An experimental technique for computing parameter–uniform error estimates for numerical solutions of singular perturbation problems, with an application to Prandtl's problem at high Reynolds number *

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Abstract. *In this paper we describe an experimental technique for computing realistic values of the parameter–uniform order of convergence and error constant in the maximum norm associated with a parameter–uniform numerical method for solving singularly perturbed problems. We employ the technique to compute Reynolds–uniform error bounds in the maximum norm for the numerical solutions generated by a fitted–mesh upwind finite difference method applied to Prandtl's problem arising from laminar flow past a thin flat plate. Thus we illustrate the efficiency of the technique for finding realistic parameter–uniform error bounds in the maximum norm for the approximate solutions generated by numerical methods for which no theoretical error analysis is available.*

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1 Introduction

The numerical solutions and their maximum pointwise errors for standard numerical methods applied to singularly perturbed problems depend on the singular pertur-

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bation parameter ε and the number N of mesh points. The error in the numerical solution increases as ε decreases up to zero. Worse still, for such numerical methods, there are always values of ε for which the maximum pointwise error grows as the mesh is refined. Such behaviour of errors is normally regarded as unacceptable in computational practice and, in particular, for numerical solutions of singularly perturbed problems. On the other hand, if the method is ε–uniform the maximum pointwise errors satisfy ε–uniform error bounds and decrease independently of ε as N grows. For an ε–uniform method the following theoretical error bound is appropriate: there exist positive constants $N_0,\ C = C(N_0)$ and $p = p(N_0)$, all independent of N and ε, such that for all $N \geq N_0$

$$\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N} \leq C_p N^{-p} \tag{1}$$

where u_ε is the exact solution of the continuous problem, U_ε^N is a numerical approximation on a piecewise–uniform fitted mesh Ω_ε^N (for details see, e.g., [1]), $\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N}$ is the maximum pointwise error of the numerical method on the mesh Ω_ε^N, C_p is the ε–uniform error constant and p is the ε–uniform order of convergence. The quantities p and C_p are called the ε–uniform error parameters.

Theoretical error analysis often does not give sharp estimates for the ε–uniform order of convergence p and practically never gives a realistic estimate of the ε–uniform error constant C_p. An underestimate of the former and an overestimate of the latter may mistakenly be taken to imply that the ε–uniform method is of no practical use. Thus, to obtain realistic ε–uniform error estimates, we are forced to adopt an experimental technique. Moreover, in many applied problems, for which the theoretical analysis of ε-uniform convergence is at present unavailable, an experimental technique may be the only option available for estimating the error. In practice, for a specific computation, the values of N and ε lie in finite ranges $R_N = \{N : \underline{N} \leq N \leq \overline{N}\}$ and $R_\varepsilon = [\underline{\varepsilon}, \overline{\varepsilon}]$ respectively, and so we want to determine estimates of p and its associated C_p such that the ε–uniform error bound $C_p N^{-p}$ of the form (1) is as small as possible, for all values of N and ε in the given ranges. In what follows we describe an experimental technique for computing realistic estimates of the ε–uniform error parameters p and C_p for a specific problem, and we illustrate its efficiency by estimating the maximum pointwise ε–uniform error in numerical solutions of Prandtl's problem for flow past a flat plate at high Reynolds number.

2 An experimental technique for computing parameter–uniform error estimates

An experimental technique for estimating the ε–uniform order of convergence p, and the ε–uniform error constant C_p, for a specific numerical method for a given singularly perturbed problem, is described here for cases when the numerical solutions can be computed for several values of N and ε. As is typical for techniques of this type, the arguments are heuristic. We assume that, on appropriate meshes Ω_ε^N, the piecewise linear interpolants $\overline{U}_\varepsilon^N$ of the numerical solutions U_ε^N have been determined. Then, for all integers N satisfying $N,\, 2N \in R_N$ and for a finite set of values $\varepsilon \in R_\varepsilon$, the maximum pointwise two–mesh differences

$$D_\varepsilon^N = ||U_\varepsilon^N - \overline{U}_\varepsilon^{2N}||_{\Omega_\varepsilon^N} \tag{2}$$

are computed. From these values the ε–uniform maximum pointwise two–mesh differences

$$D^N = \max_{\varepsilon \in R_\varepsilon} D_\varepsilon^N \tag{3}$$

are formed for each available value of N satisfying $N,\, 2N \in R_N$. Approximations to the ε–uniform order of local convergence are defined, for all $N,\ 4N \in R_N$, by

$$p^N = \log_2 \frac{D^N}{D^{2N}} \tag{4}$$

and we take the computed ε–uniform order of convergence to be

$$p^* = \min_N p^N. \tag{5}$$

Note that

$$D_\varepsilon^N = ||U_\varepsilon^N - \overline{U_\varepsilon^{2N}}||_{\Omega_\varepsilon^N} \geq \Big| \, ||U_\varepsilon^N - u_\varepsilon||_{\Omega_\varepsilon^N} - ||u_\varepsilon - \overline{U_\varepsilon^{2N}}||_{\Omega_\varepsilon^N} \Big| \approx CN^{-p}(1-2^{-p}),$$

which is used to motivate the following definitions. Corresponding to the value of p^* in (5) we calculate the quantities

$$C_{p^*}^N = \frac{D^N N^{p^*}}{1-2^{-p^*}} \tag{6}$$

and we take the computed ε–uniform error constant to be

$$C_{p^*}^* = \max_N C_{p^*}^N. \tag{7}$$

The above definitions of the computed error parameters p and C_p supercede the similar definitions given in [2].

We can use the above technique in the following way. Suppose that we apply it to a problem from a problem class for which it is known theoretically that the method is ε–uniform. This yields realistic values $p^* > 0$ and $C_{p^*}^*$ from which we derive the error bound $C_{p^*}^* N^{-p^*}$. From this bound we can determine an appropriate choice of the value N, which guarantees that the method generates numerical solutions of any prerequisite accuracy. It also enables us to generate an error table for the numerical solutions, we can replace the unknown exact solution, in the expression for the error, by a numerical solution of known guaranteed accuracy generated on a sufficiently fine mesh. Such an error table gives us information about the actual, as opposed to the asymptotic, convergence behaviour of the numerical approximations to the solution of a specific problem.

On the other hand, if we are dealing with a specific problem for which it is not known if the method is ε–uniform or not, then the algorithm can yield a value $p^* > 0$ or $p^* \le 0$. If $p^* > 0$ we conclude that, in practice, the method is ε–uniform for this problem for the particular range of ε and N used in the computations. Also we can generate an error table as before. If $p^* \le 0$, then we conclude that the method is unlikely to be ε–uniform for any problem class that contains this problem. In the event that we are dealing with a problem for which we already know either the exact solution or an approximate solution with arbitrary guaranteed accuracy, then the above procedure enables us to compute an error table for the numerical solutions generated by the numerical method applied to this specific problem.

3 Prandtl's problem and its numerical solution

In this section we introduce a classical problem of fluid mechanics, for the numerical solutions of which by direct methods there is, at present, no theoretical error analysis in the maximum norm. This is Prandtl's problem arising from steady laminar flow of an incompressible fluid past a thin flat semi–infinite plate $P = \{(x,0) \in I\!R^2 : x \ge 0\}$ for all Reynolds numbers for which there is no separation from the plate. We are interested in flow on that region where parabolic boundary layers occur. Since the vertical component of the velocity tends to infinity as we approach the leading edge, we choose the computational domain $D = (a, A) \times (0, B)$, where a, A and B are fixed positive numbers independent of ε. In non–dimensional form Prandtl's problem is

$$(P_\varepsilon) \begin{cases} \text{Find } \mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon) \text{ such that for all } (x,y) \in D \\ -\varepsilon \dfrac{\partial^2 u_\varepsilon(x,y)}{\partial y^2} + \mathbf{u}_\varepsilon \cdot \nabla u_\varepsilon(x,y) = 0 \\ \nabla \cdot \mathbf{u}_\varepsilon(x,y) = 0 \\ \mathbf{u}_\varepsilon = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{B} \\ u_\varepsilon = u_\mathrm{P} \quad \text{on} \quad \Gamma_\mathrm{L} \cup \Gamma_\mathrm{T} \end{cases}$$

where $\varepsilon = 1/Re$, Γ_B, Γ_L and Γ_T are the bottom, left and top sides of D respectively. The boundary data u_P are the values of the self–similar solution of Prandtl's problem (see [3] for details). The solution of (P_ε) can be found by using a solution of the Blasius problem. In what follows we denote the Blasius solution of (P_ε) by $\mathbf{u}_\mathrm{B} = (u_\mathrm{B}, v_\mathrm{B})$.

We discretize (P_ε) with a direct numerical method composed of a standard upwind finite difference operator on an appropriate piecewise–uniform fitted mesh [4] condensing in the parabolic boundary layer. We take the piecewise–uniform fitted rectangular mesh $\Omega_\varepsilon^\mathbf{N}$ to be the tensor product of one dimensional meshes. That is, $\overline{\Omega}_\varepsilon^\mathbf{N} = \overline{\Omega}_u^{N_x} \times \overline{\Omega}_\varepsilon^{N_y}$ where $\mathbf{N} = (N_x, N_y)$, $\overline{\Omega}_u^{N_x}$ is a uniform mesh with N_x mesh intervals on the interval $[a, A]$ of the x–axis; $\overline{\Omega}_\varepsilon^{N_y}$ is a piecewise–uniform mesh with

N_y mesh intervals on the interval $[0, B]$ of the y–axis, such that the subinterval $[0, \sigma]$ and the subinterval $[\sigma, B]$ are both subdivided into $\frac{1}{2}N_y$ uniform mesh intervals. We choose

$$\sigma = \min\{\frac{1}{2}B,\ \sqrt{\varepsilon}\ln N\}.$$

This leads to the following nonlinear system of upwind finite difference equations for the approximate velocity $\mathbf{U}_\varepsilon = (U_\varepsilon, V_\varepsilon)$

$$(P_\varepsilon^{\mathbf{N}}) \begin{cases} \text{Find } \mathbf{U}_\varepsilon = (U_\varepsilon, V_\varepsilon) \text{ such that for all mesh points } (x_i, y_j) \in \Omega_\varepsilon^{\mathbf{N}} \\ -\varepsilon\delta_y^2 U_\varepsilon(x_i, y_j) + \big(\mathbf{U}_\varepsilon \cdot \mathbf{D}^-\big)U_\varepsilon(x_i, y_j) = 0 \\ \big(\mathbf{D}^- \cdot \mathbf{U}_\varepsilon\big)(x_i, y_j) = 0 \\ \mathbf{U}_\varepsilon = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B}} \\ U_\varepsilon = U_{\mathrm{B}}^{\mathbf{N}^\star} \quad \text{on} \quad \Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{T}} \end{cases}$$

where $\mathbf{D}^- = (D_x^-, D_y^-)$ is the first order backward difference operator and $U_{\mathrm{B}}^{\mathbf{N}^\star}$ is Blasius' approximation, computed on the fitted mesh with $N^\star$ subintervals (in each coordinate direction), to the self–similar solution of Prandtl's problem from [3]. In computations $N^\star$ is taken sufficiently large to guarantee approximations of the required accuracy. Note that $U_{\mathrm{B}}^{\mathbf{N}^\star}$ is an ε–uniform approximation to u_{P}.

Since $(P_\varepsilon^{\mathbf{N}})$ is a nonlinear finite difference method, we use the following ε–uniform nonlinear solver for computing ε–uniformly convergent approximations to its solution

$$(A_\varepsilon^{\mathbf{N}}) \begin{cases} \text{With the boundary condition } \ U_\varepsilon^{M_i} = U_{\mathrm{B}}^{\mathbf{N}^\star} \ \text{ on } \ \Gamma_{\mathrm{L}}, \\ \text{for each } i,\ 1 \le i \le N_x, \text{ use the initial guess } \ \mathbf{U}_\varepsilon^0|_{X_i} = \mathbf{U}_\varepsilon^{M_{i-1}}|_{X_{i-1}} \\ \text{and for } \ m = 1, \dots, M_i \ \text{ solve the following} \\ \text{two point boundary value problem for } \ U_\varepsilon^m(x_i, y_j) \\ (-\varepsilon\delta_y^2 + \mathbf{U}_\varepsilon^{m-1} \cdot \mathbf{D}^-)U_\varepsilon^m(x_i, y_j) = 0, \quad 1 \le j \le N_y - 1 \\ \text{with the boundary conditions } \ U_\varepsilon^m = U_{\mathrm{B}}^{\mathbf{N}^\star} \ \text{ on } \ \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{T}}, \\ \text{and the initial guess } \ V_\varepsilon^0|_{X_1} = 0. \\ \text{Then solve the following initial value problem for } V_\varepsilon^m(x_i, y_j) \\ (\mathbf{D}^- \cdot \mathbf{U}_\varepsilon^m)(x_i, y_j) = 0 \\ \text{with the initial condition } \ V_\varepsilon^m = 0 \ \text{ on } \ \Gamma_{\mathrm{B}}. \\ \text{Continue to iterate between the equations for } \mathbf{U}_\varepsilon^m \text{ until } m = M_i\,, \\ \text{where } M_i \ \text{ is such that} \\ \max\big(\big|U_\varepsilon^{M_i} - U_\varepsilon^{M_i - 1}\big|_{X_i},\ \frac{1}{\sqrt{\varepsilon}}\big|V_\varepsilon^{M_i} - V_\varepsilon^{M_i-1}\big|_{X_i} \big) \le tol. \end{cases}$$

where $X_i = \{(x_i, y_j),\ 1 \le j \le N_y\}$. Henceforth, for notational simplicity, we suppress explicit mention of the iteration superscript M_i, thus $\mathbf{U}_\varepsilon$ denotes the solution generated by $(A_\varepsilon^{\mathbf{N}})$. Since there are no known theoretical results concerning the convergence in the maximum norm of the solutions $\mathbf{U}_\varepsilon$ of $(P_\varepsilon^{\mathbf{N}})$ to the solution $\mathbf{u}_\varepsilon$ of (P_ε), we have no theoretical estimate of the pointwise error $(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)(x_i, y_j)$ and we are forced to adopt an experimental technique.

4 Application of the experimental technique

In the computations in this section we take

$$a = 0.1, \quad A = 1.1, \quad B = 1,$$

$N_x = N_y = N$, $tol = 10^{-6}$ and $N^\star = 8192$ which guarantees sufficiently small ε–uniform error bounds. The results of applying the experimental error analysis technique of §2 to the numerical solutions generated in §3 are summarized in Table 1 for the errors in the approximations U_ε of the horizontal velocity u_ε and in Table 2 for the errors in the approximations $\frac{1}{\sqrt{\varepsilon}}V_\varepsilon$ of the scaled vertical velocity $\frac{1}{\sqrt{\varepsilon}}v_\varepsilon$.

Table 1: The values of D^N, p^N, p^* and $C_{p^*}^N$ given by the algorithm in §2 for U_ε obtained by the numerical method $(A_\varepsilon^{\mathbf{N}})$ for various values N

	Number of intervals N					
	8	16	32	64	128	256
D^N	0.135D+00	0.721D-01	0.313D-01	0.171D-01	0.838D-02	0.467D-02
p^N	0.91	1.20	0.87	1.03	0.84	$p^* = 0.84$
$C_{0.84}^N$	$C_{p^*}^* = 1.75$	1.68	1.30	1.27	1.12	1.12

Table 2: The values of D^N, p^N, p^* and $C_{p^*}^N$ given by the algorithm in §2 for $\frac{1}{\sqrt{\varepsilon}}V_\varepsilon$ obtained by the numerical method $(A_\varepsilon^{\mathbf{N}})$ for various values of N

	Number of intervals N					
	8	16	32	64	128	256
D^N	0.353D+01	0.183D+01	0.957D+00	0.525D+00	0.275D+00	0.156D+00
p^N	0.95	0.94	0.87	0.93	0.82	$p^* = 0.82$
$C_{0.82}^N$	$C_{p^*}^* = 44.8$	41.0	37.9	36.7	33.9	33.9

From the entries in Tables 1 and 2 respectively, we see that the values of p^N and $C_{p^*}^N$ are stabilized with growing N. We conclude that, for all $N \ge N_0$ with $N_0 = 8$, the errors $U_\varepsilon - u_\varepsilon$ and $\frac{1}{\sqrt{\varepsilon}}(V_\varepsilon - v_\varepsilon)$ satisfy the following ε–uniform bounds

$$\|U_\varepsilon - u_\varepsilon\|_{\overline{\Omega}} \;\le\; 1.75\, N^{-0.84}.$$

and

$$\frac{1}{\sqrt{\varepsilon}}\|V_\varepsilon - v_\varepsilon\|_{\overline{\Omega}} \;\le\; 44.8\, N^{-0.82}.$$

In the first row of Tables 3 and 4 we give values of these upper bounds for various values of N. In the second row of each of these tables we quote from [3, Chapter 11] the values of the ε–uniform maximum pointwise differences $\|U_\varepsilon - U_B^{8192}\|$ and $\frac{1}{\sqrt{\varepsilon}}\|V_\varepsilon - V_B^{8192}\|$, respectively, which are known to be accurate estimates (correct to 4 and 2 significant decimal figures, respectively) of the actual errors. Comparing the first and second rows in each table, we see that each entry in the first row is greater than the corresponding entry in the second row, but never more than twice its value. We conclude from this that the values in the first rows are realistic upper bounds for the corresponding values in the second rows. It should be noted that ratios between the values of $\|U_\varepsilon - U_B^{8192}\|$ and $1.75\, N^{-0.84}$ from Table 3, and between $\frac{1}{\sqrt{\varepsilon}}\|V_\varepsilon - V_B^{8192}\|$ and $44.8\, N^{-0.82}$ from Table 4 are stabilized with growing N.

It is remarkable that these realistic upper bounds are obtained by applying the simple experimental error analysis technique of the present paper to the numerical solutions generated by the above direct numerical method applied to this nonlinear system of partial differential equations.

Table 3: Computed ε–uniform maximum pointwise error bound $C^*_{p^*} N^{-p^*}$ for $\|U_\varepsilon - u_\varepsilon\|$ from the algorithm compared to the computed error from a fine computed Blasius solution for various values of N

	Number of intervals N						
	8	16	32	64	128	256	512
$1.75\, N^{-0.84}$	0.305	0.170	0.095	0.053	0.030	0.017	0.009
$\|U_\varepsilon - U_B^{8192}\|$	0.220	0.115	0.062	0.034	0.019	0.011	0.006

Table 4: Computed ε–uniform maximum pointwise error bound $C^*_{p^*} N^{-p^*}$ for $\frac{1}{\sqrt{\varepsilon}}\|V_\varepsilon - V_B^{8192}\|$ from the algorithm compared to the computed error from a fine computed Blasius solution for various values of N

	Number of intervals N						
	8	16	32	64	128	256	512
$44.8\, N^{-0.82}$	8.14	4.61	2.61	1.48	0.838	0.475	0.269
$\frac{1}{\sqrt{\varepsilon}}\|V_\varepsilon - V_B^{8192}\|$	4.24	2.67	1.54	0.893	0.523	0.309	0.183

5 Conclusion

An experimental technique to compute realistic values of the parameter–uniform order of convergence and error constant in the maximum norm associated with a parameter–uniform numerical method for solving singularly perturbed problems was described. It was then used to determine Reynolds–uniform error bounds in the maximum norm for the numerical solutions generated by a fitted–mesh upwind finite difference method applied to Prandtl's problem arising from laminar flow past a thin flat plate. We have thus illustrated the efficiency of the technique for finding realistic parameter–uniform error bounds when no theoretical error analysis is available.

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