On spectral pictures

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Abstract *The "spectral picture" of a bounded operator on a Banach space consists of its essential spectrum together with a mapping from its holes to the group of integers, obtained by taking the Fredholm index. In this note we abstract this from the Calkin algebra to a general Banach algebra, replacing the integers with the quotient of the group of invertibles by its connected component of the identity.*

By a *spectrum* K we shall understand, in the first instance, a nonempty compact subset $K \subseteq \mathbf{C}$ of the complex plane: this works because every compact set is the spectrum of something. If $K \subseteq \mathbf{C}$ is a spectrum then so is its topological boundary ∂K and so is its *connected hull*

0.1 $$\eta K = K \cup \bigcup \{H : H \in \mathrm{Hole}(K)\},$$

where [4],[8] we write $\mathrm{Hole}(K)$ for the (possibly empty) set of bounded components of the complement of K in $\mathbf{C}$: thus $\mathbf{C} \setminus \eta K$ is the unique unbounded component of $\mathbf{C} \setminus K$.

1. Definition *By a "spectral picture" we shall understand an ordered pair (K, ν) in which K is a spectrum and ν is a mapping from $\mathrm{Hole}(K)$ to the integers $\mathbf{Z}$.*

If $K \subseteq \mathbf{C}$ is a spectrum and if $f : U \to \mathbf{C}$ is a continuous mapping whose domain $U \subseteq \mathbf{C}$ includes K then it is clear that $f(K)$ is again a spectrum, where of course

1.1 $$f(K) = \{f(\lambda) : \lambda \in K\}.$$

We shall pay particular attention to functions

1.2 $$f \in \mathrm{Holo}(\eta K),$$

for which $U \supseteq \eta K$ is open in $\mathbf{C}$ and on which f is *holomorphic*. Recall ([8] Proposition 2.2)

1.3 $$\partial f(K) \subseteq f\lambda(\partial K) \text{ and } f(\eta K) \subseteq \eta f(K);$$

also if $L \in \mathrm{Hole}\ f(K)$ and $H \in \mathrm{Hole}\ K$ then ([1] Proposition 3.1, Lemma 3.5)

1.4 $$\text{either } L \cap f(H) = \emptyset \text{ or } L \subseteq f(H).$$

2. Definition *If (K, ν) is a spectral picture and if $f \in \mathrm{Holo}(\eta K)$ then*

2.1 $$f(K, \nu) = (f(K), \nu_f),$$

where for each hole $L \in \mathrm{Hole}\ f(K)$ we set

2.2 $$\nu_f(L) = \sum \{N_f(L, H)\nu(H) : L \subseteq f(H)\},$$

where if $\mu \in L \subseteq f(H)$ the equation $f(\lambda) = \mu$ has exactly $N_f(L, H)$ solutions $\lambda \in H$:

2.3 $$N_f(L, H) = \#\{\lambda \in H : f(\lambda) = \mu\} = \#\ f^{-1}(\mu) \cap H.$$

Of course, via Rouché's theorem from complex analysis, the number $N_f(L, H)$ is independent of the choice of $\mu \in L$. If $L \in \mathrm{Hole}\ f(K)$ is not a subset of $f(H)$ for any hole $H \in \mathrm{Hole}(K)$ then we interpret the right hand side of (2.2) as the integer 0 .

The fundamental example of a spectral picture comes from Fredholm theory:

3. Example *If X is a Banach space and $A = B(X)/K(X)$ the Calkin algebra on X and $T \in B(X)$ is a bounded linear operator on X then take*

3.1 $$a = [T]_{K(X)}\ ;\ K = \sigma_A(a) = \sigma_{ess}(T)$$

and for each $H \in \mathrm{Hole}(K)$ define

3.2 $$\nu(H) = \mathrm{index}(T - \lambda I) \text{ with } \lambda \in H.$$

Fredholm theory guarantees that $\nu(H)$ is well-defined (independent of the choice of $\lambda \in H$); then according to [1](3.7) or [9](8.8) the spectral picture of $f(T)$ is the image, in the sense of (2.1), of the spectral picture of T. Indeed writing $f(z) - \mu = g(z) \prod_j (z - \lambda_j)$ argue

3.3 $$\nu_f(L) = \mathrm{index}(f(T) - \mu I) = \mathrm{index}\ g(T) + \sum_j \mathrm{index}(T - \lambda_j I).$$

For a more general version of all this look at the *abstract index group* [2] of a Banach algebra, the quotient

3.4 $$\kappa(A) = A^{-1}/\mathrm{Exp}(A) = A^{-1}/A_0^{-1}$$

of the invertible group by its connected component of the identity [4],[8],[7]

3.5 $$A_0^{-1} = \mathrm{Exp}(A) \equiv \{e^{c_1}e^{c_2}\dots e^{c_k} : k \in \mathbf{N}, c \in A^k\}.$$

Now a "spectral landscape" for a Banach algebra A is an ordered pair (K,ν) in which K is a spectrum and ν is a mapping from $\mathrm{Hole}(K)$ to the abstract index group $\kappa(A)$; as a favour we ask that the cosets $\nu(H)$ mutually commute. Then the "spectral landscape" of an element $a \in A$ is what we would expect:

4. Definition *The spectral landscape of an element $a \in A$ of a Banach algebra A is the ordered pair $(K,\nu) = (\sigma_A(a), \iota_\sigma(a))$ where $\iota_\sigma(a) : \mathrm{Hole}(\sigma(a)) \to \kappa(A)$ is defined by setting*

4.1 $$\iota_\sigma(a)(\lambda) = \mathrm{Exp}(A)(a-\lambda) \textit{ if } \lambda \in H \in \mathrm{Hole}(A).$$

The image of a spectral landscape (K,ν) by a polynomial $f : \mathbf{C} \to \mathbf{C}$, or more generally a holomorphic function $f \in \mathrm{Holo}(\eta K)$, will be the ordered pair $(f(K),\nu_f)$, where

4.2 $$\nu_f(L) = \prod\{\nu(H)^{N_f(L,H)} : L \subseteq f(H)\}.$$

The spectral mapping theorem follows by the argument of (3.3):

5. Theorem *If $a \in A$ and $f \in \mathrm{Holo}(\eta\sigma(a))$ then the spectral landscape of $f(a)$ is the image of the spectral landscape of a:*

5.1 $$(\sigma f(a), \iota_\sigma f(a)) = (f\sigma(a), (\iota_\sigma a)_f).$$

Proof. If $f : U \to \mathbf{C}$ is holomorphic on an open set $U \supseteq \eta\sigma(a)$ containing the spectrum and all its holes and if $\mu \in L \in \mathrm{Hole}\ \sigma f(a)$ then

5.2 $$f(z) - \mu = g(z)\prod_j (z-\lambda_j),$$

where $g : U \to \mathbf{C}$ is holomorphic and nonvanishing on $\eta\sigma(a)$, giving

5.3 $$(f(a)-\mu)\mathrm{Exp}(A) = \prod_j \mathrm{Exp}(A)(a-\lambda_j) \bullet$$

The spectral landscape also co-operates with passage to a subalgebra: if $T : A \to B$ is a homomorphism then

5.4 $$T\ A^{-1} \subseteq B^{-1},$$

and if T is also bounded then

5.5 $$T\ \mathrm{Exp}(A) \subseteq \mathrm{Exp}(B) :$$

thus there is induced $\kappa(T) : \kappa(A) \to \kappa(B)$. If finally T is also bounded below, so that effectively A is a closed subalgebra of B, then for arbitrary $a \in A$

5.6 $$\partial\sigma_A(a) \subseteq \sigma_B(Ta) \subseteq \sigma_A(a) \subseteq \eta\sigma_B(Ta)$$

with

5.7 $$\mathrm{Hole}\ \sigma_A(a) \subseteq \mathrm{Hole}\ \sigma_B(Ta).$$

Now the behaviour of the spectral landscape under passage to a closed subalgebra $A \subseteq B$ is that for arbitrary $a \in A$

5.8 $$\iota_{B\sigma}(Ta)J = \kappa(T)\iota_{A\sigma}(a),$$

where J is the restriction mapping.

If for example $A = B(X)$ for a Banach space X then the spectral picture of an operator $a \in A$ is what we might call the *essential spectral landscape* of the element a, or rather the combination (K, ν) where $K = \sigma_{ess}(a)$ and $\nu = \mathrm{Index} \circ \iota_\sigma^{ess}$: there is a well defined mapping $\mathrm{Index} : \kappa(B(X)/K(X)) \to \mathbf{Z}$ from the abstract index group of a Calkin algebra to the integers. The actual spectral landscape of $a \in A$ maps holes in the actual spectrum of A into the quotient group $\kappa(A)$: when $A = B(X)$ for a Hilbert space X then $A^{-1} = \mathrm{Exp}(A)$ is connected, so that this becomes trivial.

In real life the "spectral picture" [11] needs to be augmented by the addition of "pseudo-holes", and the group of integers extended to include ∞ and $-\infty$. Thus the spectral landscape might be seen as the superposition of a "left" and a "right" landscape, while the abstract index group becomes the intersection of a left and a right semigroup:

6. Definition *If A is a Banach algebra then the abstract left index semi-group of A is the quotient*

6.1 $$\kappa_{left}(A) = A^{-1}_{left}/\mathrm{Exp}(A) = \{\mathrm{Exp}(A)a : a \in A^{-1}_{left}\},$$

while the abstract right index semi-group of A is the quotient

6.2 $$\kappa_{right}(A) = A^{-1}_{left}/\mathrm{Exp}(A) = \{a\mathrm{Exp}(A) : a \in A^{-1}_{right}\}.$$

We should remark that if $1 \in G \subseteq H$, G a subgroup of the semigroup H, then the set of left cosets $\{xG : x \in H\}$ forms a partition of H, in the sense that any two are either disjoint or coincide; the same is true of right cosets. When we specialise to the semigroup $H = A^{-1}_{left} \subseteq A$ of left invertibles then the left cosets are subsets of the right:

7. Theorem *If A is a Banach algebra then $\kappa_{left}(A)$ and $\kappa_{right}(A)$ are semigroups, with the discrete topology.*

Proof. Suppose $x'x = 1$: then if $0 \neq \lambda \in \mathbf{C}$

7.1 $$xA^{-1}x' \subseteq A^{-1} + \lambda(1 - xx') \subseteq x'A^{-1}x :$$

the inverse of $xax' - \lambda(1 - xx')$ is $xa^{-1}x' - \lambda^{-1}(1 - xx')$. Also (cf [7] Theorem 7.11.2)

7.2 $$\textit{the sets } A^{-1} \textit{ and } A^{-1}_{left} \setminus A^{-1} \textit{ are open in } A :$$

if $a \in A^{-1}$ then $\{a(1 - x) : \|x\| < 1\} \subseteq A^{-1}$ and if $a \in A^{-1}_{left} \setminus A^{-1}$ then $\{a(1 - x) : \|x\| < 1\} \subseteq A^{-1}_{left} \setminus A^{-1}$.

From the first part of (7.1) it follows

7.3 $$x'x = 1 \Longrightarrow xA^{-1} \subseteq A^{-1}x.$$

From (7.3) we are able to successfully multiply right cosets to form the semigroup A^{-1}_{left}/A^{-1}, which by (7.2) acquires the discrete topology. All this holds equally well with the generalized exponentials $\mathrm{Exp}(A)$ in place of A^{-1}: for example (cf [7] (7.11.3.4)) $xe^cx' + 1 - xx' = e^{xcx'}$. In addition

7.4 $$\mathrm{Exp}(A) \textit{ is the connected component of } 1 \textit{ in } A^{-1}_{left} \bullet$$

Now a "left spectral landscape" on a Banach algebra A will be an ordered pair (K, ν) in which K is a spectrum and ν is a mapping from $\mathrm{Hole}(K)$ to the left abstract index semigroup $\kappa_{left}(A)$:

8. Definition *The left spectral landscape of $a \in A$ is the ordered pair (K, ν) where $K = \sigma_A^{left}(a)$ is the left spectrum of a in A and $\nu = \iota_\sigma^{left} : \mathrm{Hole}(K) \to \kappa_{left}(A)$ takes right cosets:*

8.1 $$\nu(H) = \mathrm{Exp}(A)(a - \lambda) \textit{ if } \lambda \in H \in \mathrm{Hole}(K).$$

Continuity and the discrete topology ensure that ν is well defined. When we specialise to the Calkin algebra $A = B(X)/K(X)$ then there are well-defined mappings $\mathrm{Index} : \kappa_{left}(A) \to \mathbf{Z} \cup \{-\infty\}$ and $\mathrm{Index} : \kappa_{right}(A) \to \mathbf{Z} \cup \{\infty\}$.

If $T \in HBL(A, B)$ has closed range then with

8.2 $$\omega_T(a) = \bigcap_{Td=0} \sigma_A(a + d),$$

there is inclusion [4],[8]

8.3 $$\partial\omega_T(a) \subseteq \sigma_B(Ta) \subseteq \omega_T(a) \subseteq \eta\sigma_B(Ta)$$

and hence

8.4 $$\text{Hole } \omega_T(a) \subseteq \text{Hole } \sigma_B(Ta).$$

The analogue of the abstract index semigroup is now the quotient

8.5 $$\kappa_T(A) = (A^{-1} + T^{-1}(0))/\text{Exp}(A).$$

Whether or not the subgroup $\text{Exp}(A)$ is normal in the semigroup $A^{-1} + T^{-1}(0)$ there is a well-defined mapping, if $a \in A$,

8.6 $$\iota_\omega^T(a) : \text{Hole } \omega_T(a) \to \kappa_T(A)$$

given by setting

8.7 $$\iota_\omega^T(a)(H) = (a - \lambda)\text{Exp}(A) \textit{ if } \lambda \in H \in \text{Hole } \omega_T(a).$$

If we were to attempt to extend the spectral landscape concept to systems $a \in A^X$ of Banach algebra elements we might proceed along the following lines. Analagous to the semigroup of left invertibles, we take

8.9 $$A_{left}^{-X} = \{a \in A^X : 1 \in \sum_{x\in X} Aa_x\},$$

with the corresponding definition of A_{right}^{-X}: thus the left spectrum of $a \in A^X$ becomes

8.10 $$\sigma_A^{left}(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin A_{left}^{-X}\}.$$

Now instead of the abstract left index semigroup we have the following object:

8.11 $$\kappa_{left}(A^X) = A_{left}^{-X}/\text{Exp}(A) = \{\text{Exp}(A)a : a \in A_{left}^{-X}\},$$

collecting right cosets. Emboldened, we offer

9. Definition *If X is an arbitrary set then the left spectral picture of $a \in A^X$ is the ordered pair (K, ν) where $K = \sigma_A^{left}(a)$ and $\nu = \iota_\sigma^{left}(a) : \text{Hole}(K) \to \kappa_{left}(A^X)$ takes right cosets:*

9.1 $$\nu(H) = \text{Exp}(A)(a - \lambda) \textit{ if } \lambda \in H \in \text{Hole}(K).$$

The most obvious flaw in this construction is that in higher dimensions the left spectrum may not often have "holes". The formula (4.2) is unlikely to extend, since Rouchés theorem does not; indeed the number of $\lambda \in H$ for which $f(\lambda) = \mu$ need not be finite at all.

Our attitude to the “spectral picture” [2] has been the same as to “spectral projections” [5],[6]: the spectral projection associated with an isolated point of the spectrum of a bounded operator is given by a Cauchy integral, but in most texts has no definition. By exploring [5] what this definition should be we found ourselves creeping up behind the “Drazin inverse”, or a slight generalization thereof: in particular we found that something which by definition commuted with the operator could be proved to double commute. This and the abstracted concept of “quasi-polar” Banach algebra elements seem to be interesting in their own right. In the same way the “spectral picture” of a bounded operator is built from its essential spectrum and the Fredholm index, but the discussion does not seem to pause to explain what sort of beast we are dealing with.

References

1. C. Bosch, C. Hernandez, E. de Oteyza and C. Pearcy, *Spectral pictures of functions of operators*, J. Op. Th. **8** (1982) 391-400.
2. B. Chevreau, *On the spectral picture of an operator*, Jour. Op. Th. **4** (1980) 119-132.
3. R.G. Douglas, *Banach algebra techniques in operator theory*, Academic Press 1972.
4. R.E. Harte, *The exponential spectrum in Banach algebras*, Proc. Amer. Math. Soc. **58** (1976) 114-118.
5. R.E. Harte, *Spectral projections*, Irish Math. Soc. Newsletter **14** (1984) 10-15.
6. R.E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Roy. Ir. Acad. **85A** (1985) 151-176.
7. R.E. Harte, *Invertibility and singularity*, Dekker New York 1988.
8. R.E. Harte and A.W. Wickstead, *Boundaries, hulls and spectral mapping theorems*, Proc. Roy. Ir. Acad. **81A** (1981) 201-208.
9. C. Hernandez and E. de Oteyza, *La figura espectral de operadores*. Mono. del Inst. Mat. **17** UNAM 1986.
10. J.M. Howie, *An introduction to semigroup theory*, Academic Press New York 1976.
11. C. Pearcy, *Some recent developments in operator theory*, Amer. Math. Soc. Providence 1978.