P_i|| \leq 1$ for all i.

We claim that $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$. To see this, suppose that $x_0 \in \cap_{i=1}^{\infty} P_i^{-1}(0)$ and that $x_n \to x_0$ weakly; without loss of generality, $||x_n|| \leq 1$ for every n. Noting that there is a constant $C > 0$ such that $\sum_{j \in F_i} |\Phi(\sigma_j(x))||u_j^*(\sigma_j(x))|^{N-r-n_i} \leq C$ for every i and every x, $||x|| \leq 1$, we obtain that for every n and M,

$$|P(x_n)| \leq C \left(\sum_{i=1}^{M} \frac{|P_i(x_n)|}{2^i} + \sum_{i=M+1}^{\infty} \frac{|P_i(x_n)|}{2^i} \right).$$

Thus, if we first choose M to ensure that the second term is small independent of n and then let $n \to \infty$ so that $P_i(x_n) \to 0$ for $i = 1, ..., M$, then $P(x_n)$ will be small for all large n.

Conversely, let $x \in C_P$ and fix $i = i_0$. Since $(x + u_l)_{l \in F_{i_0}} \to x$ weakly as $l \to \infty$, $P(x+u_l) \to P(x)$. Now, $P(x+u_l) =$

$$= \sum_{i=1}^{\infty} \frac{P_i(x+u_l)}{2^i} \left(\sum_{j \in F_i} \Phi(\sigma_j(x)) u_j^*(\sigma_j(x))^{N-r-n_i} \right) +$$
$$+ \frac{P_{i_0}(x+u_l)}{2^{i_0}} [\Phi(\sigma_l(x)+u_l) u_l^*(\sigma_l(x)+u_l)^{N-r-n_{i_0}}$$
$$-[\Phi(\sigma_l(x)) u_l^*(\sigma_l(x))^{N-r-n_{i_0}}].$$

The first summand converges to $P(x)$, and so the second summand must tend to 0. Now, since $P_{i_0}(x+u_l) \to P_{i_0}(x)$ and

$$[\Phi(\sigma_l(x)+u_l)u_l^*(\sigma_l(x)+u_l)^{N-r-n_{i_0}}$$

$$-[\Phi(\sigma_l(x))u_l^*(\sigma_l(x))^{N-r-n_{i_0}}] \not\to 0,$$

it must be that $P_{i_0}(x)=0$. Since i_0 was arbitrary, it follows that $x \in \cap_{i=1}^{\infty} P_i^{-1}(0)$. □

One instance of the use of Proposition 15 can be seen by considering $E = \ell_p$ in which case $r =$. As we noted in Proposition 12, any closed subspace $S \subset \ell_2$ can be written as $S = \cap_{i=1}^{\infty}\phi_i^{-1}(0)$, and so Proposition 15 shows that $S = C_P$ for some $P \in \mathcal{P}({}^{[p]+1}\ell_p)$.

We conclude by discussing questions 1, 2, and 3 in more detail, in the case that E has unconditional FDD.

Note that in the definition of P in the above proof, if we replace each exponent $N-r-n_i$ by $N-r-n_i+1$, we obtain an $N+1-$homogeneous polynomial Q such that $C_Q = \cap_{i=1}^{\infty} P_i^{-1}(0)$. Consequently, we have the following result.

Corollary 16. Let E have an unconditional FDD and $r-1 = \sup\{n \in \mathbb{N} : \mathcal{P}({}^nE) = \mathcal{P}_{wsc}({}^nE)\} \in \mathbb{N}$. Suppose that N and $P \in \mathcal{P}({}^NE)$ are such that either $C_P = \emptyset$ or $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$, where each $P_i \in \mathcal{P}_{wsc}({}^{n_i}E)$ is chosen so that $n_i \in \{1, ..., N-r\}$. Then there exists $Q \in \mathcal{P}({}^{N+1}E)$ such that $C_P = C_Q$.

A similar method provides a strengthening of Proposition 8, valid for either real or complex spaces.

Corollary 17. Suppose that E has an unconditional FDD and that $r-1 = \sup\{n \in \mathbb{N} : \mathcal{P}({}^nE) = \mathcal{P}_{wsc}({}^nE)\} \in \mathbb{N}\}$. Let P and Q be $N-$homogeneous polynomials. If $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$ and $C_Q = \cap_{j=1}^{\infty} Q_j^{-1}(0)$, where each of the P_i and Q_j are weakly sequentially continuous polynomials of degree at most $N-r$, then there exists $R \in \mathcal{P}({}^NE)$ such that $C_R = C_P \cap C_Q$.

Proof. Let $\{F_i : i \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into a collection of pairwise disjoint infinite sets. With the same notation as in the proof of Proposition 15, let

$$R(x) \equiv \sum_{i=1}^{\infty}\left(\frac{P_i(x)}{2^i}\sum_{j\in F_{2i}} \Phi(\sigma_j(x))u_j^*(\sigma_j(x))^{N-r-n_i}\right) +$$

$$+\sum_{i=1}^{\infty}\left(\frac{Q_i(x)}{2^i}\sum_{j\in F_{2i-1}} \Phi((\sigma_j(x))u_j^*(\sigma_j(x))^{N-r-m_i}\right).$$

An argument similar to that used to prove Proposition 15 shows that $R \in \mathcal{P}({}^NE)$ and $C_R = C_P \cap C_Q$. □

A similar argument yields a partial answer to Question 3:

Corollary 18. Suppose that E has an unconditional FDD and that $r-1 = \sup\{n \in \mathbb{N} : \mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)\} \in \mathbb{N}\}$, and let $P, Q \in \mathcal{P}(^NE)$. Suppose also that $C_P = \cap_{i=1}^{\infty} P_i^{-1}(0)$ and $C_Q = \cap_{i=1}^{\infty} Q_i^{-1}(0)$, with each P_i and Q_j an n_i, respectively m_j, homogeneous *weakly sequentially continuous* polynomial, where all n_i and m_j are at most $N-r$. Then there is $R \in \mathcal{P}(^{2N+r}E)$ such that $C_R = C_P \cup C_Q$.

Proof. Let $\{F_{i,j}\}_{i,j}$ be a doubly-indexed partition of $\mathbb{N}$ where each $F_{i,j}$ is infinite. Then, the $2N+r-$homogeneous polynomial

$$R(x) \equiv \sum_{i,j} \left(\frac{P_i(x)Q_j(x)}{2^{i+j}} \sum_{l \in F_{i,j}} \Phi(\sigma_l(x)) u_l^*(\sigma_l(x))^{2N-n_i-m_j} \right)$$

is such that $C_R = C_P \cup C_Q$. □

We conclude with several comments about the possibility of extending Questions 2 and 3. First, there is no difficulty in extending Proposition 8 and Corollaries 17 and 18 to the situation in which the polynomials P and Q have different degrees. A more interesting problem is whether our results extend to infinite intersections and unions. Specifically, given a sequence of $N-$homogeneous polynomials P_i on E, are there polynomials R such that $C_R = \cap_i C_{P_i}$ and $C_R = \cup_i C_{P_i}$? It is not difficult to see that both Proposition 8 and Corollary 17 can be modified so as to be valid for infinite intersections. On the other hand, the problem for infinite unions cannot have an affirmative solution as stated, since $\cup_i C_{P_i}$ is not closed in general. In fact, there is usually no R such that $C_R = \overline{\cup_i C_{P_i}}$. To see this, take for example the sequence $(P_j) \subset \mathcal{P}(^3\ell_2)$ given by $P_j(x) = \gamma_j(x) \sum_{i=1}^{\infty} x_i^2$ where $\gamma_j(x) \equiv x_j - jx_1$. Then $C_{P_j} = \{x \in \ell_2 \ : \ x_j = jx_1\} = \gamma_j^{-1}(0)$. Suppose that there is a polynomial R such that $C_R = \overline{\cup_i C_{P_i}}$. By Proposition 4, since each $\gamma_j^{-1}(0) \subset C_R$, it follows that $C_R = E$. However, $\overline{\cup_{i=1}^{\infty} C_{P_i}} \neq E$ since $e_1 \notin \overline{\cup_{i=1}^{\infty} C_{P_i}}$, which shows that no such polynomial R exists.

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