Reynolds–uniform numerical method for Prandtl's problem with suction–blowing based on Blasius' approach *

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Abstract. We construct a new numerical method for computing *reference* numerical solutions to the self–similar solution to the problem of incompressible laminar flow past a thin flat plate with suction–blowing. The method generates global numerical approximations to the velocity components and their scaled derivatives for arbitrary values of the Reynolds number in the range $[1, \infty)$ on a domain including the boundary layer but excluding a neighbourhood of the leading edge. The method is based on Blasius' approach. Using an experimental error estimate technique it is shown that these numerical approximations are pointwise accurate and that they satisfy pointwise error estimates which are independent of the Reynolds number for the flow. The Reynolds–uniform orders of convergence of the reference numerical solutions, with respect to the number of mesh subintervals used in the solution of Blasius' problem, is at least 0.86 and the error constant is not more than 80. The number of iterations required to solve the nonlinear Blasius problem is independent of the Reynolds number. Therefore the method generates reference numerical solutions with ε–uniform errors of any prescribed accuracy.

1 Introduction

The numerical solution of singularly perturbed boundary value problems, for which the solutions exhibit boundary layers, gives rise to significant difficulties. The errors in the numerical solutions of such problems generated by classical numerical methods depend on the value of the singular perturbation parameter

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ε, and can be large for small values of ε [2]. For representative classes of singular perturbation problems special methods have been constructed and shown theoretically to generate numerical approximations that converge ε–uniformly. Also, numerical experiments have confirmed the efficacy of such methods in practice [2]. Singularly perturbed boundary value problems, for which the solutions exhibit boundary layers, frequently arise in flow problems with large Reynolds number Re. In such problems the small parameter $\varepsilon = Re^{-1}$. The discretization of such problems gives rise to nonlinear finite difference methods for which there is no known ε–uniform error analysis in the maximum norm. For this reason an experimental method for justifying ε–uniform convergence is the only remaining possibility. To make use of such a technique, especially for large Re, it is essential to have a known ε–uniform reference solution which approximates the exact solution to any prescribed accuracy. For flow problems with boundary layers there is usually no known analytic solution that can be used as a reference solution, and the same is true even for problems with a self–similar solution. Thus the task of constructing a reference *numerical* solution with ε–uniform errors of any prescribed accuracy arises from a wide class of flow problems.

An example of such a problem is flow past a flat plate with suction–blowing, for all Reynolds numbers for which the flow remains laminar and no separation occurs. For this problem it is important to construct a numerical method for which the pointwise errors in the scaled numerical solutions and their scaled derivatives are independent of the Reynolds number. In the present paper we consider the associated Prandtl problem of flow past a flat plate with suction–blowing. For large values of the Reynolds number the solution of this problem exhibits parabolic boundary layers in the neighbourhood of the plate, outside a neighbourhood of the leading edge. At the leading edge new singularities appear due to the incompatibilities of the problem data at the leading edge. Therefore,in the present paper we construct a numerical method which generates Reynolds–uniform reference numerical approximations to the scaled velocity components and their scaled derivatives for arbitrary values of the Reynolds number in a finite rectangular domain including the boundary layer but excluding a neighbourhood of the leading edge. This numerical method is based on the numerical solution of the related Blasius problem on the positive semi–axis. The accuracy of the numerical approximations depends on only the number of mesh subintervals N used for the solution of the Blasius problem. Our method is a development of that described in [2] for flow past a flat plate without suction–blowing.

2 Formulation of the Problem

We are required to find the solution, and its derivatives, of Prandtl's problem for incompressible flow past a semi–infinite flat plate $P = \{(x, 0) \in \Re^2 : x \geq 0\}$ with suction–blowing in a bounded domain $\overline{D}$, which adjoins the plate and contains the boundary layer.

Prandtl's problem on the cut plane $\Omega = \Re^2 \backslash P$ is described as follows

$$
(P_P)\begin{cases}
\text{Find } \mathbf{u_P} = (u_P, v_P) \text{ such that for all } (x,y) \in \Omega \\
\mathbf{u_P} \text{ satisfies the differential equations} \\
\frac{-1}{Re}\frac{\partial^2 u_P(x,y)}{\partial^2 y} + \mathbf{u_P}\cdot\nabla u_P(x,y) = 0 \\
\nabla\cdot\mathbf{u_P}(x,y) = 0 \\
\text{with the boundary conditions} \\
u_P(x,0) = 0, v_P = v_0(x) \text{ for all } x \ge 0 \\
\lim_{|y|->\infty}\mathbf{u_P}(x,y) = \lim_{x->-\infty}\mathbf{u_P}(x,y) = (1,0), \text{ for all } x \in \Re
\end{cases}
$$

where $v_0(x)$ is the vertical component of the suction–blowing velocity. This is a nonlinear system of equations for the unknown components u_P, v_P of the velocity $\mathbf{u}_P$. The solution at all points in the open half plane to the left of the leading edge is $\mathbf{u_P} = (1,0)$. For special choices of the function v_0 the solution of (P_P) is self–similar, see (3) below.

Note that in Prandtl's problem, even without suction–blowing, the vertical component of the velocity tends to infinity as we approach the leading edge. To avoid this singularity, we choose the computational domain $D = (a,A)\times(0,B)$ where a, A and B are fixed positive numbers independent of Re. Our aim is to construct a method for finding reference numerical approximations to the self–similar solution and its derivatives of problem (P_P) for arbitrary $Re \in [1,\infty)$ with error independent of Re.

We now describe conditions under which the solution of (P_P) is self–similar. Using the approach of Blasius, see [1], for example, a solution $\mathbf{u_P} = (u_P, v_P)$ of (P_P) can be written in the form

$$u_P(x,y) \equiv u_B(x,y) = f'(\eta) \tag{1}$$

$$v_P(x,y) \equiv v_B(x,y) = \sqrt{\frac{1}{2xRe}}(\eta f'(\eta) - f(\eta)) \tag{2}$$

where

$$\eta = y\sqrt{Re/2x}$$

and the function f is the solution of the problem

$$
(P_B)\begin{cases}
\text{Find a function f } \in C^3([0,\infty)) \text{ such that for all } \eta \in (0,\infty) \\
f'''(\eta) + f(\eta)f''(\eta) = 0 \\
\text{with the boundary conditions} \\
f(0) = f_0,\quad f'(0) = 0,\quad \lim_{\eta->\infty} f'(\eta) = 1.
\end{cases}
$$

(P_B) is known as Blasius' problem and $\mathbf{u}_B = (u_B, v_B)$ is known as the Blasius solution of (P_P). The existence and uniqueness of a solution to this third order nonlinear ordinary differential equation is discussed in [1]. Positive values of f_0 correspond to suction, while negative values of f_0 represent blowing, and f_0 is related to v_0 in (P_P) by the formula (see for example [3])

$$v_0(x) = -f_0\sqrt{1/2xRe}. \tag{3}$$

The first order derivatives of the velocity components u_P and v_P are given by

$$\frac{\partial u_P}{\partial y}(x,y) = \frac{\partial u_B}{\partial y}(x,y) = \frac{\eta}{y} f''(\eta) \tag{4}$$

$$\frac{\partial v_P}{\partial y}(x,y) = \frac{\partial v_B}{\partial y}(x,y) = \frac{\eta}{2x} f''(\eta) \tag{5}$$

$$\frac{\partial u_P}{\partial x}(x,y) = -\frac{\partial v_P}{\partial y}(x,y) \tag{6}$$

$$\frac{\partial v_P}{\partial x}(x,y) = \frac{\partial v_B}{\partial x}(x,y) = -\frac{1}{2x}[v_B + \sqrt{\frac{1}{2xRe}}\eta^2 f''(\eta)] \tag{7}$$

From (1), (2), (4), (5), (6) and (7) we see that to find the velocity components u_P and v_P, and their first order derivatives, it is necessary to know $f'(\eta),\ \eta f'(\eta) - f(\eta),\ \eta f''(\eta)$ and $\eta^2 f''(\eta)$ for all $\eta \in [0,\infty)$. We also observe from these relations that, when Re is large, v_P and $\frac{\partial v_P}{\partial x}$ are small and $\frac{\partial u_P}{\partial y}$ is large. Therefore, in order to have values of order unity, we use the following scaled components: $\sqrt{Re}v_P$, $\sqrt{Re}\frac{\partial v_P}{\partial x}$, and $\frac{1}{\sqrt{Re}}\frac{\partial u_P}{\partial y}$.

In the next section numerical approximations to the solution of (P_B), and its first order derivatives, are constructed on the semi-infinite domain $[0,\infty)$.

3 Numerical Solution of Blasius' Problem

To find u_P and v_P and their first order derivatives we have to solve (P_B) for f and its derivatives on the semi-infinite domain $[0,\infty)$. This is not a trivial matter, since numerical solutions can be obtained at only a finite number of mesh points. For this reason, for each value of the parameter $L \in [1,\infty)$, we introduce the following problem on the finite interval $(0, L)$

$$(P_{B,L}) \begin{cases} \text{Find a function } f_L \in C^3(0,L) \text{ such that for all } \eta \in (0,L) \\ f_L'''(\eta) + f_L(\eta) f_L''(\eta) = 0 \\ \text{with the boundary conditions} \\ f_L(0) = f_0,\ \ f_L'(0) = 0,\ \ f_L'(L) = 1. \end{cases}$$

The collection of all such problems forms a one-parameter family of problems related to (P_B), where the interval length L is the parameter of the family. Because the values of f_L, f'_L and f''_L are needed at all points of $[0,\infty)$, we introduce the following extrapolations

$$f''_L(\eta) = 0, \text{for all } \ \eta \geq L \tag{8}$$
$$f'_L(\eta) = 1, \text{for all } \ \eta \geq L \tag{9}$$
$$f_L(\eta) = (\eta - L) + f_L(L), \text{for all } \ \eta \geq L. \tag{10}$$

To solve (P_B), we first obtain a numerical solution F_L of $(P_{B,L})$ on the finite interval $(0,L)$ for an increasing sequence of values of L. Then, we extrapolate F_L to the semi-infinite domain $[0,\infty)$. The sequence of values of L is defined as follows. For each even number $N \geq 4$ define $L_N = lnN$ (see [2] for motivation for this choice of L_N) and consider the corresponding finite interval $[0, L_N]$. On $[0, L_N]$ a uniform mesh $\overline{I}_u^N = \{\eta_i : \eta_i = iN^{-1}lnN, 0 \leq i \leq N\}_0^N$ with N mesh subintervals is constructed. Then numerical approximations F_L, D^+F_L, $D^+D^+F_L$ to f_L, f'_L, f''_L respectively, are determined at the mesh points in $\overline{I}_u^N$ using the following non-linear finite difference method

$$(P_{B,L}^N)\begin{cases} \text{Find F on } \bar{I}_u^N \text{ such that, for all } \eta_i \in I_u^N, 2 \leq i \leq N-1, \\ \\ \delta^2(D^-F)(\eta_i) + F(\eta_i)D^+(D^-F)(\eta_i) = 0 \\ \\ F(0) = f_0 \quad D^+F(0) = 0, \text{ and } \quad D^0F(\eta_{N-1}) = 1. \end{cases}$$

We note that, in order to simplify the notation, we have dropped explicit use of the indices L and N. Thus, we denote the solution of $P_{B,L}^N$ by F instead of F_L^N.

Since $(P_{B,L}^N)$ is non-linear, we use the following iterative solver to compute its solution

$$(A_B^N)\begin{cases} \text{For each integer } \ m, 1 \leq m \leq M, \text{ find } F^m \text{ on } I_u^N \text{ such that , for all } \eta_i \in I_u^N \\ \\ \delta^2(D^-F^m)(\eta_i) + F^{m-1}(\eta_i)D^+(D^-F^m)(\eta_i) - D^-(F^m - F^{m-1})(\eta_i) = 0 \\ \\ F^m(0) = f_0, \quad D^+F^m(0) = 0, \text{ and } D^0F^m(\eta_{N-1}) = 1 \\ \\ \text{with the starting values for all mesh points } \eta_i \in \overline{I}_u^N \\ \\ F^0(\eta_i) = \eta_i. \end{cases}$$

Algorithm (A_B^N) involves the solution of a sequence of linear problems, with one linear problem for each value of the iteration index m. The total number of iterations M is taken to be $M = 8lnN$ The motivation for this choice of M is described in [2]. It is important to note the crucial property that M is independent of the Reynolds number Re. The final output of algorithm (A_B^N)

is denoted by F, where again we simplify the notation by omitting explicit mention of the total number of iterations M. We follow the same criterion as in [2] to determine F on the finest required mesh as the "exact" solution.The corresponding value of N is denoted by N_0.

To ensure that F, D^+F and D^+D^+F are defined at all points of each mesh $\overline{I}_u^N$ the following values are assigned: $D^+F(\eta_N) = 1$, $D^+D^+F(\eta_{N-1}) = 0$, $D^+D^+F(\eta_N) = 0$. We then define F, D^+F and D^+D^+F at each point of $[0, L_N]$ using piecewise linear interpolation of the values at the mesh points of $\overline{I}_u^N$. The resulting interpolants are denoted by $\overline{F}$, $\overline{D^+F}$ and $\overline{D^+D^+F}$ respectively.

In order to define $\overline{F}$, $\overline{D^+F}$ and $\overline{D^+D^+F}$ at each point $\eta \in [0, \infty)$ the following extrapolations, analogous to (8), (9) and (10), are introduced

$$\overline{D^+D^+F}(\eta) = 0, \text{for all } \eta \in [L_N, \infty) \tag{11}$$

$$\overline{D^+F}(\eta) = 1, \text{for all } \eta \in [L_N, \infty) \tag{12}$$

$$\overline{F}(\eta) = \overline{F}(L_N) + (\eta - L_N), \text{for all } \eta \in [L_N, \infty). \tag{13}$$

The values of $\overline{F}$, $\overline{D^+F}$ and $\overline{D^+D^+F}$, respectively, are the required numerical approximations to f, f', f'' of the Blasius solution and its derivatives at each point of $[0, \infty)$.

4 Numerical Experiments for Blasius' Problem

In [3] a limiting value for suction is found at $f_0 = 7.07$ and for blowing at $f_0 = -0.875745$. In numerical experiments to illustrate the proposed technique, we take the representative values $f_0 = 3$ and $f_0 = 6$ for suction; $f_0 = -0.25$ and $f_0 = -0.5$ for blowing.

We want to determine error estimates for the approximations $\overline{F}$, $\overline{D^+F}$ and $\overline{D^+D^+F}$ to f, f' and f'', respectively, for all $N \geq 2048$. Consequently, we take $I_u^{N_0}$, where $N_0 = 65536$, to be the finest mesh on which we solve Blasius' problem. Using the experimental numerical technique described in [2] we determine the following computed error estimates

$f_0 = 3$

$$\|\overline{F} - f\|_{[0,\infty)} \leq 2.505 N^{-0.86}$$
$$\|\overline{D^+F} - f'\|_{[0,\infty)} \leq 1.452 N^{-0.86}$$
$$\|\overline{D^+D^+F} - f''\|_{[0,\infty)} \leq 20.427 N^{-0.84}$$

$f_0 = 6$

$$\|\overline{F} - f\|_{[0,\infty)} \leq 2.635 N^{-0.86}$$
$$\|\overline{D^+F} - f'\|_{[0,\infty)} \leq 2.925 N^{-0.86}$$
$$\|\overline{D^+D^+F} - f''\|_{[0,\infty)} \leq 65.927 N^{-0.81}$$

$f_0 = -0.25$

$$\|\overline{F} - f\|_{[0,\infty)} \leq 1.066N^{-0.86}$$
$$\|\overline{D^+F} - f^{'}\|_{[0,\infty)} \leq 0.202^{-0.86}$$
$$\|\overline{D^+D^+F} - f^{''}\|_{[0,\infty)} \leq 0.453N^{-0.86}$$

$f_0 = -0.5$

$$\|\overline{F} - f\|_{[0,\infty)} \leq 0.603N^{-0.85}$$
$$\|\overline{D^+F} - f^{'}\|_{[0,\infty)} \leq 0.345N^{-0.87}$$
$$\|\overline{D^+D^+F} - f^{''}\|_{[0,\infty)} \leq 0.488N^{-0.86}.$$

Similarly, the computed error estimates for the approximations $\eta\overline{D^+F}(\eta) - \overline{F}(\eta)$, $\eta\overline{D^+D^+F}(\eta)$ and $\eta^2\overline{D^+D^+F}(\eta)$ to $(\eta f^{'} - f)(\eta)$, $\eta f^{''}(\eta)$ and $\eta^2 f^{''}(\eta)$, respectively, for all $N \geq 2048$, are

$f_0 = 3$

$$\|(\eta\overline{D^+F} - \overline{F}) - (\eta f^{'} - f)\|_{[0,\infty)} \leq 2.505N^{-0.86}$$
$$\|\eta(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 1.8N^{-0.85}$$
$$\|\eta^2(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 0.7N^{-0.86}$$

$f_0 = 6$

$$\|(\eta\overline{D^+F} - \overline{F}) - (\eta f^{'} - f)\|_{[0,\infty)} \leq 2.635N^{-0.86}$$
$$\|\eta(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 3.297N^{-0.85}$$
$$\|\eta^2(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 0.745N^{-0.86}$$

$f_0 = -0.25$

$$\|(\eta\overline{D^+F} - \overline{F}) - (\eta f^{'} - f)\|_{[0,\infty)} \leq 1.066N^{-0.86}$$
$$\|\eta(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 1.178N^{-0.86}$$
$$\|\eta^2(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 3.275N^{-0.86}$$

$f_0 = -0.5$

$$\|(\eta\overline{D^+F} - \overline{F}) - (\eta f^{'} - f)\|_{[0,\infty)} \leq 1.228N^{-0.86}$$
$$\|\eta(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 1.670N^{-0.86}$$
$$\|\eta^2(\overline{D^+D^+F} - f^{''})\|_{[0,\infty)} \leq 5.952N^{-0.86}.$$

We see from the above computed error estimates that, in all cases and at each point of $[0,\infty)$, the orders of convergence with respect to N, the number of mesh intervals used to solve Blasius' problem, are not less than 0.81. Similarly, in all cases, the error constants are at most 65.927. The worst cases occur for $f_0 = 6$.

5 Numerical Experiments for Prandtl's Problem

In this section we find reference numerical solutions of Prandtl's problem and computed error estimates for the scaled numerical solutions and their derivatives. In all of the numerical computations we use the specific values $a = 0.1, A = 1.1, B = 1.0$.

We construct the approximations $\mathbf{U}_B = (U_B, V_B)$ of the velocity components $\mathbf{u}_B$ of the self–similar solution of Prandtl's problem (P_P) by substituting the approximate expressions $\overline{F}$ and $\overline{D^+F}$ for f and f' respectively, into (1) and (2). Thus, for each (x, y) in the open quarter plane $\{(x, y) : x > 0, y > 0)\}$ we have

$$U_B(x, y) = \overline{D^+F}(\eta) \tag{14}$$

$$V_B(x, y) = \sqrt{\frac{1}{2xRe}}(\eta D^+F(\eta) - F(\eta)) \tag{15}$$

We call $\mathbf{U}_B = (U_B, V_B)$ the reference numerical solutions of the self–similar solution of Prandtl's problem (P_P).

We now assume that error estimates, for the scaled approximations $(U_B, \sqrt{Re}V_B)$ to $(u_P, \sqrt{Re}v_P)$, of the form

$$\|U_B - u_P\|_{\overline{\Omega}} \le C_1 N^{-p_1}$$

$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} \le C_2 N^{-p_2}$$

are valid for all $N > N_0$ where $p_1 > 0, p_2 > 0$, and the constants N_0, p_1, p_2, C_1, C_2 are independent of the total number of iterations M and the number of mesh intervals N used in the numerical solution of Blasius' problem.

The errors in the x-component U_B and the scaled y-component $\sqrt{Re}V_B$ of the velocity corresponding to $M \ge 8lnN$ satisfy

$$\|U_B - u_P\|_{\overline{\Omega}} = \|\overline{D^+F} - f'\|_{[0,\infty)}$$

$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} = \sqrt{Re}\|\sqrt{\frac{1}{2xRe}}[(\eta\overline{D^+F}(\eta) - \overline{F}(\eta)) - (\eta f' - f)]\|_{[0,\infty)}$$

$$\le \sqrt{5}\|(\eta\overline{D^+F}(\eta) - \overline{F}(\eta)) - (\eta f' - f)\|_{[0,\infty)}.$$

Then, using the experimental numerical technique described in [2] and the computed error estimates for the numerical solutions of Blasius' problem in the previous section, we obtain for all $N \ge 2048$ the following computed error estimates for the reference numerical solutions of Prandtl's problem

$f_0 = 3$

$$\|U_B - u_P\|_{\overline{\Omega}} \le 1.452N^{-0.86}$$
$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} \le 5.601N^{-0.86}$$

$f_0 = 6$

$$\|U_B - u_P\|_{\overline{\Omega}} \leq 2.925 N^{-0.86}$$
$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} \leq 5.89 N^{-0.86}$$

$f_0 = -0.25$

$$\|U_B - u_P\|_{\overline{\Omega}} \leq 0.202 N^{-0.86}$$
$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} \leq 2.38 N^{-0.86}$$

$f_0 = -0.5$

$$\|U_B - u_P\|_{\overline{\Omega}} \leq 0.345 N^{-0.87}$$
$$\sqrt{Re}\|V_B - v_P\|_{\overline{\Omega}} \leq 1.35 N^{-0.86}.$$

We see from these computed error estimates that, in all cases, the orders of convergence with respect to N, the number of mesh intervals used to solve Blasius' problem, are at least 0.86. Similarly, in all cases, the error constants are at most 5.89. The worst case occurs for $f_0 = 6$.

Substituting the appropriate expressions into (4), (5), (6) and (7) we obtain the approximations $D_x U_B, D_y U_B, D_x V_B, D_y V_B$ to the first order derivatives of the velocity components of the self–similar solution of Prandtl's problem (P_P), where

$$D_y U_B(\eta(x,y)) = \frac{\eta}{y}\overline{D_\eta^+ D_\eta^+ F}(\eta)$$

$$D_y V_B(\eta(x,y)) = \frac{\eta}{2x}\overline{D_\eta^+ D_\eta^+ F}(\eta)$$

$$D_x U_B(\eta(x,y)) = -D_y V_B(\eta(x,y))$$

$$D_x V_B(\eta(x,y)) = -\frac{1}{2x}(V_B + \sqrt{\frac{1}{2xRe}}\eta^2 \overline{D_\eta^+ D_\eta^+ F}(\eta)).$$

From the computed error estimates for the numerical solutions of Blasius' problem, in the previous section, we obtain for all $N \geq 2048$ the following computed error estimates for the reference scaled discrete derivatives of the velocity components

$$\frac{1}{\sqrt{Re}}\|D_y U_B - \frac{\partial u_P}{\partial y}\|_{\overline{\Omega}} = \sqrt{\frac{1}{2x}}\|\overline{D_\eta^+ D_\eta^+ F(\eta)} - f^{''}(\eta)\|_{[0,\infty)}$$

$$\leq \sqrt{5}\|\overline{D_\eta^+ D_\eta^+ F(\eta)} - f^{''}(\eta)\|_{[0,\infty)}$$

$$\|D_y V_B - \frac{\partial v_P}{\partial y}\|_{\overline{\Omega}} = \|D_x U_B - \frac{\partial u_P}{\partial x}\|$$

$$= \frac{\eta}{2x}\|\overline{D_\eta^+ D_\eta^+ F(\eta)} - f''(\eta)\|$$

$$\sqrt{Re}\|D_x V_B - \frac{\partial v_P}{\partial x}\|_{\overline{\Omega}} = \frac{\sqrt{Re}}{2x}(\|V_B - v_B\| + \sqrt{\frac{1}{2xRe}}\eta^2\|\overline{D_\eta^+ D_\eta^+ F(\eta)} - f^{''}(\eta)\|)$$

$$\leq \frac{1}{2x}(\sqrt{Re}\|V_B - v_B\| + \sqrt{\frac{1}{2x}}\eta^2\|\overline{D_\eta^+ D_\eta^+ F(\eta)} - f''(\eta)\|).$$

Then, for all $N \geq 2048$ we obtain the following estimates

$f_0 = 3$

$$\frac{1}{\sqrt{Re}}\|D_y U_B - \frac{\partial u_P}{\partial y}\|_{\overline{\Omega}} \leq 45.676N^{-0.86}$$
$$\|D_y V_B - \frac{\partial v_P}{\partial y}\|_{\overline{\Omega}} \leq 9N^{-0.85}$$
$$\sqrt{Re}\|D_x V_B - \frac{\partial v_P}{\partial x}\|_{\overline{\Omega}} \leq 35.831N^{-0.86}$$

$f_0 = 6$

$$\frac{1}{\sqrt{Re}}\|D_y U_B - \frac{\partial u_P}{\partial y}\|_{\overline{\Omega}} \leq 147.42N^{-0.86}$$
$$\|D_y V_B - \frac{\partial v_P}{\partial y}\|_{\overline{\Omega}} \leq 16.49N^{-0.85}$$
$$\sqrt{Re}\|D_x V_B - \frac{\partial v_P}{\partial x}\|_{\overline{\Omega}} \leq 37.78N^{-0.86}$$

$f_0 = -0.25$

$$\frac{1}{\sqrt{Re}}\|D_y U_B - \frac{\partial u_P}{\partial y}\|_{\overline{\Omega}} \leq 1.01N^{-0.86}$$
$$\|D_y V_B - \frac{\partial v_P}{\partial y}\|_{\overline{\Omega}} \leq 5.89N^{-0.86}$$
$$\sqrt{Re}\|D_x V_B - \frac{\partial v_P}{\partial x}\|_{\overline{\Omega}} \leq 48.52N^{-0.86}$$

$f_0 = -0.5$

$$\frac{1}{\sqrt{Re}}\|D_y U_B - \frac{\partial u_P}{\partial y}\|_{\overline{\Omega}} \leq 1.09N^{-0.86}$$
$$\|D_y V_B - \frac{\partial v_P}{\partial y}\|_{\overline{\Omega}} \leq 8.35N^{-0.86}$$
$$\sqrt{Re}\|D_x V_B - \frac{\partial v_P}{\partial x}\|_{\overline{\Omega}} \leq 73.3N^{-0.86}.$$

We see from these computed error estimates that, in all cases, the orders of convergence with respect to N, the number of mesh intervals used to solve Blasius' problem, are at least 0.85. Similarly, in all cases, the error constants are at most 73.3. The worst order of convergence occurs for $f_0 = 6$ and the worst error constant for $f_0 = -0.5$.

Remark on Navier-Stokes' Problem It is well known that incompressible flow past a plate $P = \{(x, 0) \in \Re^2 : x \geq 0\}$ with suction–blowing in the domain $D = \Re^2 \backslash P$ is governed by the Navier-Stokes equations

$$(P_{NS}) \begin{cases} \text{Find } \mathbf{u_{NS}} = (u_{NS}, v_{NS}),\ p_{NS} \text{ such that for all } (x, y) \in D \\ \mathbf{u_{NS}} \text{ satisfies the differential equations} \\ -\frac{1}{Re}\triangle \mathbf{u_{NS}} + \mathbf{u_{NS}} \cdot \nabla \mathbf{u_{NS}} = -\frac{1}{\rho}\nabla p_{NS} \\ \nabla \cdot \mathbf{u}_{NS} = 0 \\ \text{with the boundary conditions} \\ u_{NS}(x, 0) = 0, v_{NS} = v_0(x) \text{ for all } x \geq 0 \\ lim_{|y|->\infty}\mathbf{u_{NS}}(x, y) = lim_{x->-\infty}\mathbf{u_{NS}}(x, y) = (1, 0),\ \text{for all } x \in \Re \end{cases}$$

where $\mathbf{u_{NS}}$ is the velocity of the fluid, Re is the Reynolds number, ρ is the density of the fluid and p is the pressure. This is a nonlinear system of equations for the unknowns $\mathbf{u}_{NS}$, p_{NS}. It is known that the solution of (P_P) is a good approximation to the solution of (P_{NS}) in a subdomain excluding the leading edge region, provided that the flow remains laminar and no separation occurs. Moreover, as Re increases the difference between the solutions of problems (P_P) and (P_{NS}) decreases. This means that the reference solution of Prandtl's problem is the leading term in the solution of the above Navier–Stokes' problem.

6 Conclusion

For the problem of incompressible laminar flow past a thin flat plate with suction–blowing we construct a new numerical method for computing *reference* numerical solutions to the self–similar solution of the related Prandtl problem. The method generates global numerical approximations to the velocity components and their scaled derivatives for arbitrary values of the Reynolds number in the range $[1, \infty)$ on a domain including the boundary layer but excluding a neighbourhood of the leading edge. The method is based on Blasius' approach. Using an experimental error estimate technique it is shown that these numerical approximations are pointwise accurate and that they satisfy pointwise error estimates which are independent of the Reynolds number for the flow. The Reynolds–uniform orders of convergence of the reference numerical solutions, with respect to the number of mesh subintervals used in the solution of Blasius' problem, is at least 0.86 and the error constant is not more than 80. The number of iterations required to solve the nonlinear Blasius problem is independent of the Reynolds number. Therefore the method generates reference numerical solutions with ε–uniform errors of any prescribed accuracy.

References

1. H. Schlichting, *Boundary Layer Theory* , 7th Edition, McGraw Hill, 1951.
2. P. Farrell, A. Hegarty, J.J.H. Miller, E. O' Riordan, G.I. Shishkin, *Robust Computational Techniques for Boundary Layers* , Series in Applied Mathematics and Mathematical Computation (Eds. R J Knops and K W Morton), Chapman and Hall/CRC Press, 2000
3. D.F. Rogers, *Laminar Flow Analysis* , Cambridge University Press, 1992.
4. B. Gahan, J. J. H. Miller, G. I. Shishkin: Accurate numerical method for Blasius' problem for flow past a flat plate with mass transfer. Trinity College Dublin, Maths. Dept. Preprint TCDMATH 00–05.
5. D.J. Acheson, *Elementary Fluid Dynamics* , Oxford: Clarendon, 1990.