A ROBUST LAYER–RESOLVING NUMERICAL METHOD FOR A FREE CONVECTION PROBLEM

Jocelyn Étienne, John J.H. Miller, Grigorii I. Shishkin

Abstract. We consider free convection near a semi-infinite vertical flat plate. This problem is singularly perturbed with perturbation parameter Gr, the Grashof number. Our aim is to find numerical approximations of the solution in a bounded domain, which does not include the leading edge of the plate, for arbitrary values of $Gr \geq 1$. Thus, we need to determine values of the velocity components and temperature with errors that are Gr–independent. We use the Blasius approach to reformulate the problem in terms of two coupled non-linear ordinary differential equations on a semi– infinite interval. A novel iterative numerical method for the solution of the transformed problem is described and numerical approximations are obtained for the Blasius solution functions, their derivatives and the corresponding physical velocities and temperature. The numerical method is Gr–uniform in the sense that error bounds of the form C_pN^{-p}, where C_p and p are independent of the Gr, are valid for the interpolated numerical solutions. The numerical approximations are therefore of controllable accuracy.

Keywords and Phrases: Robust method, Layer-resolving, Boundary layer, Fitted mesh, Free convection, Coupled nonlinear equations

1 THE FREE CONVECTION PROBLEM

A free or natural convection flow occurs when a fluid at rest, subjected to a body force such as gravity, is near an object at a different temperature. The heat transfer between the object and the fluid causes an increase or a decrease in the fluid density at the surface of the object, and thus generates an unbalancing body force. The fluid near the surface is accelerated, and a boundary layer develops. We study this problem for a two-dimensional, steady flow near a semi-infinite flat plate. This involves an interesting and typical system of singularly perturbed partial differential equations.

Our goal is to construct a numerical method for this problem in a bounded domain that does not include the leading edge of the plate. Because the solution we seek is self-similar, using Blasius' approach, we can reduce the problem to the numerical solution of a coupled system of nonlinear ordinary differential equations. We require that the numerical approximations generated by our method converge to the exact solution with an order of convergence that is independent of the Grashof number Gr for all $Gr \geq 1$. We refer to a numerical method with this property as a layer–resolving method. No standard numerical method exists, which fulfills this requirement.

1.1 PHYSICAL DESCRIPTION

We consider a semi-infinite vertical flat plate in an incompressible fluid. We assume that the density of the fluid varies linearly with the temperature and that its other properties are constant. The plate is heated to the temperature θ_1, while the fluid temperature away from the plate is θ_∞.
For $\theta_1 > \theta_\infty$, the heat transfer into the fluid decreases its density in a small region around the plate, resulting in an upward motion of the fluid. Since motion in the fluid results only from this heat transfer, we assume that the fluid away from the surface is not affected by this upward motion.
The governing equations are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= g\bar{\beta}(\theta - \theta_\infty) + \nu\frac{\partial^2 u}{\partial y^2} \\ u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} &= \alpha\frac{\partial^2 \theta}{\partial y^2} \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{array}{lll} v = u = 0, & \theta = \theta_1 & \text{for} \quad y = 0 \\ u \to 0, & \theta \to \theta_\infty & \text{for} \quad y \to \infty \end{array}$$

When we non-dimensionalise these equations we obtain

$$\begin{aligned} \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} &= 0 \\ -\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} &= Gr \cdot \hat{\theta} \\ -\frac{1}{Pr}\frac{\partial^2 \hat{\theta}}{\partial \hat{y}^2} + \hat{u}\frac{\partial \hat{\theta}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{\theta}}{\partial \hat{y}} &= 0 \end{aligned} \tag{2}$$

with the boundary conditions

$$\begin{array}{lll} \hat{v} = \hat{u} = 0, & \hat{\theta} = 1 & \text{for} \quad \hat{y} = 0 \\ \hat{u} \to 0, & \hat{\theta} \to 0 & \text{for} \quad \hat{y} \to \infty \end{array}$$

where the Grashof and Prandtl numbers have their usual definitions

$$Gr = \frac{g\bar{\beta}L^3(\theta_1 - \theta_\infty)}{\nu^2} \tag{3}$$

$$Pr = \frac{\nu}{\alpha} \tag{4}$$

1.2 BLASIUS' FORMULATION

The problem is now transformed using Blasius' technique to a one dimensional problem. For a complete description of this we refer the reader to [2].
The transformed problem involves the unique dimensionless variable

$$\eta = \left(\frac{Gr}{4}\right)^{\frac{1}{4}} \frac{\hat{y}}{\hat{x}^{1/4}}$$

and two dimensionless functions f and t, which are related to the physical velocities and temperature through the following relations

$$\hat{\theta}(\hat{x}, \hat{y}) = t(\eta) \tag{5}$$

$$\hat{u}(\hat{x}, \hat{y}) = 4\left(\frac{Gr}{4}\right)^{\frac{1}{2}} \hat{x}^{\frac{1}{2}} f'(\eta) \tag{6}$$

$$\hat{v}(\hat{x}, \hat{y}) = \left(\frac{Gr}{4}\right)^{\frac{1}{4}} \frac{1}{\hat{x}^{\frac{1}{4}}} (\eta f'(\eta) - 3f(\eta)) \tag{7}$$

In terms of these functions, the governing equations become

$$t'' + 3Pr \cdot ft' = 0 \tag{8}$$

$$f''' + 3ff'' - 2f'^2 + t = 0 \tag{9}$$

with the boundary conditions

$$f(0) = f'(0) = 0, \qquad t(0) = 1$$
$$f'(\eta \to \infty) \to 0, \qquad t(\eta \to \infty) \to 0$$

This is again a singularly perturbed problem. Our aim is to find numerical approximations of the velocity components and temperature in a bounded domain, which does not include the leading edge of the plate. Because we want this solution for arbitrary values of $Gr \geq 1$, we need to determine numerical approximations to the solution of the above problem at each point η of the semi–infinite interval $I = (0, \infty)$.

2 LAYER-RESOLVING METHOD FOR BLASIUS' PROBLEM

The equations obtained by Blasius are posed on the semi-infinite interval I, and there are boundary conditions at infinity. It is obvious that the problem cannot be solved numerically in this form. A standard alternative approach is to satisfy these boundary conditions by using an iterative method involving additional boundary conditions at $\eta = 0$ (see [2]).

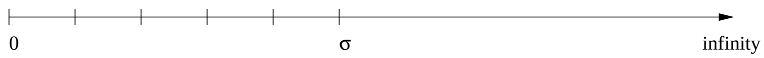

Figure 1: Mesh on semi-infinite domain for Blasius' problem.

Here we use the method described in [1, Chap.11], which yields a solution on the whole of I. We divide I into two subintervals, $[0, L]$ and $[L, +\infty)$. On the first we define a discrete problem on a uniform mesh, and on the second we define an affine extension of each function using the boundary conditions. Thus, for $\overline{T}$, the interpolated function of the discrete approximation of t, we have $\overline{T}(\eta \to +\infty) = 0$ and therefore we take $\overline{T}(\eta \geq L) = 0$. Similarly, for $\overline{F}$ we know that $\frac{\partial}{\partial \eta} f(\eta \to +\infty) = 0$ and we take $\overline{F}(\eta > L) = F(L)$, where this latter value is obtained from the solution of the discrete problem on $[0, L]$.
In order to guarantee that the method is Gr–uniform, a careful choice of the point L is of course crucial. We take $L_N = \ln N$ and on the subinterval $[0, L_N]$ we define the uniform mesh $I^N = \{x_i = iN^{-1} \ln N : 0 \leq i \leq N\}$. This choice is motivated by the discussion in [1] for a simpler problem. The computations described in what follows show that in practice the resulting method is L–uniform.
We introduce the discrete problem

$$(P_{L_N}) \begin{cases} \forall i \in \{1 \cdots N-1\},\ \delta^2 T_i + 3Pr \cdot F_i D^- T_i = 0 \\ \forall i \in \{2 \cdots N-1\},\ D^- \delta^2 F_i + 3F_i \delta^2 F_i - 2(D^- F_i)^2 = -T_i \\ \text{with} \\ \quad F_0 = 0, \quad D^+ F_0 = 0, \quad D^- F_N = 0 \\ \quad T_0 = 1, \quad T_N = 0 \end{cases}$$

where D^+ and D^- are the forward and backward first order finite difference operators, δ^2 is the centred second order operator and, for any mesh function G, $G_i = G(x_i)$ for all $x_i \in I^N$.

We need to linearize these equations. The natural first attempt is the linearization

$$\delta^2 T^m + 3Pr \cdot F^{m-1} D^- T^m = 0 \tag{10}$$
$$D^- \delta^2 F^m + 3F^{m-1} \delta^2 F^m - 2(D^- F^m)^2 = -T^m \tag{11}$$

which we iterate until uniform convergence is achieved for a given tolerance. But, for a fixed N, the iterates do not converge as m grows. In fact we have

$$\lim_{m\to\infty} F^{2m} \neq \lim_{m\to\infty} F^{2m+1} \text{ and } \lim_{m\to\infty} T^{2m} \neq \lim_{m\to\infty} T^{2m+1}$$

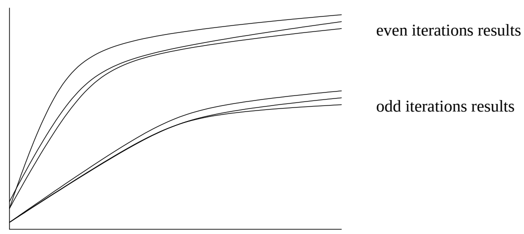

Figure 2: Oscillations of F function (sketch)

Figure 2 shows a sketch of the oscillations of the iterates F^m.
To prevent these oscillations we use previously computed values of F by introducing the auxiliary variables

$$F_c{}^{m-1} = \frac{1}{2}F^{m-1} + \frac{1}{2}F_c{}^{m-2}. \tag{12}$$

It is clear that $F_c{}^{m-1}$ depends on all previously computed values of F. Since it is much less subject to oscillation than F^{m-1}, we use it to replace F^{m-1} in equation (10). The resulting method yields good results for all physically relevant values of the Prandtl number.

3 CONVERGENCE OF THE METHOD

We use the above method to compute approximations for values of N in the range 128 to 32768, and work in quadruple-precision in order to obtain significant error bounds. We study the convergence of the resulting sequence of solutions, and their first and second derivatives, using the experimental error analysis technique described in [1].
All of the computations in this section are carried out for $Pr = 0.72$, which is the value of the Prandtl number for air. Other experiments within a physically relevant range of Pr yield similar results.
As in [1], for any mesh functions G^N on the mesh I^N, we define the maximum pointwise error

$$E^N = \left\| G^N - \overline{G}^{N_{\max}} \right\|_{I^N}$$

the two–mesh difference

$$\bar{D}^N = \left\| \overline{G}^N - \overline{G}^{2N} \right\|_{I^N \cup I^{2N}}$$

and the order of convergence

$$\bar{p}^N = \log_2 \frac{\bar{D}^N}{\bar{D}^{2N}}$$

where $\overline{G}^N$ is the interpolated function corresponding to the G^N mesh function and $N_{\max} = 32768$ is the largest value of N used in the computations. The computed values of the error parameters C and p are defined in an analogous way to those in [1]. From the numerical results in the first and last two rows of Tables 1–3 we see that, in practice, the method is robust and layer–resolving in the sense that it is L–uniform and that the L–uniform order of convergence of the numerical approximations of f and t, and their derivatives, is better than 0.78 for all $N \geq 512$.

N	128	256	512	1024	2048	4096	8192	16384
$E^N(F)$	0.020684	0.010497	0.005228	0.002543	0.001201	0.000547	0.000236	0.000092
$E^N(T)$	0.005699	0.002864	0.001672	0.000938	0.000510	0.000268	0.000134	0.000061
$\bar{D}^N(F)$	0.014051	0.006856	0.003334	0.001606	0.000761	0.000353	0.000159	0.000069
$\bar{D}^N(T)$	0.003695	0.001468	0.000738	0.000429	0.000242	0.000134	0.000073	0.000040
$\bar{p}^N(F)$	1.035344	1.039840	1.053815	1.077516	1.109775	1.150155	1.200088	1.264512
$\bar{p}^N(T)$	1.331509	0.991703	0.781964	0.825135	0.853144	0.872730	0.886969	0.897438

N	128	256	512	1024	2048	4096	8192	16384
$E^N(D^+F)$	0.008547	0.003847	0.001754	0.000800	0.000363	0.000161	0.000070	0.000057
$E^N(D^+T)$	0.006304	0.003760	0.002147	0.001190	0.000643	0.000337	0.000176	0.000087
$\bar{D}^N(D^+F)$	0.020668	0.012189	0.007557	0.004147	0.002438	0.001319	0.000735	0.000395
$\bar{D}^N(D^+T)$	0.007780	0.004259	0.002350	0.001287	0.000701	0.000380	0.000205	0.000110
$\bar{p}^N(D^+F)$	0.761882	0.689707	0.865701	0.766244	0.885989	0.843121	0.896398	0.886538
$\bar{p}^N(D^+T)$	0.869365	0.857542	0.869375	0.875633	0.883201	0.890366	0.896959	0.902954

N	128	256	512	1024	2048	4096	8192	16384
$E^N(\delta^2 F)$	0.017785	0.010544	0.006069	0.003395	0.001850	0.000977	0.000491	0.000224
$E^N(\delta^2 T)$	0.007241	0.004030	0.002212	0.001197	0.000624	0.000325	0.000163	0.000066
$\bar{D}^N(\delta^2 F)$	0.042189	0.023946	0.014474	0.008000	0.004551	0.002483	0.001362	0.000735
$\bar{D}^N(\delta^2 T)$	0.011175	0.006820	0.003682	0.002133	0.001136	0.000641	0.000338	0.000187
$\bar{p}^N(\delta^2 F)$	0.817064	0.726389	0.855304	0.813816	0.873976	0.866127	0.891145	0.893042
$\bar{p}^N(\delta^2 T)$	0.712518	0.889433	0.787396	0.908475	0.825205	0.924485	0.851711	0.934954

Table 3: Computed maximum pointwise error E^N, computed two-mesh difference $\bar{D}^N$ and computed order of convergence p^N for $\overline{\delta^2 F}$ and $\overline{\delta^2 T}$ in quadruple precision arithmetic.

3.1 Computed error bounds for Blasius' functions

The results in [1] for a simpler problem suggest that we can expect error bounds of the form CN^{-p}, where C and p are independent of Gr. Considering values of $N \geq 512$, the experimental error analysis described in [1, chap.8] yields computed values for p and C. Applying this technique to the present problem we obtain the following *a posteriori* error bounds for the functions $\overline{F}$, $\overline{T}$ and their derivatives, for all $N \geq 512$

$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{F} - f\right)(\eta)\right| \leq 4.607N^{-1.054} \tag{13}$$
$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{T} - t\right)(\eta)\right| \leq 2.320N^{-0.782}$$
$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{D^+F} - f'\right)(\eta)\right| \leq 2.182N^{-0.766}$$
$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{D^+T} - t'\right)(\eta)\right| \leq 1.175N^{-0.869}$$
$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{\delta^2F} - f''\right)(\eta)\right| \leq 5.386N^{-0.814}$$
$$\max_{\eta\in[0,+\infty)} \left|\left(\overline{\delta^2T} - t''\right)(\eta)\right| \leq 1.188N^{-0.787}$$

These computed error bounds show experimentally that our numerical method is robust and layer–resolving for N in the range 512 to 32768.

3.2 Error in the physical quantities

We return now to the original non-dimensionalised problem. We want to compute the error for the velocities and temperature on a bounded sub-domain $\overline{\Omega} = [0.1, 1] \times [0, 1]$ of the non-dimensionalised semi-infinite domain. The choice of the interval $[0.1, 1]$ for the variable $\hat{x}$ is required because of the singularity in the velocity components $\hat{u}$, $\hat{v}$ and their derivatives at the point $\hat{x} = 0$.
We use the relations between these quantities and Blasius' functions described in section 1.2. We see that the velocity components $\hat{u}$ and $\hat{v}$ respectively behave like $Gr^{\frac{1}{2}}$ and $Gr^{\frac{1}{4}}$. Therefore, we need to scale the components by these factors in order to obtain quantities that are bounded uniformly with respect to the Grashof number. Graphs of the resulting approximate scaled velocity components and temperature on $[0.1, 1] \times [0, 1]$ are shown in Figures 3–5 for $Gr = 10^5$ and $N = 32768$. We see that a boundary layer in each physical variable arises on the boundary of the plate.
The corresponding scaled errors in the physical quantities are

$$\begin{aligned} &Gr^{-\frac{1}{2}} \max_{\overline{\Omega}} \left|(U - \hat{u})(\hat{x}, \hat{y})\right| \\ &= \max_{\substack{(\hat{x},\hat{y})\in\overline{\Omega} \\ \eta=\eta(\hat{x},\hat{y})}} \left|2\hat{x}^{\frac{1}{2}} \left(\overline{D^+F} - f\right)(\eta)\right| \\ &\leq 2 \max_{\substack{(\hat{x},\hat{y})\in\overline{\Omega} \\ \eta=\eta(\hat{x},\hat{y})}} \left|\left(\overline{D^+F} - f\right)(\eta)\right| \end{aligned} \tag{14}$$

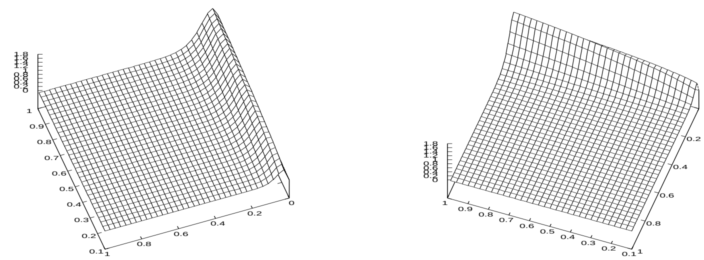

Figure 3: The approximate scaled horizontal velocity for the free convection problem on $[0.1, 1] \times [0, 1]$ with $Gr = 10^5$ generated with N=32768.

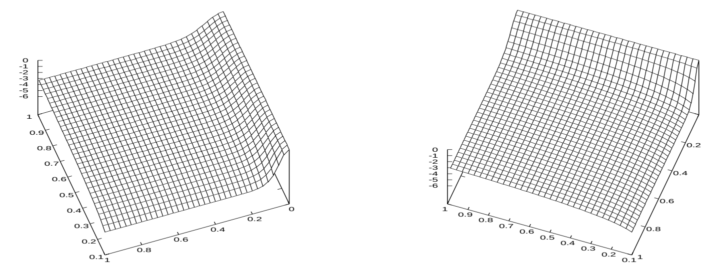

Figure 4: The approximate scaled vertical velocity for the free convection problem on $[0.1, 1] \times [0, 1]$ with $Gr = 10^5$ generated with N=32768.

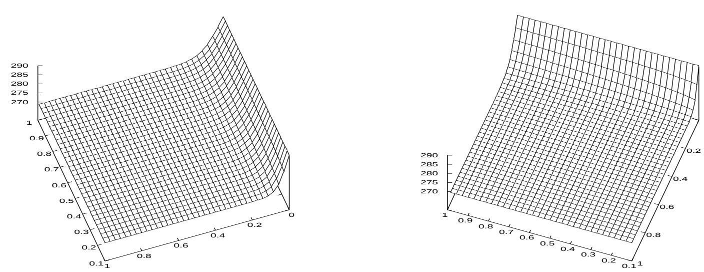

Figure 5: The approximate temperature for the free convection problem on $[0.1, 1] \times [0, 1]$ with $Gr = 10^5$ generated with N=32768.

$$Gr^{-\frac{1}{4}} \max_{\overline{\Omega}} |(V - \hat{v})(\hat{x}, \hat{y})| \tag{15}$$
$$= \max_{\substack{(\hat{x},\hat{y})\in\overline{\Omega} \\ \eta=\eta(\hat{x},\hat{y})}} \left| \frac{1}{\hat{x}^{\frac{1}{4}}\sqrt{2}} \left(\eta\overline{D^+F}(\eta) - 3\overline{F}(\eta) - (\eta f'(\eta) - 3f(\eta)) \right) \right|$$
$$\leq 1.26 \max_{\substack{(\hat{x},\hat{y})\in\overline{\Omega} \\ \eta=\eta(\hat{x},\hat{y})}} \left| \eta\overline{D^+F}(\eta) - 3\overline{F}(\eta) - (\eta f'(\eta) - 3f(\eta)) \right|$$

$$\max_{\overline{\Omega}} \left| \left(\Theta - \hat{\theta} \right)(\hat{x}, \hat{y}) \right| = \max_{\substack{(\hat{x},\hat{y})\in\overline{\Omega} \\ \eta=\eta(\hat{x},\hat{y})}} \left| \left(\overline{T} - t \right)(\eta) \right| \tag{16}$$

We see that we need to estimate the additional quantity $\eta\overline{D^+F}(\eta) - 3\overline{F}(\eta)$. The required numerical results are given in Table 4.

N	128	256	512	1024	2048	4096	8192	16384
$\bar{D}^N$	0.042154	0.020567	0.010003	0.004819	0.002283	0.001058	0.000477	0.000207
$\bar{p}^N$	1.035344	1.039840	1.053815	1.077516	1.109775	1.150155	1.200088	1.247754

Table 4: Computed two-mesh difference $\bar{D}^N$ and computed order of convergence p^N for $\eta\overline{D^+F} - 3\overline{F}$ in quadruple precision arithmetic.

With these results, and those from the previous section, we find the following computed scaled error bounds for the physical quantities

$$\begin{aligned} Gr^{-\frac{1}{2}} \|U - \hat{u}\|_{\overline{\Omega}} &\leq 4.37N^{-0.766} \\ Gr^{-\frac{1}{4}} \|V - \hat{v}\|_{\overline{\Omega}} &\leq 17.4N^{-1.053} \\ \left\| \Theta - \hat{\theta} \right\|_{\overline{\Omega}} &\leq 2.32N^{-0.782} \end{aligned} \tag{17}$$

These computed error bounds show that the boundary layers have been successfully resolved. We remark that we can use the same approach to generate similar approximations to the derivatives of the physical variables.

4 CONCLUSION

For free convection on a semi-infinite vertical flat plate, Grashof uniform numerical approximations to the velocity components and temperature have been generated in a bounded domain, which does not include the leading edge of the plate, for arbitrary values of Gr, using the Blasius formulation. Analysis of the numerical approximations shows that this numerical method is robust and layer-resolving. It follows that numerical approximations of controllable accuracy, with errors independent of the value of the Grashof number, can be computed with this method.

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Jocelyn Étienne
John J.H. Miller
Department of Mathematics
University of Dublin
Trinity College
Dublin 2,
Ireland

Grigorii I. Shishkin
Institute of Mathematics and Mechanics
Russian Academy of Sciences, Ural Branch
Ekaterinburg 620219
Russia